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Three Results on the Nonlinear Differential Equations and Differential-Difference Equations

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Abstract: We mainly study the transcendental entire solutions of the differential equation $f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$, where p_1, p_2, α_1 and α_2 are nonzero constants satisfying $\alpha_1 \neq \alpha_2$ and $P(f)$ is a differential polynomial in f of degree $n-1$. We improve Chen and Gao's results and partially answer a question proposed by Li (J. Math. Anal. Appl. 375 (2011), pp. 310–319).

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MSC: 34M05; 39A10; 39B32

1. Introduction and Main Results

In the past several decades, a great deal of mathematical effort in complex analysis has been devoted to studying differential equations, differential-difference equations and difference equations. The essential reason is penetration and application of Nevanlinna theory for the difference operator, see [1–4]. In this study, we assume readers are familiar with the standard notations and fundamental results used in the theory such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$ and the counting function $N(r, f)$, see [5–8]. Moreover, we use the notations $\rho(f)$ and $\rho_2(f)$ to denote the order and the hyper-order of f, respectively.

Many scholars recently have had tremendous interest in developing solvability and existence of solutions of non-linear differential equations and differential-difference equations in the complex plane, see [9–15].

In 2011, Li [16] considered to find all entire solutions of the following nonlinear differential equation

$$f^n(z) + P(f) = p_1 e^{\lambda z} + p_2 e^{-\lambda z} \tag{1}$$

and obtained the following result.

Theorem 1. (see [16]) *Let $n \geq 2$ be an integer, $P(f)$ be a differential polynomial in f of degree at most $n-1$ and λ, p_1, p_2 be three nonzero constants. If f is a meromorphic function of Equation (1) satisfying $N(r, f) = S(r, f)$, then there exist two nonzero constants c_1, c_2 ($c_i^n = p_i$) and a small function c_0 of f such that*

$$f = c_0 + c_1 e^{\frac{\lambda z}{n}} + c_2 e^{-\frac{\lambda z}{n}}.$$

Li [16] also investigated $p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$ for two distinct constants α_1 and α_2 instead of $p_1 e^{\lambda z} + p_2 e^{-\lambda z}$ in the right side of Equation (1) and obtained the following results.

Theorem 2. (see [16]) *Let $n \geq 2$ be an integer, $P(f)$ be a differential polynomial in $f(z)$ of degree at most $n-2$ and $\alpha_1,\ \alpha_2,\ p_1,\ p_2$ be nonzero constants satisfying $\alpha_1 \neq \alpha_2$. If $f(z)$ is a transcendental meromorphic solution of the following equation*

$$f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} \tag{2}$$

satisfying $N(r,f) = S(r,f)$, then one of the following relations holds:

(1) $f(z) = c_0(z) + c_1 e^{\frac{\alpha_1 z}{n}}$;
(2) $f(z) = c_0(z) + c_2 e^{\frac{\alpha_2 z}{n}}$;
(3) $f(z) = c_1 e^{\frac{\alpha_1 z}{n}} + c_2 e^{\frac{\alpha_2 z}{n}}$ *and* $\alpha_1 + \alpha_2 = 0$,

where $c_0(z)$ is a small function of f and constants c_1 and c_2 satisfy $c_1^n = p_1$ and $c_2^n = p_2$, respectively.

For further study, Li proposed a related question:

Question 1. *How to find the solutions of Equation (2) if* $\deg\ P(f) = n-1$*?*

The question was studied by Chen and Gao [17]. They partially answered it and obtained the following result.

Theorem 3. (see [17]) *Let $a(z)$ be a nonzero polynomial and $p_1,\ p_2,\ \alpha_1,\ \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If $f(z)$ is a transcendental entire solution of finite order of the differential equation*

$$f^2(z) + a(z)f'(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} \tag{3}$$

satisfying $N(r, \frac{1}{f}) = S(r,f)$, then $a(z)$ must be a constant and one of the following relations holds:

(1) $f(z) = c_1 e^{\frac{\alpha_1 z}{2}}$, $ac_1\alpha_1 = 2p_2$ *and* $\alpha_1 = 2\alpha_2$;
(2) $f(z) = c_2 e^{\frac{\alpha_2 z}{2}}$, $ac_2\alpha_2 = 2p_1$ *and* $\alpha_2 = 2\alpha_1$,

where c_1 and c_2 are constants satisfying $c_1^2 = p_1$ and $c_2^2 = p_2$, respectively.

