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Informational Updates and the Derivative Pricing Kernel

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Abstract: It is common in financial markets for market makers to offer prices on derivative instruments even though they are uncertain about the underlying asset's value. This paper studies the mathematical problem that arises as a result. Derivatives are priced in the risk-neutral framework, so as the market maker acquires more information about the underlying asset, the change of measure for transition to the risk-neutral framework (the pricing kernel) evolves. This evolution takes a precise form when the market maker is Bayesian. It is shown that Bayesian updates can be characterized as additional informational drift in the underlying asset's stochastic process. With Bayesian updates, the change of measure needed for pricing derivatives is two-fold: the first change is from the prior probability measure to the posterior probability measure, and the second change is from the posterior probability measure to the risk-neutral measure. The relation between the regular pricing kernel and the pricing kernel under this two-fold change of measure is characterized.

Keywords: pricing kernel; Bayesian learning; derivatives; risk-neutral measure; Girsanov transform; mathematical finance

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1. Introduction

Imagine a market maker who has to price a derivative instrument on an underlying asset even as she is learning about the asset's value. This paper studies the mathematical problem of how the derivative pricing kernel evolves as the market maker updates her belief about the underlying asset.

The market maker (she/her) is presumed to be a Bayesian agent, so she uses the Bayes' formula to update her belief about the underlying asset's value. The underlying asset is assumed to obey a given stochastic process prior to the Bayesian update, but after the update, the stochastic process of the asset changes. Our first concern is to examine this change. The usual technique to analyze the change in a stochastic process under a change of measure is the Girsanov class of theorems, but the situation is complicated in our case by the fact that, technically, the prior measure and the posterior measure live on different spaces. The posterior measure inhabits a product space formed from the prior measure and conditionalizing random variable's state spaces. A Bayesian update bridges these spaces and guarantees that the posterior measure is absolutely continuous with respect to the prior measure in an appropriate sense. This allows us to employ a version of the Girsanov theorem to transport the underlying asset's stochastic process from the prior measure to the posterior measure. The change of measure generates additional quadratic covariation, and we provide a characterization of this extra informational drift from Bayesian update in Theorem 1.

Derivatives are priced under a risk-neutral measure when hedging is continuous and markets are arbitrage-free, but the everyday world of trading only generates the physical probability measure. In finance theory, one therefore undertakes a change of measure to move from the world of the physical probability measure to the risk-neutral world. This change of measure is termed the *pricing kernel*. With a Bayesian update, the change of measure is two-fold: the first change of measure moves us from the prior probability measure to the posterior probability measure, and a second change of measure then moves us from the posterior measure to the risk-neutral measure. We characterize the relation between the regular change of measure to the risk-neutral world and this two-fold change of measure to the risk-neutral world in Theorem 2.

From an applied point of view, the contribution of the paper is that it shows how one may incorporate Bayesian informational updates in the underlying into the risk-neutral derivative pricing framework. The paper clarifies a number of mathematical subtleties that arise in the process, and the technique espoused in the paper has many potential applications in derivative modeling. In the actual world of trading, derivative price-setters need to constantly update their priors as they incorporate new information about the world into their models. Such updates are most commonly modeled as Bayesian updates. These updates change the pricing kernel that market makers use to price derivative instruments, and the approach in this paper provides a conceptual framework to address such changes.

Related literature: Although the domain of application is different, this paper has close connections to the literature on the enlargement of filtrations in probability theory and mathematical finance. For the most part, the focus in that literature has been on finding conditions under which, for two filtrations $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ satisfying $\mathcal{F}_t \subset \mathcal{G}_t$, any $\mathcal{F}$ martingale remains a $\mathcal{G}$ semi-martingale. In the standard interpretation, $\mathcal{F}_t$ represents the filtration of the market while $\mathcal{G}_t$ is the filtration of an insider, i.e., an investor who possesses extra information not known to the market. The first well-known condition in the literature for preserving the semi-martingale property under filtration enlargement was Jacod's criterion in Jacod [1] (see also Jacod [2], Jeulin [3], Jeulin and Yor [4]). Over the years, the set of conditions has been expanded, and Mansuy and Yor [5], Akasmit and Jeanblanc [6], Jeanblanc [7] and Protter [8] provide comprehensive surveys of the field.

