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On the Zeros of the Differential Polynomial $\varphi(z)f^2(z)f'(z)^2 - 1$

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Abstract: In this study, the value distribution of the differential polynomial $\varphi f^2 f'^2 - 1$ is considered, where f is a transcendental meromorphic function, $\varphi(\not\equiv 0)$ is a small function of f by the reduced counting function. This result improves the existed theorems which obtained by Jiang (Bull Korean Math Soc 53: 365-371, 2016) and also give a quantitative inequality of $\varphi f f' - 1$.

Keywords: meromorphic function; differential polynomials; value distribution; small functions

MSC: 30D35; 26D10

1. Introduction and Results

In this paper, we assumed that the reader is familiar with the notations of Nevanlinna theory (see, e.g., [1,2]).Let $f(z)$ and $\alpha(z)$ be two meromorphic functions in the complex plane. If $T(r,\alpha) = S(r,f)$, then $\alpha(z)$ is called a small function of $f(z)$.

Definition 1. *Reference [2] Let k be a positive integer. For any constant a in the complex plane we denote by $N_{k)}(r,1/(f-a))$ the counting function of those a-points of f whose multiplicities are not great than k, by $N_{(k}(r,1/(f-a))$ the counting function of those a-points of f whose multiplicities are not less than k, by $N_k(r,1/(f-a))$ the counting function of those a-points of f with multiplicity k, and denote the reduced counting function by $\overline{N}_{k)}(r,1/(f-a))$, $\overline{N}_{(k}(r,1/(f-a))$ and $\overline{N}_k(r,1/(f-a))$, respectively.*

Definition 2. *If z_0 is a pole of $f(z)$ with multiplicity l, then we say $\omega(f,z_0) = l$, $\overline{\omega}(f,z_0) = 1$. Otherwise, $\omega(f,z_0) = \overline{\omega}(f,z_0) = 0$.*

Clearly, for p meromorphic functions, we have

$$\omega(\Pi_{j=1}^{p} f_j, z_0) \leq \sum_{j=1}^{p} \omega(f_j, z_0), \tag{1}$$

and when $f_j \not\equiv 0 (\forall j = 1,2,...,p)$, we have

$$\omega(\Pi_{j=1}^{p} f_j, z_0) = \sum_{j=1}^{p} \omega(f_j, z_0). \tag{2}$$

Definition 3. *Reference [2] Let $f(z)$ be a transcendental meromorphic function. The deficiency of a complex number a with respect to $f(z)$ is defined by*

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,\frac{1}{f-1})}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)}.$$

It is easy to see $0 \leq \delta(a,f) \leq 1$.

Definition 4. *Reference [2] If the coefficients of differential polynomials $M[f]$ are a_j, $j = 0, 1, ..., n$, which satisfy $m(r, a_j) = S(r, f)$, then $M[f]$ is called a quasi-differential polynomials in f.*

In 1959, Hayman proved the following theorem.

Theorem 1. (see [3]) *Let f be a transcendental meromorphic function, $n(\geq 3)$ be an integer. Then $\phi = f^n f'$ has infinitely many zeros for finite non-zero complex value a.*

Moreover, Hayman [4] conjectured that Theorem 1 remains valid for the cases $n = 1, 2$. In 1979, Mues [5] confirmed the case $n = 2$ and the conjecture was proved by Bergweiler-Eremenko [6] in 1995 and independently by H.H. Chen and M.L. Fang [7].

Naturally, we will ask that if the constant a is replaced by a small function of $\alpha(z)$, what is the distributions of zeros of $ff' - \alpha$? Many scholars have studied this problem.

In 1994, Q. D. Zhang proved the following two results:

Theorem 2. (see [8]) *Let f be a transcendental meromorphic function, $\alpha(\not\equiv 0, \infty)$ is a small function and $\delta(\infty; f) > \frac{7}{9}$, then $ff' - \alpha$ has infinitely many zeros.*

Theorem 3. (see [8]) *Let f be a transcendental meromorphic function, $a(\not\equiv 0, \infty)$ is a small function and $2\delta(0; f) + \delta(\infty; f) > 1$, then $ff' - \alpha$ has infinitely many zeros.*

In 1997, W. Bergweiler proved the following special case when f is of finite order and α is a polynomial:

Theorem 4. (see [9]) *If f is a transcendental meromorphic function of finite order and α is a non-vanishing polynomial, then $ff' - \alpha$ has infinitely many zeros.*

In order to achieve the desired result, there are some conditions for the zeros or poles of f in Theorem 2 and Theorem 3. Except for the order of f, there is no other conditional constraint in Theorem 4, but the result is only valid for the polynomial.

Yu deals with the general situation of the small functions and proved the following result:

Theorem 5. (see [10]) *Let f be a transcendental meromorphic function and $\alpha(\not\equiv 0, \infty)$ be a small function. Then $ff' - \alpha$ and $ff' + \alpha$ at least one has infinitely many zeros.*

Remark 1. Note that the proof of Theorem 5 requires the conclusion of Theorem 2, this is, the proof only holds under the condition $\delta(\infty; f) \leq 7/9$. In this paper, we will use a new way to get a quantitative description of Theorem 5 (see [11–13]). In fact, we prove the following result.