Now, we remove the condition that $f(z)$ is a finite-order function, improve Theorem 3 and obtain the following result.

Theorem 4. *Let $a(z)$ be a nonzero polynomial and $p_1,\ p_2,\ \alpha_1,\ \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. Suppose that $f(z)$ is a transcendental entire solution of the differential Equation (3) satisfying $N(r, \frac{1}{f}) = S(r,f)$. Then $a(z)$ must be a constant and one of the following relations holds:*

(1) $f(z) = c_1 e^{\frac{\alpha_1 z}{2}}$, $ac_1\alpha_1 = 2p_2$ *and* $\alpha_1 = 2\alpha_2$;
(2) $f(z) = c_2 e^{\frac{\alpha_2 z}{2}}$, $ac_2\alpha_2 = 2p_1$ *and* $\alpha_2 = 2\alpha_1$,

where c_1 and c_2 are constants satisfying $c_1^2 = p_1$ and $c_2^2 = p_2$, respectively.

Next we consider the general case in Question 1 and obtain the following theorem.

Theorem 5. *Let $n \geq 2$ be an integer. Suppose that $P(f)$ is a differential polynomial in $f(z)$ of degree $n-1$ and that α_1, α_2, p_1 and p_2 are nonzero constants such that $\alpha_1 \neq \alpha_2$. If $f(z)$ is a transcendental meromorphic solution of the differential Equation (2) satisfying $N(r,f) = S(r,f)$, then $\rho(f) = 1$ and one of the following relations holds:*

(1) $f(z) = c_1 e^{\frac{\alpha_1 z}{n}}$ *and* $c_1^n = p_1$;
(2) $f(z) = c_2 e^{\frac{\alpha_2 z}{n}}$ *and* $c_2^n = p_2$, *where c_1 and c_2 are constants;*

(3) $T(r,f) \leq N_{1)}(r,\frac{1}{f}) + T(r,\varphi) + S(r,f)$, *where* $N_{1)}(r,\frac{1}{f})$ *denotes the counting function corresponding to simple zeros of* f *and* $\varphi(\not\equiv 0)$ *is equal to* $\alpha_1\alpha_2 f^2 - n(\alpha_1+\alpha_2)ff' + n(n-1)(f')^2 + nff''$.

Three examples are shown to illustrate the cases (1)–(3) of Theorem 5.

Example 1. *Let* $f(z) = e^z$ *be an entire solution of the differential equation*

$$f^2(z) + f'(z) = e^{2z} + e^z,$$

where $c_1 = 1$ *and* $p_1 = 1$. *It implies the case (1) occurs.*

Example 2. *Let* $f(z) = 2e^{2z}$ *be an entire solution of the differential equation*

$$f^2(z) + \tfrac{1}{8}f''(z) = e^{2z} + 4e^{4z},$$

where $c_2 = 2$ *and* $p_2 = 4$. *It implies case (2) occurs.*

Example 3. *Let* $f(z) = e^z - 1$ *be an entire solution of the differential equation*

$$f^2(z) + (f'-1) = e^{2z} - e^z.$$

We can easily verify the inequality $T(r,f) \leq N_{1)}(r,\frac{1}{f}) + T(r,\varphi) + S(r,f)$, *where* $\varphi = 2f^2 - 6ff' + 2(f')^2 + 2ff'' = 2$. *It implies that case (3) occurs.*

Remark 1. *From Theorem 4 and Example 3, we conjecture that case (3) in Theorem 5 can be removed if* $N(r,1/f) = S(r,f)$.

In [18], Wang and Li investigated the following differential-difference equation

$$f^n(z) + q(z)f^{(k)}(z+c) = ae^{ibz} + de^{-ibz} \tag{4}$$

and obtained the existence of entire solutions when $n \geq 3$.