The type of conditions that we shall rely on in this paper was first studied in Föllmer and Imkeller [9] in the context of enlargement of filtrations (see also Buckdahn and Föllmer [10]). Föllmer and Imkeller [9] obtained the product space on which the conditional probability measure—obtained after a filtration enlargement—lives, and defined the decoupling measure on this space. They were working in a setting with a one-time (or initial) enlargement of filtration with a random variable and derived the Radon–Nikodym derivative process for the setting. Later, Ankirchner [11] and Ankirchner, Dereich and Imkeller [12] expanded the approach by defining an absolute continuity condition directly on the product space that allowed them to deal with the case of multi-period (or progressive) enlargement of filtration for general stochastic bases.

In the current paper, instead of the viewpoint of an insider as in the literature above, we take the perspective of a Bayesian market maker who has to set derivative prices even as she is learning about the underlying asset's value. Unlike an insider who may have a distinct filtration from the market, a Bayesian market maker's filtration *is the market filtration*, and it expands using the Bayes' formula each time she learns. Furthermore, the emphasis on the pricing kernel means that our main goal is to understand the change to the risk-neutral measure, unlike the literature cited above, where the focus is on deciphering the trading behavior of an insider. More citations to the literature are included in the body of the paper.

Organization of the paper: The paper is organized as follows. Section 2 introduces the basic setup and notation of the paper. The problem facing the market maker is defined, along with other preliminaries that are used throughout the paper. Section 3 contains the main results of the paper. Proposition 1 obtains the absolute continuity condition that is satisfied under Bayesian update. Next, building on the proposition, Theorem 1 describes how a semi-martingale transforms under a Bayesian update. It is shown that the change of measure generates an information drift, and a characterization for the drift is obtained in this theorem and its corollaries. Theorem 2 then derives the change in the derivative pricing kernel that results from a Bayesian update in the underlying asset's value. The Bayesian update leads to an additional covariation term in the semi-martingale, and this covariation must be subtracted out under the change of measure to the risk-neutral world.

Section 4 provides a simple Brownian Bridge-based example to illustrate the results of the paper. Section 5 concludes the paper.

2. Basic Setup

We are modeling the pricing process employed by a market maker who has to set the price of a derivative instrument in a financial market even as she continues to learn about the underlying asset's value. In mathematical finance, *pricing kernels* represent valuation operators for financial instruments in dynamic, stochastic settings (see Hansen and Renault [13]). If the discounted stochastic payoff from an instrument at some future time $T > t$ is known to be $\bar{p}_T$, the price of the instrument at t is $p_t = \mathbb{E}_t[\psi_{t,T}\bar{p}_T]$, with $\psi_{t,T}$ representing the pricing kernel. In the case of a financial derivative in an arbitrage-free market, the price is the risk-neutral expectation of the instrument's payoff. So, $\psi_{t,T}$ is the Radon–Nikodym derivative process representing the ratio of the risk-neutral and physical probability measures.

Our interest is in the interaction between the pricing kernel and learning in the context of derivative market making. To this end, let $(\Omega, \mathcal{F}, P)$ be a probability space that constitutes the uncertainty about the underlying from the perspective of the market maker, and let $(\mathcal{F}_t)_{t\geq 0}$ be a right continuous filtration that represents her information sets as she incorporates additional information into her model with the passage of time.