Theorem 6. *Let f be a transcendental meromorphic function and $\varphi(\not\equiv 0)$ be a small function. Then*

$$T(r, f) < 6\overline{N}(r, \frac{1}{\varphi f^2 f'^2 - 1}) + S(r, f). \tag{3}$$

Corollary 1. *Let f be a transcendental meromorphic function and $\alpha(\not\equiv 0)$ be a small function of f. Then*

$$T(r, f) < 6\overline{N}(r, \frac{1}{ff' + \alpha}) + 6\overline{N}(r, \frac{1}{ff' - \alpha}) + S(r, f). \tag{4}$$

From the corollary, we can obtain Theorem 5.

Recently, Y. Jiang obtained the following inequality:

Theorem 7. (see [14]) *Let f be a transcendental meromorphic function, let $\varphi(\not\equiv 0)$ be a small function and $n(\geq 2)$ be an integer. Then*

$$T(r,f) < (3+\frac{6}{n-1})N(r,\frac{1}{\varphi f^2(f')^n-1})+S^*(r,f), \tag{5}$$

where $S^*(r,f)=o(T(r,f))$ as $r\to\infty, r\notin E^*$, E^* is a set of logarithmic density 0.

If $n=2$, Theorem 6 improves the conclusion of Theorem 7. Not only is the coefficient 9 reduced to 6, but also the counting function is replaced by a reduced counting function. We conjecture the coefficient $3+\frac{6}{n-1}$ can be reduced to 6 for $n\geq 2$ in Theorem 7.

2. Lemmas

In order to prove our result, we need the following lemma.

Lemma 1. (see [15]) *Let f be a non-constant meromorphic function in the complex plane, let $Q_1[f], Q_2[f]$ be quasi-differential polynomials in f, satisfying $f^nQ_1[f]=Q_2[f]$. If the total degree of Q_2 is inferior or equal to n, then*

$$m(r,Q_1[f])=S(r,f).$$

Notations:

$$F(z)=\varphi(z)f^2(z)(f'(z))^2-1, \tag{6}$$

$$h(z)=\frac{F'(z)}{f(z)}=2\varphi(z)[f(z)f'(z)f''(z)+(f'(z))^3]+\varphi'(z)f(z)(f'(z))^2, \tag{7}$$

$$\phi(z)=\frac{h(z)}{F(z)}=\frac{1}{f(z)}\cdot\frac{F'(z)}{F(z)}, \tag{8}$$

$$\begin{aligned} G(z) &= 20(\frac{F'(z)}{F(z)})^2+24(\frac{F'(z)}{F(z)})'-39\frac{F'(z)}{F(z)}\frac{h'(z)}{h(z)}+18(\frac{h'(z)}{h(z)})^2-18(\frac{h'(z)}{h(z)})' \\ &+\frac{15}{4}\frac{\varphi'(z)}{\varphi(z)}\frac{F'(z)}{F(z)}-\frac{9}{4}\frac{\varphi'(z)}{\varphi(z)}\frac{h'(z)}{h(z)}-9(\frac{\varphi'(z)}{\varphi(z)})'-\frac{15}{8}(\frac{\varphi'(z)}{\varphi(z)})^2. \end{aligned} \tag{9}$$

Lemma 2. *Let $f(z)$ be a transcendental meromorphic function and let $\varphi(z)(\not\equiv 0)$ be a small function. Then $\varphi f^2f'^2$ is not equivalent to a constant.*

Proof. Suppose $\varphi f^2f'^2\equiv C$, where C is a constant.

Obviously, $C\neq 0$. Then

$$\frac{1}{f^4}\equiv\frac{\varphi}{C}(\frac{f'}{f})^2,\quad \frac{1}{f^2f'^2}\equiv C\varphi.$$

Therefore,

$$\begin{aligned} m(r,\frac{1}{f}) &\leq \frac{1}{4}m(r,\frac{1}{C}\varphi(\frac{f'}{f}))^2 \\ &\leq \frac{1}{4}m(r,\varphi)+\frac{1}{4}m(r,(\frac{f'}{f})^2)+O(1)=S(r,f), \\ N(r,\frac{1}{f}) &\leq N(r,\frac{1}{f^2f'^2}) \\ &= N(r,\frac{1}{C}\varphi)=S(r,f). \end{aligned}$$

From the above, we have $T(r,f)=S(r,f)$. It is a contradiction. Hence the proof of Lemma 2 is completed. □

Lemma 3. *Let f be a transcendental meromorphic function, and let $\varphi(z)(\not\equiv 0)$ be a small function of f. Then*

$$\begin{aligned} 4T(r,f) &\leq \overline{N}(r,f)+3\overline{N}(r,\frac{1}{f}) \\ &+ \overline{N}(r,\frac{1}{\varphi f^2 f'^2-1})-N_0(r,\frac{1}{(\varphi f^2 f'^2)'})+S(r,f), \end{aligned} \tag{10}$$

where $N_0(r,\frac{1}{(\varphi f^2 f'^2)'})$ denotes the counting function of the zeros of $(\varphi f^2 f'^2)'$, which are not zeros of $f(\varphi f^2 f'^2-1)$.