In 2018, Chen and Gao went far to study Equation (4) with $n = 2$. They obtained the following theorem.

Theorem 6. (see [17]) *Let* $a(z)$ *be a nonzero polynomial,* $k \geq 0$ *be an integer and* $p_1,\ p_2,\ \lambda,\ c$ *be nonzero constants. If* $f(z)$ *is a transcendental entire solution of finite order of the differential-difference equation*

$$f^2(z) + a(z)f^{(k)}(z+c) = p_1e^{\lambda z} + p_2e^{-\lambda z}, \tag{5}$$

then $a(z)$ *must be a constant and one of the following relations holds:*

(1) $f(z) = \pm\frac{i}{2}a(\frac{\lambda}{2})^k + c_1e^{\frac{\lambda z}{2}} + c_2e^{-\frac{\lambda z}{2}}$ *and* $e^{\lambda c} = -1$, *when* k *is odd;*
(2) $f(z) = \pm\frac{1}{2}a(\frac{\lambda}{2})^k + c_1e^{\frac{\lambda z}{2}} + c_2e^{-\frac{\lambda z}{2}}$ *and* $e^{\lambda c} = 1$, *when* k *is even and* $k > 0$, *where* a, c_1 *and* c_2 *are constants with* $\frac{1}{64}a^4(\frac{\lambda}{2})^{4k} = p_1p_2$ *and* $c_i^2 = p_i$ $(i = 1,\ 2)$;
(3) $f(z) = \pm\frac{1}{2}a + c_1e^{\frac{\lambda z}{2}} + c_2e^{-\frac{\lambda z}{2}}$ *and* $e^{\lambda c} = 1$, *when* $k = 0$, *where* a, c_1 *and* c_2 *are constants with* $\frac{1}{64}a^4 = p_1p_2$ *or* $\frac{9}{64}a^4 = p_1p_2$ *and* $c_i^2 = p_i$ $(i = 1,\ 2)$.

For the right side of Equations (4) and (5), a question to be raised is how to find the existence of solutions if $e^{\lambda z}$ and $e^{-\lambda z}$ can be replaced by a linear combination of $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ for two distinct constants α_1 and α_2. We consider the question and obtain the following result.

Theorem 7. *Let α_1, α_2, p_1, p_2 and h be nonzero constants satisfying $\alpha_1 \neq \alpha_2$. Suppose that $k \geq 0$ and $n \geq 2$ are integers and that $q(z)$ is a nonzero polynomial. If $f(z)$ is a transcendental entire solution with $\rho_2(f) < 1$ of the differential-difference equation*

$$f^n(z) + q(z)f^{(k)}(z+h) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{6}$$

then we have $\rho(f) = 1$, $q(z)$ must be a constant and one of the following relations holds:

(1) $f(z) = c_1 e^{\frac{\alpha_1 z}{n}}$, $qc_1\left(\frac{\alpha_1}{n}\right)^k e^{\frac{\alpha_1 h}{n}} = p_2$, $\alpha_1 = n\alpha_2$ *and* $c_1^n = p_1$;
(2) $f(z) = c_2 e^{\frac{\alpha_2 z}{n}}$, $qc_2\left(\frac{\alpha_2}{n}\right)^k e^{\frac{\alpha_2 h}{n}} = p_1$, $\alpha_2 = n\alpha_1$ *and* $c_2^n = p_2$;
(3) *If $n = 2$, we have $T(r,f) \leq N_{1)}(r,1/f) + T(r,\varphi) + S(r,f)$, where $N_{1)}(r,1/f)$ and φ are the same as defined in Theorem 5. If $n = 3$, we have $T(r,f) = N_{1)}(r,1/f) + S(r,f)$. If $n \geq 4$, we only have the cases (1) and (2).*

Next we give three examples to show existence of solutions of Equation (6).

Example 4. *Let $f(z) = e^z$. Then f is a transcendental entire solution of the following differential-difference equation*

$$f^3(z) + f'(z + 2\pi i) = e^{3z} + e^z,$$

where $\alpha_1 = 3 = 3\alpha_2$, $c_1 = 1$, $q = 1$ and $p_1 = p_2 = 1$. Thus, case (1) occurs.