To be concrete, we shall focus on an instant s when the market maker acquires information about a random variable G on $(\Omega, \mathcal{F})$ that she did not know before, so that

$$\mathcal{F}_{s+} = \mathcal{F}_s \vee \sigma(G), \tag{1}$$

where $\vee$ represents the logical Or. In terms of the filtration, we obtain, figuratively, a fork at s. The actual filtration progresses from $\mathcal{F}_s$ to $\mathcal{F}_{s+}$ as indicated in Equation (1). However, one could consider a counterfactual scenario: what would happen had the market marker not acquired the information about the random variable at time s? The filtration of the market maker in this counterfactual scenario is termed the *counterfactual filtration*, and we shall represent it by $({}_G\mathcal{F}_t)_{t\geq 0}$. This filtration has the property

$${}_G\mathcal{F}_{s+} = \mathcal{F}_s. \tag{2}$$

In terms of informational content, other than the effects that accrue from the acquisition of random variable G at s, filtrations $({}_G\mathcal{F}_t)$ and $(\mathcal{F}_t)$ are indistinguishable. That is,

$${}_G\mathcal{F}_q = \mathcal{F}_q \text{ for } q < s; \;\; {}_G\mathcal{F}_t \vee \sigma(G) = \mathcal{F}_t \text{ for } t \geq s. \tag{3}$$

We shall assume that $({}_G\mathcal{F}_t)$ is the filtration of a standard Brownian motion, so that the martingale representation theorem shall apply where needed.

The underlying asset return follows a stochastic process $X = (X_t)_{t\geq 0}$ from the perspective of the market maker, and X is assumed to be a continuous semi-martingale with respect to the filtration $({}_G\mathcal{F}_t)_{t\geq 0}$. Thus,

$$X_t = M_t + A_t \tag{4}$$

under $({}_G\mathcal{F}_t)$, where $M = (M_t)$ is a continuous local martingale and $A = (A_t)$ is a finite variation process. That is to say, had the market maker not acquired the information about G at time t, the underlying asset's return would be a continuous semi-martingale from her perspective. Without loss of generality, we shall assume $X_0 = 0$, so that $M_0 = A_0 = 0$.

Since G is a random variable, it is a function from Ω to $\mathbb{R}$. So, acquiring information about G shall imply a restriction on the range of the random variable. By a slight abuse of notation, we shall represent the outcome of information acquisition as $G \in \gamma$, $\gamma \subset \mathbb{R}$. As a result of acquiring the information, the market maker's probability measure is updated to P^γ

$$(\gamma, E) \mapsto P^\gamma(E), \quad \gamma \subset \mathbb{R}, \; E \in \mathcal{F}. \tag{5}$$

The market maker's initial measure P is labeled her *prior measure*. It is the probability measure that she uses before she acquires the information. P^γ is the market maker's *posterior measure*. It is the probability measure that she uses after acquiring the information.

We shall maintain throughout that the market maker is a Bayesian. This means that the change from measure P to P^γ shall constitute a Bayesian update. So, for an event $E \in \mathcal{F}$, by Bayes' theorem, we shall have

$$P^\gamma(E) = P(E|G \in \gamma) = P(E)\frac{P(G \in \gamma|E)}{P(G \in \gamma)}. \tag{6}$$

In this paper, we shall use a version of the Bayesian update for the market maker that also includes her filtration (see Equation (8) below). Our aim is to understand the properties of the update and its effect on the pricing kernel for derivative instruments on the underlying X.

The venue in which the market maker operates is arbitrage-free, which implies that a derivative must be priced under the risk-neutral measure when the market maker hedges continuously. The risk-free rate of return in the economy will be denoted by r. Thus, the drift of stochastic processes representing risky asset returns must match $R = \int r\mathrm{d}t$ under the risk-neutral measure.