Proof. Let

$$\frac{1}{f^4} \equiv \frac{\varphi f^2 f'^2}{f^4} - \frac{(\varphi f^2 f'^2)'}{f^4}\frac{\varphi f^2 f'^2-1}{(\varphi f^2 f'^2)'},$$

We have

$$\begin{aligned} 4m(r,\tfrac{1}{f}) &= m(r,\frac{1}{f^4}) \\ &\leq m(r,\frac{\varphi f^2 f'^2-1}{(\varphi f^2 f'^2)'})+m(r,\varphi\frac{f^2 f'^2}{f^4})+m(r,\frac{(\varphi f^2 f'^2)'}{f^4})+O(1) \\ &\leq N(r,\frac{(\varphi f^2 f'^2)'}{\varphi f^2 f'^2-1})-N(r,\frac{\varphi f^2 f'^2-1}{(\varphi f^2 f'^2)'})+S(r,f) \\ &= N(r,(\varphi f^2 f'^2)')+N(r,\frac{1}{\varphi f^2 f'^2-1})-N(r,\frac{1}{(\varphi f^2 f'^2)'})- \\ &\quad N(r,\varphi f^2 f'^2-1)+S(r,f) \\ &= \overline{N}(r,f)+N(r,\frac{1}{\varphi f^2 f'^2-1})-N(r,\frac{1}{(\varphi f^2 f'^2)'})+S(r,f). \end{aligned} \tag{11}$$

Hence

$$\begin{aligned} 4T(r,f) &= 4m(r,\frac{1}{f})+4N(r,\frac{1}{f})+O(1) \\ &= \overline{N}(r,f)+4N(r,\frac{1}{f})+N(r,\frac{1}{\varphi f^2 f'^2-1})-N(r,\frac{1}{(\varphi f^2 f'^2)'})+S(r,f). \end{aligned} \tag{12}$$

Let

$$N(r,\frac{1}{(\varphi f^2 f'^2)'}) = N_{000}(r,\frac{1}{(\varphi f^2 f'^2)'})+N_{00}(r,\frac{1}{(\varphi f^2 f'^2)'})+N_0(r,\frac{1}{(\varphi f^2 f'^2)'}), \tag{13}$$

where $N_{000}(r,\frac{1}{(\varphi f^2 f'^2)'})$ denotes the counting function of the zeros of $(\varphi f^2 f'^2)'$, which come from the zeros of $\varphi f^2 f'^2-1$, $N_{00}(r,\frac{1}{(\varphi f^2 f'^2)'})$ denotes the counting function of the zeros of $(\varphi f^2 f'^2)'$, which come from the zeros of f. Then we obtain

$$N(r,\frac{1}{\varphi f^2 f'^2-1})-N_{000}(r,\frac{1}{(\varphi f^2 f'62)'})=\overline{N}(r,\frac{1}{\varphi f^2 f'^2-1}). \tag{14}$$

Suppose that z_0 is a zero of f with multiplicity q and the pole of φ with multiplicity of t.

Case I. Suppose that $t \leq 4q-3$. If $q=1$, then z_0 is a zero of $(\varphi f^2 f'^2)'$ with multiplicity at least $1-t$; if $q \geq 2$, then z_0 is a zero of $(\varphi f^2 f'^2)'$ with multiplicity at least $4q-3-t$.

Case II. Suppose that $t \geq 4q-2$. Then z_0 is at most the pole of φ^2.

Hence we have

$$4N(r,\frac{1}{f})-N_{00}(r,\frac{1}{(\varphi f^2f'^2)'}) \quad \begin{aligned} &\leq 2N_{1)}(r,\tfrac{1}{f})+\overline{N}_{1)}(r,\tfrac{1}{f})+3\overline{N}_{(2}(r,\tfrac{1}{f})+N(r,\varphi^2) \\ &= 2N_{1)}(r,\tfrac{1}{f})+\overline{N}(r,\tfrac{1}{f})+2\overline{N}_{(2}(r,\tfrac{1}{f})+S(r,f). \end{aligned} \tag{15}$$

Combining (12)–(15), we have

$$4T(r,f) \quad \leq \quad \overline{N}(r,f)+3\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{\varphi f^2f'^2-1})-N_0(r,\frac{1}{(\varphi f^2f'^2)'}).$$

This completes the proof of the Lemma 3. □

Lemma 4. *Under the hypotheses of Theorem 6, for any $z_0 \in \mathbb{C}$, we have*

$$\omega(\tfrac{1}{f},z_0)+\omega(\tfrac{1}{h},z_0) \leq \omega(\tfrac{1}{fh},z_0)+\omega(\varphi,z_0)+\omega(\tfrac{1}{\varphi},z_0). \tag{16}$$

Proof. This proof is divided into three Cases:

Case 1. $f(z_0) \neq 0,\infty$. If $h(z_0) \neq 0$, then $\omega(\frac{1}{f},z_0)+\omega(\frac{1}{h},z_0)=0$. If $h(z_0)=0$, from (2), then we get

$$\omega(\tfrac{1}{f},z_0)+\omega(\tfrac{1}{h},z_0)=\omega(\tfrac{1}{fh},z_0).$$

Hence the inequality (16) holds.