Example 5. *Let $f(z) = \sqrt{2}e^z$. Then f is a transcendental entire solution of the following differential-difference equation*

$$f^2(z) + \sqrt{2}f^{(3)}(z + 2\pi i) = 2e^z + 2e^{2z},$$

where $\alpha_2 = 2 = 2\alpha_1$, $c_2 = \sqrt{2}$, $q = \sqrt{2}$ and $p_1 = p_2 = 2$. Thus, case (2) occurs.

Example 6. *Let $f(z) = e^z - 1$. Then f is a transcendental entire solution of the following equation*

$$f^2(z) + f(z + \pi i) = e^{2z} - 3e^z.$$

A routine computation yields $T(r,f) \leq N_{1)}(r,\frac{1}{f}) + T(r,\varphi) + S(r,f)$, where $\varphi = 2f^2 - 6ff' + 2(f')^2 + 2ff'' = 2$. Thus, case (3) occurs.

Example 7. *Let $f(z) = e^z + e^{-z}$. Then f is a transcendental entire solution of the following differential-difference equation*

$$f^3(z) + f''(z + \pi i) = e^{3z} + e^{-3z}.$$

A routine computation yields $T(r,f) = N_{1)}(r,\frac{1}{f}) + S(r,f)$.

Remark 2. *From Examples 6 and 7, we conjecture that case (3) in Theorem 7 can be removed if $N(r,1/f) = S(r,f)$ for $n = 2,3$.*

Remark 3. *In Theorem 3, our result holds for $\alpha_1 \neq \alpha_2$. However, if $\alpha_1 + \alpha_2 = 0$, we just know the solutions satisfy case (3) for $n = 2,3$. The expression of solutions can be obtained when $n = 2$ in Theorem 6.*

2. Some Lemmas

In this section, we introduce several lemmas to prove three theorems.

Lemma 1. (see [5]) *Let $f(z)$ be an entire function and k be a positive integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).$$

Lemma 2. (see [3]) *Let $c \in \mathbb{C} \backslash \{0\}$, $\varepsilon > 0$ and $f(z)$ be a meormorphic function of $\rho_2(f) < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\rho_2(f)-\varepsilon}}\right)$$

outside of an exceptional set of finite logarithmic measures.

Lemma 3. (see [8]) *Suppose that $f_1(z)$, $f_2(z), \cdots, f_n(z) (n \geq 2)$ are meromorphic functions and that $g_1(z)$, $g_2(z), \cdots, g_n(z) (n \geq 2)$ are entire functions satisfying the following conditions:*

(1) $f_1(z)e^{g_1(z)} + f_2(z)e^{g_2(z)} + \cdots + f_n(z)e^{g_n(z)} \equiv 0$;
(2) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;
(3) *For* $1 \leq j \leq n$ *and* $1 \leq h < k \leq n$, $T(r, f_j(z)) = o\big(T(r, e^{g_h(z) - g_k(z)})\big)(r \to \infty, r \notin E)$, *where* $E \subset [1, \infty)$ *is a finite linear measure or finite logarithmic measure.*

Then $f_j(z) \equiv 0$ $(j = 1, 2, \cdots, n)$.

Applying Lemmas 1 and 2 to Theorem 2.3 of [19], we get the following lemma, which is a version of the difference analogue of the Clunie lemma.

Lemma 4. *Let f be a transcendental meromorphic solution of $\rho_2(f) < 1$ of a difference equation of the form*

$$H(z, f)P(z, f) = Q(z, f),$$

where $H(z, f)$, $P(z, f)$, $Q(z, f)$ are difference polynomials in f such that the total degree of $H(z, f)$ in f and its shifts is n, and that the corresponding total degree of $Q(z, f)$ is $\leq n$. If $H(z, f)$ contains just one term of maximal total degree, then for any $\varepsilon > 0$

$$m(r, P(z, f)) = S(r, f)$$

possibly outside of an exceptional set of finite logarithmic measure.