Throughout, $[X, X]$ shall denote the quadratic variation of process X, and $[X, Y]$ shall denote the quadratic covariation of processes X and Y. We shall use the notation $\mathscr{E}(L)$ to denote the stochastic exponential of process L, i.e.,

$$\mathscr{E}(L)_t = \exp\left(L_t - \frac{1}{2}[L, L]_t\right). \tag{7}$$

3. Pricing Kernel Updates Resulting from Informational Updates

Let us begin the analysis by examining the informational update more closely. At time s, the market maker acquires the information $G \in \gamma$, where G is a random variable on $(\Omega, \mathcal{F})$. Given that the agent is Bayesian, we may write

$$P^\gamma(E|_G\mathcal{F}_t) = P(E|_G\mathcal{F}_t)\frac{P(G \in \gamma|_G\mathcal{F}_t, E)}{P(G \in \gamma|_G\mathcal{F}_t)}, \quad \gamma \subset \mathbb{R}, \; E \in \mathcal{F}, \; t \geq s. \tag{8}$$

Throughout the paper, a Bayesian update is a reference to this informational update. In Equation (8), P is the prior probability measure and P^γ is the posterior probability measure. Thus, the prior measure conditions on the counterfactual information set $_G\mathcal{F}_t$ while the posterior measure conditions on the actual information set $\mathcal{F}_t$, which includes information about the random variable.

One of the features of a Bayesian update is that null sets under the prior continue to remain null sets under the posterior. This follows from the form of Equation (8).

Proposition 1. *The posterior probability measure P^γ is absolutely continuous with respect to the prior probability measure P under the Bayesian update. That is,*

$$P^\gamma \ll P \;\; on \;\; \mathcal{F} \;\; for \;\; P - a.e. \; \gamma \subset \mathbb{R}. \tag{9}$$

Proof. For an event $E \in \mathcal{F}$ and a random variable G on $(\Omega, \mathcal{F})$ we must have that Equation (8) must be defined, and satisfied, if P^γ is obtained from P through a Bayesian update. In particular, this means that if $P(E|_G\mathcal{F}_t) = 0$, then $P^\gamma(E|_G\mathcal{F}_t) = 0$ for all $t \geq s$. This gives the result in (9). □

The absolute continuity condition under a Bayesian update in Proposition 1 is a variant of well-known results in the literature on the enlargement of filtrations. Jacod [1] was among the first to formulate an absolute continuity condition under the filtration enlargement that preserves the semi-martingale property of stochastic processes. The specific form of absolute continuity that is most suited to our setting is the one in Föllmer and Imkeller [9], and Proposition 1 checks that it holds for the Bayesian update. Note that the posterior is a conditional probability measure and technically it lives in the space $(\Omega \times \mathbb{R}, {}_G\mathcal{F} \otimes \mathcal{B})$. It is not hard to see that the absolute continuity in (1) is equivalent to the condition that the joint distribution $P_{(X,G)}$ is absolutely continuous with respect to the product measure $P \otimes P_G$ on ${}_G\mathcal{F} \otimes \mathcal{B}$. Studies like Ankirchner [11], and Ankirchner, Dereich and Imkeller [12] show how one may use very general techniques if one works directly in the product space. However, for the purposes of this paper, it is expedient to work with Ω, and change the filtration from ${}_G\mathcal{F}$ to $\mathcal{F}$.

Next, we would like to understand how the underlying asset's stochastic process X transforms under a Bayesian update. Recall that $\mathscr{E}(L)$ denotes the stochastic exponential of process L (see Equation (7)).