Case 2. $f(z_0)=0$. By $\omega(\frac{\varphi'}{\varphi},z_0) \leq 1$, we have

$$\tfrac{h}{\varphi}=2[ff'f''+(f')^3]+\tfrac{\varphi'}{\varphi}f(f')^2 \neq \infty (z=z_0).$$

From this, (1) and (2), we have

$$\begin{aligned} \omega(\tfrac{1}{f},z_0)+\omega(\tfrac{1}{h},z_0) \quad &\leq \quad \omega(\tfrac{1}{f},z_0)+\omega(\tfrac{\varphi}{h},z_0)+\omega(\tfrac{1}{\varphi},z_0) \\ &= \quad \omega(\tfrac{1}{f}\cdot\tfrac{\varphi}{h},z_0)+\omega(\tfrac{1}{\varphi},z_0) \\ &\leq \quad \omega(\tfrac{1}{fh},z_0)+\omega(\varphi,z_0)+\omega(\tfrac{1}{\varphi},z_0). \end{aligned}$$

Hence the inequality (16) holds.

Case 3. $f(z_0)=\infty$. Suppose $l=\omega(f,z_0)$, $l_1=\max\{\omega(\varphi,z_0),\omega(\frac{1}{\varphi},z_0)\}$. In the following, we divide into two Subcases:

Subcase 3.1. Let $1 \leq l \leq l_1$. Then

$$\begin{aligned} \omega(\tfrac{1}{f},z_0)+\omega(\tfrac{1}{h},z_0) \quad &= \quad \omega(\tfrac{1}{fh}\cdot f,z_0) \leq \omega(\tfrac{1}{fh},z_0)+\omega(f,z_0) \\ &\leq \quad \omega(\tfrac{1}{fh},z_0)+\omega(\varphi,z_0)+\omega(\tfrac{1}{\varphi},z_0). \end{aligned}$$

Subcase 3.2. Let $l > l_1 \geq 0$. Using the Laurent series of $\frac{h}{f^3\varphi}=2[\frac{f'f''}{f^2}+(\frac{f'}{f})^3]+\frac{\varphi'}{\varphi}\cdot(\frac{f'}{f})^2$ at the point z_0, we obtain the coefficient of $(z-z_0)^{-3}$:

$$a_{-3}=-2l^2(l+1)-2l^3-l^2(l_1+1)<0.$$

Thus $\omega(\frac{f^3\varphi}{h},z_0)=0$. Therefore,

$$\begin{aligned} \omega(\tfrac{1}{f},z_0)+\omega(\tfrac{1}{h}) \quad &= \quad \omega(\tfrac{f^3\varphi}{h}\cdot\tfrac{1}{f^3}\cdot\tfrac{1}{\varphi},z_0) \\ &\leq \quad \omega(\tfrac{f^3\varphi}{h},z_0)+3\omega(\tfrac{1}{f},z_0)+\omega(\tfrac{1}{\varphi},z_0)=\omega(\tfrac{1}{\varphi},z_0). \end{aligned}$$

Hence the inequality (16) holds.

This completes the proof of Lemma 4. □

Lemma 5. *Under the hypotheses of Theorem 6, if $z_0 \in \mathbb{C}$ and $G(z_0) = 0$, then*

$$\omega(\phi, z_0) \leq 2\omega(\varphi, z_0) + \omega(\frac{1}{\varphi}, z_0), \tag{17}$$

$$\omega(\frac{1}{F}, z_0) \leq \omega(\frac{1}{h}, z_0) + 2\omega(\varphi, z_0) + \omega(\frac{1}{\varphi}, z_0). \tag{18}$$

Proof. First, we prove the following inequality

$$\overline{\omega}(\frac{1}{F}, z_0) \leq \omega(\varphi, z_0) + \omega(\frac{1}{\varphi}, z_0). \tag{19}$$

Obviously, if $F(z_0) \neq 0$, then the inequality (19) holds. Now let $\omega(\frac{1}{F}, z_0) = l(l \geq 1)$ and $\varphi(z_0) \neq 0, \infty$. Since $\varphi(z_0)f^2(z_0)f'^2(z_0) = F(z_0) + 1 = 1$, then $f(z_0) \neq 0, \infty$. Thus, z_0 is a zero of $h(z) = \frac{F'(z)}{f(z)}$ with multiplicity $l - 1$ (if $l = 1$ and $h(z_0) \neq 0, \infty$). Using the Laurent series of $G(z)$ at the point z_0, we obtain the coefficient of $(z - z_0)^{-2}$:

$$b_{-2} = 20l^2 - 24l - 39l(l-1) + 18(l-1)^2 + 18(l-1) = -l(l+3) < 0.$$

It contradicts with $G(z_0) = 0$. Hence z_0 is a zero or a pole of $\varphi(z_0)$. This implies that the inequality (19) holds.

In order to prove (17), we will divide two Cases.

Case 1. $f(z_0) \neq 0$.

Suppose that $F(z_0) \neq \infty$. By (19), we have

$$\begin{aligned} \omega(\phi, z_0) &= \omega(\tfrac{1}{f} \cdot \tfrac{F'}{F}, z_0) \leq \omega(\tfrac{1}{f}, z_0) + \omega(\tfrac{F'}{F}, z_0) \\ &= \overline{\omega}(\tfrac{1}{F}, z_0) \leq \omega(\varphi, z_0) + \omega(\tfrac{1}{\varphi}, z_0). \end{aligned}$$

Suppose that $F(z_0) = \infty$. If $f(z_0) = \infty$ and $\frac{1}{f} \cdot \frac{F'}{F} \neq \infty (z = z_0)$, then

$$\omega(\phi, z_0) = \omega(\frac{1}{f} \cdot \frac{F'}{F}, z_0) = 0.$$

If $f(z_0) \neq \infty$, since $\varphi(z_0)f^2(z_0)f'^2(z_0) = F(z_0) + 1 = \infty$, then we get

$$\omega(\varphi, z_0) \geq 1.$$

Therefore,

$$\omega(\phi, z_0) \leq \omega(\frac{1}{f}, z_0) + \omega(\frac{F'}{F}, z_0) = 1 \leq \omega(\varphi, z_0).$$

Case 2. $f(z_0) = 0$.