3. Proof of Theorem 4

Proof. Denote $P_1(f) := a(z)f'(z)$. Suppose $f(z)$ be a transcendental entire solution of Equation (3).

Differentiating Equation (3), we obtain

$$2ff' + P_1' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{7}$$

Eliminating $e^{\alpha_2 z}$ from Equations (3) and (7) gives

$$\alpha_2 f^2 - 2ff' + \alpha_2 P_1 - P_1' = (\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \tag{8}$$

Differentiating Equation (8) yields

$$2\alpha_2 ff' - 2(f')^2 - 2f'f'' + \alpha_2 P_1' - P_1'' = \alpha_1(\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \tag{9}$$

It follows from Equations (8) and (9) that

$$\varphi = Q,$$

where

$$\varphi = \alpha_1\alpha_2 f^2 - 2(\alpha_1+\alpha_2)ff' + 2(f')^2 + 2ff''$$

and

$$Q = -\alpha_1\alpha_2 P_1 + (\alpha_1+\alpha_2)P_1' - P_1''.$$

Here we distinguish two cases below.

Case 1. $\varphi \not\equiv 0$.

Similar to the proof of Theorem 3 [17], we can obtain a contradiction.

Case 2. $\varphi \equiv 0$.

By taking $n = 2$, we use the method of Case 1 of Theorem 5 to obtain $t_1 = \frac{\alpha_1}{2}$ and $t_2 = \frac{\alpha_2}{2}$, where $t_i = \frac{f'}{f}(i = 1, 2)$.

Now if $t_1 = \frac{\alpha_1}{2}$, then $f(z) = c_1 e^{\frac{\alpha_1 z}{2}}$, where c_1 is a constant satisfying $c_1^2 = p_1$. Substituting these formulas into Equation (3), we have $a(z)c_1\alpha_1 = 2p_2$ and $\alpha_1 = 2\alpha_2$, where $a(z)$ must be a constant. Set $a := a(z)$.

Similarly, if $t_2 = \frac{\alpha_2}{2}$, then we have $f(z) = c_2 e^{\frac{\alpha_2 z}{2}}$, $ac_2\alpha_2 = 2p_1$ and $\alpha_2 = 2\alpha_1$, where c_2 is a constant satisfying $c_2^2 = p_2$. $\square$

4. Proof of Theorem 5

Proof. Assume that $f(z)$ is a transcendental meromorphic solution of Equation (2) with $N(r, f) = S(r, f)$.

A differential polynomial $P(f)$ with $\deg P(f) = n-1$ can be written in the following form

$$P(f) = \sum_{i=1}^{n-1} a_i M_i(f) = a_1 M_1(f) + a_2 M_2(f) + \cdots + a_{n-1} M_{n-1}(f),$$

where a_i are the small functions of f and $M_i(f) = f^{n_{0i}}(f')^{n_{1i}} \cdots (f^{(k)})^{n_{ki}}$ are the differential monomials such that $\deg M_i(f) = n_{0i} + n_{1i} + \cdots + n_{ki} = i \le n-1$.

We can represent $P(f)$ as

$$P(f) = \frac{a_1 M_1(f)}{f} f + \frac{a_2 M_2(f)}{f^2} f^2 + \cdots + \frac{a_{n-1} M_{n-1}(f)}{f^{n-1}} f^{n-1}.$$

By Lemma 1, we derive

$$m\left(r, \frac{a_i M_i(f)}{f^i}\right) = m\left(r, \frac{a_i f^{n_{0i}}(f')^{n_{1i}} \cdots (f^{(k)})^{n_{ki}}}{f^i}\right) = S(r, f)$$

for $1 \le i \le n-1$. Furthermore, we have

$$m(r, P(f)) \le (n-1)m(r, f) + S(r, f).$$

Since $N(r, f) = S(r, f)$

$$T(r, P(f)) \le (n-1)T(r, f) + S(r, f) \tag{10}$$

holds.