Theorem 1. *The underlying asset's return process continues to be a continuous semi-martingale under a Bayesian update. Specifically, $X = M + A$ under P transforms to $X = M^\gamma + A^\gamma$ under P^γ, where M^γ is a continuous local martingale and A^γ is a finite variation process under P^γ. Let $(Z_t^\gamma)_{t \geq 0}$ be the continuous martingale with*

$$Z_t^\gamma = \frac{\mathrm{d}P^\gamma}{\mathrm{d}P}\bigg|_{{}_G\mathcal{F}_t},$$

and let L^γ be the unique continuous local martingale such that $Z_t^\gamma = \mathscr{E}(L^\gamma)_t$. Then,

$$M^\gamma = M - [M, L^\gamma]. \tag{10}$$

Proof. Since ${}_G\mathcal{F}_t$ is a standard Brownian filtration, by the martingale representation theorem Z_t^γ is a continuous martingale, and there exists a unique continuous local martingale such that $Z_t^\gamma = \mathscr{E}(L^\gamma)_t$. Since X is a continuous local martingale under P, and P^γ is absolutely continuous with respect to P, the Lenglart–Girsanov theorem and its corollary apply (see Theorem 42 and corollaries, Chapter III in Protter [14]). Thus,

$$M - \frac{1}{Z_-^\gamma} \cdot [M, Z^\gamma]$$

is a P^γ local martingale. The definition of L implies that $\frac{1}{Z_-^\gamma} \cdot [M, Z^\gamma] = [M, L^\gamma]$, P^γ-a.s., so that $M - [M, L^\gamma]$ is a P^γ local martingale. By the Kunita–Watanabe inequality (see Corollary to Theorem 42, Chapter III in Protter [14]), there exists a bounded $\mathcal{F}_t$ predictable process $(\alpha_t^\gamma)_{t \geq 0}$ such that

$$[M, L^\gamma] = \alpha^\gamma \cdot [M, M] \tag{11}$$

which implies that $[M, L^\gamma]$ has finite variation. Thus, M is the sum of M^γ and a finite variation process. Since $X = M + A$, we can then write

$$X = M^\gamma + [M, L^\gamma] + A = M^\gamma + A^\gamma. \tag{12}$$

This completes the proof. □

Theorem 1 belongs to the family of Girsanov-type theorems that describe how the semi-martingale decomposition changes under a change of measure. In the case of Theorem 1, the change of measure is from the prior to the posterior, and the absolute continuity in Proposition 1 enables us to apply a version of the Lenglart–Girsanov theorem that applies to continuous local martingales. What we find is that, as a result of the Bayesian update, the underlying asset acquires an additional drift. Prior to the update, the drift in X was just A, but after the Bayesian update, it becomes $A + [M, L^\gamma]$. This is because the martingale M generates the quadratic covariation $[M, L^\gamma]$ under the Bayesian update which adds to the original drift. Ankirchner, Dereich and Imkeller [12] use the label *information drift* for such additional drift generated under filtration enlargement. In our setting, the information drift resulting from the Bayesian update is $[M, L^\gamma]$. In fact, there is an alternative form for the information drift that we used in the proof of Theorem 1 that is quite useful in applications. Let us record the alternative form in a Corollary.

Corollary 1. (Theorem 1). *Under Bayesian update, the underlying asset's return process acquires an additional drift of*

$$[M, L^\gamma] = \alpha^\gamma \cdot [M, M] \tag{13}$$

where $(\alpha_t^\gamma)_{t\geq 0}$ *is an* $\mathcal{F}_t$ *predictable process.*

Proof. The existence of the $\mathcal{F}_t$ predictable process $(\alpha_t^\gamma)_{t\geq 0}$ follows from an application of the Kunita–Watanabe inequality (see Corollary to Theorem 42, Chapter III in Protter [14]). □

Corollary 1 implies that the additional drift acquired by a semi-martingale under the Bayesian update may be solely expressed in terms of its quadratic variation and a deterministic process. This characterization of the information drift is useful in applications because it suppresses the dependence on the L^γ process. We may now rewrite the transformed semi-martingale under P^γ as simply

$$X = M^\gamma + \alpha^\gamma \cdot [M, M] + A \tag{14}$$

and this form is often easier to work with because it only involves variants of M.

Having characterized how the underlying asset's stochastic process transforms under a Bayesian update, we now proceed to examine how the derivative pricing kernel changes when the underlying asset updates. In other words, we seek a characterization for the evolution in the change of measure, needed for risk-neutrality, as a result of the Bayesian update.