Suppose that $\omega(\frac{1}{f}, z_0) > \omega(\varphi, z_0)$. Then $F(z_0) = \varphi(z_0)f^2(z_0)f'^2(z_0) - 1 = -1$ and $\frac{\varphi'}{\varphi} f \neq \infty$ $(z = z_0)$, therefore

$$\omega(\frac{1}{F}, z_0) = 0,$$

$$\omega(\frac{h}{\varphi}, z_0) = \omega(2ff'f'' + f'^3 + \frac{\varphi'}{\varphi} ff'^2, z_0) = 0.$$

Thus, $\omega(\phi, z_0) = \omega(\frac{1}{F} \cdot \frac{h}{\varphi} \cdot \varphi, z_0) \leq \omega(\frac{1}{F}, z_0) + \omega(\frac{h}{\varphi}, z_0) + \omega(\varphi, z_0) = \omega(\varphi, z_0)$.

Suppose that $1 \leq \omega(\frac{1}{f}, z_0) \leq \omega(\varphi, z_0)$. Then

$$\omega(\phi, z_0) \leq \omega(\frac{1}{f}, z_0) + \omega(\frac{F'}{F}, z_0) \leq \omega(\phi, z_0) + 1 \leq 2\omega(\phi, z_0).$$

Therefore, the inequality (17) holds.

In the following we begin to prove the Equation (18).

If $F(z_0) \neq 0$, then the inequality (18) obviously holds. If $F(z_0) = 0$, then from (19) we obtain

$$\begin{aligned} \omega(\tfrac{1}{F}, z_0) - \omega(\tfrac{1}{h}, z_0) &= [\omega(\tfrac{1}{F}, z_0) - \omega(\tfrac{1}{F'}, z_0)] + [\omega(\tfrac{1}{fh}, z_0) - \omega(\tfrac{1}{h}, z_0)] \\ &\leq \overline{\omega}(\tfrac{1}{F}, z_0) + \omega(\tfrac{1}{f}, z_0) \leq \omega(\tfrac{1}{f}, z_0) + \omega(\varphi, z_0) + \omega(\tfrac{1}{\varphi}, z_0). \end{aligned}$$

If $f(z_0) \neq 0$, then we have $\omega(\frac{1}{f}, z_0) = 0$.

If $f(z_0) = 0$, then from $\varphi(z_0) f^2(z_0) f'^2(z_0) = F(z_0) + 1 = 1$ we have $\omega(\frac{1}{f}, z_0) \leq \omega(\varphi, z_0)$. Hence

$$\omega(\frac{1}{F}, z_0) - \omega(\frac{1}{h}, z_0) \leq 2\omega(\varphi, z_0) + \omega(\frac{1}{\varphi}, z_0).$$

Thus, the inequality (18) holds.

This completes the proof of Lemma 5. □

Lemma 6. *Let f be a transcendental meromorphic function, and let $a_j(z) (j = 0, 1, \cdots, 5)$ be meromorphic functions, satisfying $T(r, a_j) = S(r, f)$. If*

$$a_5(z) f^5(z) + a_4(z) f^4(z) + a_3(z) f^3(z) + a_2(z) f^2(z) + a_1(z) f(z) + a_0(z) = 0,$$

then $a_j(z) \equiv 0, (j = 0, 1, \cdots, 5)$.

Proof. If $a_5(z) \not\equiv 0$, then from $f^5 = -\frac{a_4}{a_5} f^4 - \frac{a_3}{a_5} f^3 - \frac{a_2}{a_5} f^2 - \frac{a_1}{a_5} f - \frac{a_0}{a_5}$, we get

$$\begin{aligned} 5T(r, f) &\leq 4T(r, f) + T(r, \tfrac{a_4}{a_5}) + T(r, \tfrac{a_0}{a_5}) \\ &\leq 4T(r, f) + S(r, f). \end{aligned}$$

It is a contradiction. Hence $a_5 \equiv 0$. Similarly, we get $a_j \equiv 0 (j = 0, 1, \cdots, 5)$.

This completes the proof of Lemma 6. □

3. The Proof of Theorem 6

Now we begin to prove Theorem 6.

Since $F(z) = \varphi(z) f^2(z) (f'(z))^2 - 1$ and

$$h(z) = \frac{F'(z)}{f(z)} = 2\varphi(z) \{ f(z) f'(z) f''(z) + (f'(z))^3 \} + \varphi'(z) f(z) (f'(z))^2.$$

Obviously, $h(z) \not\equiv 0$. If $h(z) \equiv 0$, then $F(z) \equiv C$, where C is a constant. By Lemma 2, it is a contradiction.