By Equation (10), we obtain

$$\begin{aligned} & T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) = T(r, f^n(z) + P(f)) \\ \leq\ & T(r, f^n(z)) + T(r, P(f)) + O(1) \\ \leq\ & nT(r, f) + (n-1)T(r, f) + S(r, f) \\ =\ & (2n-1)T(r, f) + S(r, f) \end{aligned} \tag{11}$$

and

$$\begin{aligned} & T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) = T(r, f^n(z) + P(f)) \\ \geq\ & T(r, f^n(z)) - T(r, P(f)) + O(1) \\ \geq\ & nT(r, f) - (n-1)T(r, f) + S(r, f) \\ =\ & T(r, f) + S(r, f). \end{aligned} \tag{12}$$

It follow from Equations (11) and (12) that

$$T(r, f) + S(r, f) \leq T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) \leq (2n-1)T(r, f) + S(r, f),$$

which implies $\rho(f) = 1$.

We next turn to proving conclusions (1)–(3).

Differentiating Equation (2), we have

$$nf^{n-1}f' + P' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{13}$$

Eliminating $e^{\alpha_2 z}$ from Equations (2) and (13) gives

$$\alpha_2 f^n - nf^{n-1}f' + \alpha_2 P - P' = (\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \tag{14}$$

Differentiating Equation (14) yields

$$n\alpha_2 f^{n-1}f' - n(n-1)f^{n-2}(f')^2 - nf^{n-1}f'' + \alpha_2 P' - P'' = \alpha_1(\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \tag{15}$$

By Equations (14) and (15), we have

$$f^{n-2}\varphi = Q,$$

where

$$\varphi = \alpha_1\alpha_2 f^2 - n(\alpha_1 + \alpha_2) ff' + n(n-1)(f')^2 + nff'' \tag{16}$$

and

$$Q = -\alpha_1\alpha_2 P + (\alpha_1 + \alpha_2)P' - P''.$$

We still consider two cases below.

Case 1. $\varphi \equiv 0$.

Dividing with f^2 on both sides in Equation (16) and recalling $\frac{f''}{f} = \left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2$, we get a Riccati equation

$$t' + nt^2 - (\alpha_1 + \alpha_2)t + \tfrac{\alpha_1\alpha_2}{n} = 0$$

where $t = \frac{f'}{f}$. A routine computation yields two constant solutions $t_1 = \frac{\alpha_1}{n}$ and $t_2 = \frac{\alpha_2}{n}$.

Given that $t \neq t_1$ and $t \neq t_2$ hold, we have

$$\frac{1}{t_1 - t_2}\left(\frac{t'}{t - t_1} - \frac{t'}{t - t_2}\right) = -n.$$

Integrating it on both sides gives

$$\ln \frac{t - t_1}{t - t_2} = n(t_2 - t_1)z + C,\ C \in \mathbb{C},$$

which is equivalent to

$$\frac{t-t_1}{t-t_2}=e^{n(t_2-t_1)z+C}.$$

It immediately yields

$$t=t_2+\frac{t_2-t_1}{e^{n(t_2-t_1)z+C}-1}=\frac{f'}{f}.$$

Note that zeros of $e^{n(t_2-t_1)z+C}-1$ are the zeros of f. If z_0 is a zero of f with multiplicity k, then

$$k=\text{Res}\left[\frac{f'}{f},z_0\right]=\text{Res}\left[t_2+\frac{t_2-t_1}{e^{n(t_2-t_1)z+C}-1},z_0\right]=\frac{1}{n}$$

is a contradiction.

If $t_1=\frac{\alpha_1}{2}$, then $f(z)=c_1e^{\frac{\alpha_1 z}{2}}$, where c_1 is a constant satisfying $c_1^2=p_1$.

Similarly, if $t_2=\frac{\alpha_2}{2}$, then we have $f(z)=c_2e^{\frac{\alpha_2 z}{2}}$, where c_2 is a constant satisfying $c_2^2=p_2$.

Case 2. $\varphi\not\equiv 0$.