For this purpose, we must first assume that the pricing kernel exists for the prior measure P. That is, we have to define some condition that is sufficient for the existence of the Radon–Nikodym derivative for the change to the risk-neutral probability measure in the pre-update scenario. We could, if we like, directly impose the Novikov condition on L^*, with $\mathscr{E}(L^*)$ representing the Radon–Nikodym derivative for the risk-neural measure. In financial applications, however, it is more common to adopt restrictions on the process $X = M + A$ such that a Novikov-type sufficient condition is satisfied as a consequence. For instance, we might take the ratio of the squared drift to the variance of the semi-martingale process to be bounded.

Assumption 1. *The ratio* $\frac{A_t^2}{[M,M]_t}$ *is bounded.*

As Protter [14] shows (see p. 142, Chapter III) when the filtration is Brownian, Assumption 1 implies that the appropriate Novikov condition is satisfied (recall from Section 2 that $({}_G\mathcal{F}_t)$ is the filtration of a standard Brownian motion).

We are now ready to derive our main result on the change in the pricing kernel as a result of a Bayesian update in the underlying asset's value.

Theorem 2. *The pricing kernel for a derivative instrument undergoes an update when the underlying asset's return process is Bayesian updated. The derivative pricing kernel on the underlying process* $X = M + A$, $A \neq R$, *changes from*

$$Z_t^* = \left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{{}_G\mathcal{F}_t} \quad \textit{to} \quad Z_t^{\gamma *} = \left.\frac{\mathrm{d}Q^\gamma}{\mathrm{d}P^\gamma}\right|_{\mathcal{F}_t} \tag{15}$$

with $Z_t^* = \mathscr{E}(L^*)_t$ and $Z_t^{\gamma *} = \mathscr{E}(L^{\gamma *})_t$. L^* solves the equation

$$[M, L^*] = R - A \tag{16}$$

and $L^{*\gamma}$ solves the equation

$$[M^\gamma, L^{\gamma *}] = R - A - [M, L^\gamma], \tag{17}$$

and the change from L^* to $L^{\gamma *}$ is

$$L_t^{\gamma *} = (1 - \eta_t) L_t^* \tag{18}$$

where $\eta = [M, L^\gamma]/(R - A)$.

Proof. Recall that the risk-free asset follows process R, so any asset under the risk-neutral measure must have a drift of R. By assumption, the pricing kernel exists for the measure P, so probability measure Q and processes Z^* and L^* are well defined. Thus, we may apply the Lenglart–Girsanov theorem and its corollaries (see Theorem 42 and corollaries, Chapter III in Protter [14]). To compute L^*, note that by the Lenglart–Girsanov theorem, the continuous local martingale M under P transforms to the process

$$M^Q + [M, L^*] \tag{19}$$

under Q, where M^Q is a continuous local martingale and $[M, L^*]$ is a finite variation process under Q, with $Z_t^* = \mathscr{E}(L^*)_t$. Thus, the underlying stochastic process transforms from $X = M + A$ to

$$X = M^Q + [M, L^*] + A. \tag{20}$$

Since the risk-free process is R, by the definition of risk-neutrality, we must have

$$[M, L^*] + A = R$$

and this gives Equation (16).