Suppose z_0 is a simple pole of f, such that $\varphi(z_0) \neq 0, \infty$. We firstly prove $G(z_0) = 0$. Near $z = z_0$, we have

$$f(z) = \frac{a}{(z - z_0)} \left\{ 1 + b(z - z_0) + c(z - z_0)^2 + O[(z - z_0)^3] \right\} (a \neq 0),$$

and

$$\varphi(z) = A \left\{ 1 + x(z - z_0) + y(z - z_0)^2 + O[(z - z_0)^3] \right\} (A \neq 0).$$

Therefore we obtain

$$\begin{aligned} F(z) &= \varphi(z)f^2(z)(f'(z))^2 - 1 \\ &= \frac{Aa^4}{(z-z_0)^6} + \frac{Aa^4(x+2b)}{(z-z_0)^5} + \frac{Aa^4(b^2+2bx+y)}{(z-z_0)^4} + O[(z-z_0)^{-3}], \end{aligned}$$

$$h(z) = \frac{F'(z)}{f(z)} = -\frac{6Aa^3}{(z-z_0)^6} - \frac{Aa^3(5x+4b)}{(z-z_0)^5} - \frac{Aa^3(3bx+4y-6c)}{(z-z_0)^4} + O[(z-z_0)^{-3}].$$

$$\frac{\varphi'}{\varphi} = x + (2y-x^2)(z-z_0) + O[(z-z_0)^2], \tag{20}$$

$$\frac{F'}{F} = -\frac{6}{(z-z_0)} + (x+2b) + (2y-x^2-2b^2)(z-z_0) + O[(z-z_0)^2], \tag{21}$$

$$\frac{h'}{h} = -\frac{6}{(z-z_0)} + (\tfrac{5}{6}x + \tfrac{2}{3}b) + (\tfrac{25}{36}x^2 + \tfrac{4}{9}b^2 + \tfrac{1}{9}bx + 2c - \tfrac{4}{3}y)(z-z_0) + O[(z-z_0)^2], \tag{22}$$

$$\left(\frac{F'}{F}\right)^2 = \frac{36}{(z-z_0)^2} - \frac{12(x+2b)}{(z-z_0)} + (13x^2 + 28b^2 + 4bx - 24y) + O[(z-z_0)], \tag{23}$$

$$\left(\frac{F'}{F}\right)' = \frac{2k+4}{(z-z_0)^2} + (2y + 4c - x^2 - 2b^2) + O[(z-z_0)], \tag{24}$$

$$\frac{F'}{F}\frac{h'}{h} = \frac{36}{(z-z_0)^2} - \frac{(11x+16b)}{(z-z_0)} + (11x^2 + 16b^2 + 3bx + 12c - 20y) + O[(z-z_0)], \tag{25}$$

$$\left(\frac{h'}{h}\right)^2 = \frac{36}{(z-z_0)^2} - \frac{2(5x+4b)}{(z-z_0)} + (\tfrac{325}{36}x^2 + \tfrac{52}{9}b^2 + \tfrac{29}{18}bx + 24c - 16y) + O[(z-z_0)], \tag{26}$$

$$\left(\frac{h'}{h}\right)' = \frac{6}{(z-z_0)^2} + (\tfrac{25}{36}x^2 + \tfrac{4}{9}b^2 + \tfrac{1}{9}bx + 2c - \tfrac{4}{3}y] + O[(z-z_0)], \tag{27}$$

$$\frac{\varphi'}{\varphi}\frac{F'}{F} = -\frac{6x}{(z-z_0)^2} + (7x^2 + 2bx - 12y) + O[(z-z_0)], \tag{28}$$

$$\frac{\varphi'}{\varphi}\frac{h'}{h} = -\frac{6x}{(z-z_0)^2} + (\tfrac{41}{6}x^2 + \tfrac{2}{3}bx - 12y) + O[(z-z_0)], \tag{29}$$

$$\left(\frac{\varphi'}{\varphi}\right)' = (2y - x^2) + O[(z-z_0)], \tag{30}$$

$$\left(\frac{\varphi'}{\varphi}\right)^2 = x^2 + 2x(2y-x^2)(z-z_0) + (2y-x^2)^2(z-z_0)^2 + O[(z-z_0)^3]. \tag{31}$$

Substituting (23)–(31) into (9), we have

$$G(z) = O[(z-z_0)].$$

This shows $G(z_0) = 0$, which means that the simple pole of $f(z)$ is the zero of $G(z)$ except for the zeros and poles of $\varphi(z)$.

In the following, we begin to prove $G(z) \not\equiv 0$.

Suppose $G(z) \equiv 0$. From (17) and (18) of Lemma 5, we have

$$N(r,\phi) \le 2N(r,\varphi) + N(r,\tfrac{1}{\varphi}) = S(r,f), \tag{32}$$

and

$$N(r,\tfrac{1}{F}) - N(r,\tfrac{1}{h}) \le 2N(r,\varphi) + N(r,\tfrac{1}{\varphi}) = S(r,f). \tag{33}$$

By (11), we have

$$4m(r,\tfrac{1}{f}) \le \overline{N}(r,f) + N(r,\tfrac{1}{F}) - N(r,\tfrac{1}{fh}) + S(r,f). \tag{34}$$

By (16), we have

$$N(r,\tfrac{1}{f})+N(r,\tfrac{1}{h})\le N(r,\tfrac{1}{fh})+N(r,\varphi)+N(r,\tfrac{1}{\varphi}). \tag{35}$$