Equation (16) can be written as

$$\frac{1}{f^2}=\frac{1}{\varphi}\left[\alpha_1\alpha_2-n(\alpha_1+\alpha_2)\left(\frac{f'}{f}\right)+n(n-1)\left(\frac{f'}{f}\right)^2+n\left(\frac{f''}{f}\right)\right].$$

Using Lemma 1, we have

$$2m\left(r,\frac{1}{f}\right)=m\left(r,\frac{1}{f^2}\right)\le m\left(r,\frac{1}{\varphi}\right)+S(r,f). \tag{17}$$

From Equation (16), if z_0 is a multiple zero of f, then z_0 must be a zero of φ. Thus, it follows that

$$N_{(2}\left(r,\frac{1}{f}\right)\le N\left(r,\frac{1}{\varphi}\right)+S(r,f), \tag{18}$$

where $N_{(2}(r,\frac{1}{f})$ denotes the counting function of multiple zeros of f. Equations (17) and (18) and the first fundamental theorem give

$$T(r,f)\le N_{1)}\left(r,\frac{1}{f}\right)+T(r,\varphi)+S(r,f). \tag{19}$$

□

5. Proof of Theorem 7

Proof. Assume that $f(z)$ is a transcendental entire solution with $\rho_2(f)<1$ of Equation (6). Applying Lemmas 1 and 2 to Equation (6), we have

$$\begin{aligned}
&T(r,p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z})=T(r,f^n(z)+q(z)f^{(k)}(z+h))\\
\le\ &T(r,f^n)+T(r,q(z)f^{(k)}(z+h))+O(1)\\
\le\ &T(r,f^n)+m\left(r,\tfrac{q(z)f^{(k)}(z+h)}{f(z)}\right)+m(r,f)+O(1)\\
\le\ &T(r,f^n)+m\left(r,q(z)\tfrac{f(z+h)}{f(z)}\right)+m\left(r,\tfrac{f^{(k)}(z+h)}{f(z+h)}\right)+m(r,f)+O(1)\\
\le\ &(n+1)T(r,f)+S(r,f).
\end{aligned} \tag{20}$$

On the other hand, we deduce

$$
\begin{aligned}
& T(r, p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}) = T(r, f^n(z) + q(z)f^{(k)}(z+h)) \\
\geq\ & T(r, f^n) - T(r, q(z)f^{(k)}(z+h)) + O(1) \\
\geq\ & nT(r,f) - m\left(r, \frac{q(z)f^{(k)}(z+h)}{f(z)}\right) - m(r,f) + O(1) \\
\geq\ & nT(r,f) - m\left(r, \frac{q(z)f(z+h)}{f(z)}\right) - m\left(r, \frac{f^{(k)}(z+h)}{f(z+h)}\right) - m(r,f) + O(1) \\
\geq\ & nT(r,f) - T(r,f) + S(r,f) \\
=\ & (n-1)T(r,f) + S(r,f).
\end{aligned}
\tag{21}
$$

Combining Equations (20) and (21), it follows that

$$(n-1)T(r,f) + S(r,f) \leq T(r, p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}) \leq (n+1)T(r,f) + S(r,f),$$

which implies $\rho(f) = 1$.

Denoting $P_2(f) := q(z)f^{(k)}(z+h)$ and differentiating Equation (6), we have

$$nf^{n-1}f' + P_2' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{22}$$

Eliminating $e^{\alpha_2 z}$ from Equations (6) and (22) gives

$$\alpha_2 f^n - nf^{n-1}f' + \alpha_2 P_2 - P_2' = (\alpha_2 - \alpha_1)p_1e^{\alpha_1 z}. \tag{23}$$

Differentiating Equation (23) yields

$$n\alpha_2 f^{n-1}f' - n(n-1)f^{n-2}(f')^2 - nf^{n-1}f'' + \alpha_2 P_2' - P_2'' = \alpha_1(\alpha_2 - \alpha_1)p_1e^{\alpha_1 z}. \tag{24}$$

It follows from Equations (23) and (24) that

$$f^{n-2}\varphi = Q, \tag{25}$$

where

$$\varphi = \alpha_1\alpha_2 f^2 - n(\alpha_1+\alpha_2)ff' + n(n-1)(f')^2 + nff''$$

and

$$Q = -\alpha_1\alpha_2 P_2 + (\alpha_1+\alpha_2)P_2' - P_2''.$$

Next we discuss two cases below.