To prove the validity of Equation (17), we can use arguments that mirror the ones above. The Bayesian update leads to a change of measure from P to P^γ, and this generates an additional covariation of $[M, L^\gamma]$. So, the risk-neutral measure must subtract this extra drift, in addition to A. This gives us that $L^{\gamma *}$ must satisfy

$$[M^\gamma, L^{\gamma *}] = R - A - [M, L^\gamma]. \tag{21}$$

To complete the proof, we also need to show that the Q^γ generated from the $L^{\gamma *}$ in the above equation is a genuine probability measure, and provide an argument justifying (18). (Note that in the case of the pre-update measure P, we had by assumption that the Q generated by L^* was a probability measure; Assumption 1 ensures that the pricing kernel exists for measure P.) For this, note that, since $R - A$ and $[M, L^\gamma]$ are finite variation processes, $[M, L^*] = R - A$, and $A \neq R$, we may write

$$[M, L^\gamma] = \eta \cdot [M, L^*],$$

where $(\eta_t)_{t \geq 0}$ is a bounded, predictable process. Thus, the right-hand side of (21) can be written as

$$R - A - [M, L^\gamma] = (1 - \eta) \cdot [M, L^*]. \tag{22}$$

Next, recall from Theorem 1 that $M^\gamma = M - [M, L^\gamma]$. Since $[M, L^\gamma]$ is a drift term, it does not contribute to covariation, which means that

$$[M^\gamma, L^{\gamma *}] = [M, L^{\gamma *}]. \tag{23}$$

Substituting (23) and (22) into (21), we obtain

$$[M, L^{\gamma *}] = (1 - \eta) \cdot [M, L^*],$$

which implies that the $L^{\gamma *}$ process is defined by

$$L_t^{\gamma *} = (1 - \eta_t) L_t^*.$$

Since L^* satisfies the Novikov condition and η is bounded, we must have that $L_t^{\gamma *}$ also satisfies the Novikov condition. Therefore, the $Q^{\gamma *}$ obtained by this procedure is a genuine probability measure. This completes the proof. □

Theorem 2 gives the precise form of the change in the derivative pricing kernel as a result of a Bayesian update. The Bayesian update generates quadratic covariation $[M, L^{\gamma}]$, and this extra drift must be negated by the pricing kernel for transition to the risk-neutral framework. $L^{\gamma *}$, the new pricing kernel, is obtained from L^*, the old pricing kernel, by multiplying with the process $(1 - \eta)$. The process η accounts for the impact of the Bayesian update on the pricing kernel. The difference between the terms on the right-hand side in equations in (16) and (17) is $[M, L^{\gamma}]$, and the pricing kernel $L^{\gamma *}$ needs to negate this extra drift in order to transition to the risk-neutral world.

Informally, informational updates increase the "distance" between the world of physical measure and the world of risk-neutral measure. This is the reason why the updated pricing kernel needs to bridge more distance to make the transition. This fact has a number of practical consequences for derivative trading strategies. For instance, if there are frictions that inhibit continuous hedging, the impact must be greater in an environment with informational updates and learning compared to a static environment.

Notice that the results described in Theorem 2 are defined for $A \neq R$. If the finite variation process in X is identical to R, then there is no *change* of measure needed to move to the risk-neutral world, i.e., $L^* = 0$, as we are already in the risk-neutral world. Bayesian updates in the risk-neutral world are of a qualitatively different nature: any informational update in the risk-neutral world must affect R, the risk-free rate process. In fact, we can describe the update precisely using Theorem 1.

Corollary 2. (Theorem 1). *When $P = Q$, the underlying asset's return process is $X = M + R$. In this case, a Bayesian update changes the risk-free process from*

$$R \quad to \quad R + [M, L^{\gamma}]. \tag{24}$$

Proof. This follows from Equation (10) in Theorem 1. □

Economic theories postulate a number of reasons for an update of this nature: the risk-free rate may change when factors like the inflation expectation, or wage expectation, or the temporal consumption preferences of participants in the economy change.