From (34) and (35), we have

$$4m(r,\tfrac{1}{f})\le \overline{N}(r,f)+N(r,\tfrac{1}{F})-N(r,\tfrac{1}{f})-N(r,\tfrac{1}{h})+S(r,f). \tag{36}$$

From (33) and (36), we have

$$3m(r,\tfrac{1}{f})\le N(r,\tfrac{1}{F})-N(r,\tfrac{1}{h})+S(r,f)=S(r,f). \tag{37}$$

From (32) and (37), we have

$$\begin{aligned} T(r,\phi) &= m(r,\phi)+N(r,\phi)=m(r,\tfrac{1}{f}\cdot\tfrac{F'}{F})+N(r,\phi)\\ &\le m(r,\tfrac{1}{f})+m(r,\tfrac{F'}{F})+N(r,\phi)=S(r,f). \end{aligned} \tag{38}$$

By (8), we have

$$\tfrac{F'}{F}=\phi f, \tag{39}$$

and

$$\tfrac{h'}{h}=\tfrac{F'}{F}+\tfrac{\phi'}{\phi}=\phi f+\tfrac{\phi'}{\phi}. \tag{40}$$

Substituting (39) and (40) into (9), we have

$$f'=\tfrac{1}{6}\phi f^2-\tfrac{1}{2}(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})f+\tfrac{1}{16}\tfrac{P}{\phi}, \tag{41}$$

where

$$P=48(\tfrac{\phi'}{\phi})'-48(\tfrac{\phi'}{\phi})^2+6\tfrac{\varphi'}{\varphi}\tfrac{\phi'}{\phi}+24(\tfrac{\varphi'}{\varphi})'+5(\tfrac{\varphi'}{\varphi})^2. \tag{42}$$

Therefore,

$$\begin{aligned} F=\varphi f^2f'^2-1 &= \tfrac{1}{36}\varphi\phi^2f^6-\tfrac{1}{6}\varphi\phi(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})f^5+\tfrac{1}{4}\varphi[\tfrac{P}{12}+(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})^2]f^4\\ &- \tfrac{\varphi P}{16\phi}(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})f^3+\tfrac{\varphi P^2}{256\phi^2}f^2-1. \end{aligned} \tag{43}$$

Differentiating (43) and combining (41), we have

$$\begin{aligned} F' &= \tfrac{1}{36}\varphi\phi^3f^7+[\tfrac{(\varphi\phi^2)'}{36}-\tfrac{\varphi\phi^2}{6}(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})]f^6\\ &+ [\tfrac{7\varphi\phi P}{288}+\tfrac{\varphi\phi}{4}(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})^2+\tfrac{\varphi\phi}{6}(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})'-\tfrac{(\varphi\phi)'}{6}(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})]f^5\\ &+ \Big(\tfrac{\varphi}{4}(2\varphi+1)(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})^3+\tfrac{\varphi'}{4}(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})^2-\tfrac{\varphi P}{24}(\varphi+\tfrac{\phi'}{24}+4)(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})\\ &- \varphi^2(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})'+\tfrac{1}{48}[\tfrac{\varphi\phi'P}{\phi}-\varphi P+(\varphi P)']\Big)f^4\\ &+ [\tfrac{\varphi P}{16\phi}(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})^2+\tfrac{P}{8\phi}(\varphi-\tfrac{1}{4})(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})-\tfrac{(\varphi P)'}{16\phi}(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})+\tfrac{3\varphi P^2}{384\phi}]f^3\\ &+ \tfrac{1}{256\phi^2}[(\varphi P^2)'-\tfrac{2\varphi\phi'P^2}{\phi}-4\varphi P^2(\tfrac{\varphi'}{2\varphi}+\tfrac{\phi'}{\phi})]f^2\\ &+ \tfrac{\varphi P^3}{2048\phi^3}f. \end{aligned} \tag{44}$$

Substituting (42) and (43) into the equality $\phi=\frac{F'}{fF}$, we have

$$a_5f^5+a_4f^4+a_3f^3+a_2f^2+a_1f+a_0=0, \tag{45}$$

where

$$a_5=\tfrac{(\varphi\phi^2)'}{36}, \tag{46}$$

$$a_4 = \frac{\varphi\phi P}{288} + \frac{\varphi\phi}{6}(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi})' - \frac{(\varphi\phi)'}{6}(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi}), \tag{47}$$

$$\begin{aligned} a_3 &= \frac{\varphi}{4}(2\varphi+1)(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi})^3 + \frac{\varphi'}{4}(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi})^2 - \frac{\varphi P}{24}(\varphi + \frac{\phi'}{24} + \frac{5}{2})(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi}) \\ &- \varphi^2(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi})(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi})' + \frac{1}{48}[\frac{\varphi\phi' P}{\phi} - \varphi P + (\varphi P)'], \end{aligned} \tag{48}$$

$$a_2 = \frac{\varphi P}{16\phi}(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi})^2 + \frac{P}{8\phi}(\varphi - \frac{1}{4})(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi}) - \frac{(\varphi P)'}{16\phi}(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi}) + \frac{\varphi P^2}{256\phi}, \tag{49}$$

$$a_1 = \frac{1}{256\phi^2}[(\varphi P^2)' - \frac{2\varphi\phi' P^2}{\phi} - 4\varphi P^2(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi})], \tag{50}$$