Case 1. $\varphi \equiv 0$.

This case can be completed by the same method as employed in Case 1 of Theorem 5. We obtain $f(z) = c_2 e^{\frac{\alpha_2 z}{n}}$, where c_2 is a constant satisfying $c_2^n = p_2$. Substituting these formulas into Equation (6), we have

$$q(z)c_2\left(\tfrac{\alpha_2}{n}\right)^k e^{\frac{\alpha_2 h}{n}} e^{\frac{\alpha_2}{n}z} - p_1e^{\alpha_1 z} = 0.$$

According to $\alpha_1 \neq \alpha_2$ and Lemma 3, we have

$$\alpha_2 = n\alpha_1 \ \text{ and } \ q(z)c_2\left(\tfrac{\alpha_2}{n}\right)^k e^{\frac{\alpha_2 h}{n}} = p_1,$$

which implies that $q(z)$ is a constant. Set $q := q(z)$.

Similarly, we proceed to obtain $f(z) = c_1 e^{\frac{\alpha_1 z}{n}}$, $qc_1\left(\tfrac{\alpha_1}{n}\right)^k e^{\frac{\alpha_1 h}{n}} = p_2$, $\alpha_1 = n\alpha_2$ and $c_1^n = p_1$.

Case 2. $\varphi \not\equiv 0$.

For $n \geq 4$, we shall derive a contradiction. In fact, Q is a difference-differential polynomial in f and its degree at most is 1. By Equation (25) and Lemma 4, we have $m(r, \varphi) = S(r, f)$ and $T(r, \varphi) = S(r, f)$. On the other hand, we can rewrite Equation (25) as $f^{n-3}(f\varphi) = Q$, which implies $m(r, f\varphi) = S(r, f)$ and $T(r, f\varphi) = S(r, f)$. If $\varphi \not\equiv 0$, then $T(r, f) = T(r, \frac{f\varphi}{\varphi}) = S(r, f)$ and this is impossible.

For $n = 3$, since Q is a difference-differential polynomial in f and its degree at most is 1, it follows from Equation (25) and Lemma 4 that $m(r, \varphi) = S(r, f)$ and

$$T(r, \varphi) = S(r, f). \tag{26}$$

We still use the same method in Case 2 of Theorem 5 to obtain the inequality of Equation (19). Equations (19) and (26) and the first fundamental theorem result in

$$T(r, f) = N_{1)}\left(r, \frac{1}{f}\right) + S(r, f).$$

For $n = 2$, we just obtain the inequality of Equation (19). □

6. Conclusions

In this study, we consider two questions. Firstly, the first question posed by Li in [16] is how to find the solutions of Equation (2) if $\deg P(f) = n - 1$. Since the degree of $P(f)$ is bigger than $n - 2$, one cannot use Clunie's lemma which is a key in the proof in Theorem 2. It is very difficult to resolve the question. Chen and Gao considered the entire solution f of Equation (2) with the order $\rho(f) < \infty$ and $N(r, 1/f) = S(r, f)$ when $n = 2$ and partially answered the question. We remove the condition that the order $\rho(f) < \infty$ by a different method and improve the result of Chen and Gao in Theorem 4. For the general case of Li's question, we use the method of Theorem 4 and give a partial answer in Theorem 5.

Secondly, motived by Theorem 2, a question to be raised is how to find the existence of solutions to Equation (5) if $e^{\lambda z}$ and $e^{-\lambda z}$ can be replaced by a linear combination of $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ for two distinct constants α_1 and α_2. We consider the general case by the similar method with Theorem 5 and give the partial solutions of Equation (6).

For further study, we conjecture that the inequality $T(r, f) \leq N_{1)}(r, \frac{1}{f}) + T(r, \varphi) + S(r, f)$ or $T(r, f) = N_{1)}(r, \frac{1}{f}) + S(r, f)$ can be removed if $N(r, 1/f) = S(r, f)$ in Theorems 5 and 7.

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