4. Example

In this section we discuss a simple example to illustrate the theory that was developed in the main body of the paper. Suppose the underlying asset return process were

$$X_t = \int_0^t \mu(\omega, s) \mathrm{d}s + \int_0^t \sigma(\omega, s) \mathrm{d}B_s,$$

with (B_t) representing a standard Brownian motion process. Suppose also that the ratio $\mu(\omega, s)/\sigma(\omega, s)$ is bounded, so that Assumption 1 is satisfied. In this case, Equation (16) in Theorem 2 gives us that

$$[M, L^*]_t = \left[\int_0^t \sigma(\omega, s) \mathrm{d}B_s, L_t^* \right] = R_t - \int_0^t \mu(\omega, s) \mathrm{d}s,$$

so we obtain that the risk-neutral measure is characterized by

$$L_t^* = -\int_0^t \frac{\mu(\omega,s)-r}{\sigma(\omega,s)} \mathrm{d}B_s;$$
$$Z_t^* = \exp\left(-\int_0^t \frac{\mu(\omega,s)-r}{\sigma(\omega,s)} \mathrm{d}B_s - \frac{1}{2}\int_0^t \left(\frac{\mu(\omega,s)-r}{\sigma(\omega,s)}\right)^2 \mathrm{d}t\right).$$

Now, suppose the random variable acquired by the market maker at s was the value of the Brownian process at some future time $\tau > s$. In this case, it is well known that (see, for instance, Chapter 4 in Akasmit and Jeanblanc [6]),

$$[M, L^\gamma] = \int_0^t \frac{B_\tau - B_s}{1-s} \mathrm{d}s, \ \ t < \tau.$$

Then, Equation (17) in Theorem 2 gives us that

$$[M^\gamma, L^{\gamma *}]_t = \left[\int_0^t \sigma(\omega,s)\mathrm{d}B_s, L_t^{\gamma *}\right] = R_t - \int_0^t \mu(\omega,s)\mathrm{d}s - \int_0^t \frac{B_\tau - B_s}{1-s}\mathrm{d}s, \ \ t < \tau.$$

Let us use the notation $\theta(s) = \frac{B_\tau - B_s}{1-s}$. Solving the equation above, we see that the updated risk-neutral measure is characterized by

$$L_t^{\gamma *} = -\int_0^t \frac{\mu(\omega,s)+\theta(s)-r}{\sigma(\omega,s)} \mathrm{d}B_s, \ \ t < \tau;$$
$$Z_t^{\gamma *} = \exp\left(-\int_0^t \frac{\mu(\omega,s)+\theta(s)-r}{\sigma(\omega,s)} \mathrm{d}B_s - \frac{1}{2}\int_0^t \left(\frac{\mu(\omega,s)+\theta(s)-r}{\sigma(\omega,s)}\right)^2 \mathrm{d}t\right), \ \ t < \tau.$$

Notice that the last four equations above are only defined for $t < \tau$. As Föllmer and Imkeller [9] show, under the posterior (decoupled) measure on the product space, the events $t \geq \tau$ have zero probability. In terms of Bayesian update, the intuition is that after the realization of the random variable becomes available, its content no longer has any informational value.

5. Conclusions

To conclude, in this paper, we described how the risk-neutral measure changes as a result of an informational update in the underlying asset process. With a Bayesian informational update, the change of measure needed to move to the risk-neutral world is two-fold: first, a change of measure to move from the prior to the posterior measure, and second, a change of measure to move from the posterior to the risk-neutral measure. In short, the Bayesian update generates an information drift, and this additional drift needs to be subtracted in order to accomplish the move to the risk-neutral world. The paper obtained a number of results to characterize this process.

The results in the paper have many potential applications in the world of derivative pricing. In today's high-frequency trading environment, derivative market makers have to contend with an unending deluge of new information as they try to ensure that their quoted prices keep pace with the ever-evolving informational landscape. However, the traditional models of derivative pricing were largely built for static informational environments. This paper is part of a larger effort to develop an array of mathematical techniques that should allow derivative traders to transport their traditional models to today's fast-paced informational landscape. In this paper, we showed how derivative pricing changes as a result of Bayesian informational updates in the underlying in a setting with continuous semi-martingale processes. An interesting future extension of the work would be to broaden the approach to cover discontinuous stochastic processes.

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