$$a_0 = \frac{\varphi P^3}{2048\phi^3} + \phi. \tag{51}$$

From the assumptions of Theorem 6, (38) and (42), we have

$$T(r,\varphi) = S(r,f), \quad T(r,\phi) = S(r,f), \quad T(r,P) = S(r,f).$$

Therefore,

$$T(r,a_j) = S(r,f) \quad (j = 1,2,3,4,5).$$

Applying Lemma 6, we have

$$a_j(z) \equiv 0 \quad (j = 1,2,3,4,5).$$

From (51) and $a_0(z) \equiv 0$, we have

$$P^6 = \frac{2048^2\phi^8}{\varphi^2}. \tag{52}$$

Therefore

$$P(z) \not\equiv 0. \tag{53}$$

From (50) and $a_1(z) \equiv 0$, we have

$$\frac{(\frac{\varphi P^2}{\phi^2})'}{\frac{\varphi P^2}{\phi^2}} = 4(\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi}).$$

Therefore

$$\frac{\varphi P^2}{\phi^2} = c\varphi^2\phi^4, \tag{54}$$

where c is a nonzero constant.

Combining (52) and (54), we have

$$\varphi^5\phi^{10} = \frac{2048^2}{c}.$$

Therefore

$$\frac{\varphi'}{2\varphi} + \frac{\phi'}{\phi} = 0. \tag{55}$$

Substituting (55) into (49), and combining $a_2(z) \equiv 0$ we have

$$P(z) \equiv 0.$$

It is a contradiction with (53), thus $G(z) \not\equiv 0$.

Differentiating the equation $F = \varphi f^2 f'^2 - 1$, we get

$$f\beta = -\frac{F'}{F}, \tag{56}$$

where

$$\beta = \varphi' f f'^2 + 2(\varphi f')^3 + 2\varphi f f' f'' - \varphi f f'^2 \frac{F'}{F}, \quad h = -\beta F.$$

Note that the poles of $G(z)$ whose multiplicities are at most two, come from the multiple poles of $f(z)$, the zeros of $F(z)$ and $h(z)$, or the zeros and the poles of $\varphi(z)$.

Next we consider the poles of $\beta^2 G$. From $h = -\beta F$, we know the zeros of h are either the zeros of F, or the zeros of β. From (56), we know that the multiple poles of f with multiplicity $q(\geq 2)$ are the zeros of β with multiplicity of $q-1$. Therefore, the poles of $\beta^2 G$ only come from the zeros of F, except for the zeros and the poles of $\varphi(z)$, and the multiplicity of $\beta^2 G$ is at most 4. Thus

$$N(r, \beta^2 G) \leq 4\overline{N}(r, 1/F) + S(r, f).$$

By (56) and Lemma 1, we have $m(r, \beta) = S(r, f)$. Note that $m(r, G) = S(r, f)$. Therefore $m(r, \beta^2 G) = S(r, f)$. From the above, we have

$$T(r, \beta^2 G) \leq 4\overline{N}(r, 1/F) + S(r, f).$$

If z_1 is a zero of f with multiplicity $p(\geq 2)$ and a pole of $\varphi(z)$ with multiplicity t, then z_1 is a zero of β with multiplicity at least $3p - 3 - t$, therefore, is a zero of $\beta^2 G$ with multiplicity at least $2(3p-3) - 2 - t = 6p - 8 - t$. Also note that the simple pole of f is the zero of $\beta^2 G$ except for the zeros and poles of φ. Hence we have

$$\begin{aligned} \overline{N}_{1)}(r, f) + 4N_1(r, \frac{1}{f}) - N(r, \varphi) &\leq N(r, \tfrac{1}{\beta^2 G}) + N(r, \varphi) + N(r, \tfrac{1}{\varphi}) \\ &\leq T(r, \beta^2 G) + S(r, f) \\ &\leq 4\overline{N}(r, \frac{1}{F}) + S(r, f), \end{aligned} \tag{57}$$

where $N_1(r, \frac{1}{f}) = N(r, \frac{1}{f}) - \overline{N}(r, \frac{1}{f})$.

From (10), we have

$$m(r, f) + N(r, f) - \overline{N}(r, f) + 3m(r, \frac{1}{f}) + 3N_1(r, \frac{1}{f}) \leq \overline{N}(r, \frac{1}{F}) + S(r, f). \tag{58}$$

Combining doubled (58) with (57), we have

$$\begin{aligned} & T(r, f) + N_{(2}(r, f) - 2\overline{N}_{(2}(r, f) + m(r, f) + 6m(r, \frac{1}{f}) + 10N_1(r, \frac{1}{f}) \\ \leq \;& 6\overline{N}(r, \frac{1}{F}) + S(r, f), \end{aligned}$$

Hence we have

$$T(r, f) < 6\overline{N}(r, \frac{1}{\varphi f^2 f'^2 - 1}) + S(r, f).$$

This completes the proof of Theorem 6.

4. The Proof of Corollary 1

Let $\varphi = \frac{1}{\alpha^2}$, $\alpha \neq 0, \infty$. Then by Theorem 6, we have

$$\begin{aligned} T(r, f) &< 6\overline{N}(r, \frac{1}{f^2 f'^2 - \alpha^2}) + S(r, f) \\ &< 6\overline{N}(r, \frac{1}{f f' + \alpha}) + 6\overline{N}(r, \frac{1}{f f' - \alpha}) + S(r, f). \end{aligned}$$

This completes the proof of Corollary 1.

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