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Alexandrov L-Fuzzy Pre-Proximities

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Abstract: In this paper, we introduce the concepts of Alexandrov L-fuzzy pre-proximities on complete residuated lattices. Moreover, we investigate their relations among Alexandrov L-fuzzy pre-proximities, Alexandrov L-fuzzy topologies, L-fuzzy upper approximate operators, and L-fuzzy lower approximate operators. We give their examples.

Keywords: complete residuated lattice; Alexandrov L-fuzzy topologies; L-lower and L-upper approximation operators; Alexandrov L-fuzzy pre-proximities

1. Introduction

Pawlak [1,2] introduced the concept of rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Ward et al. [3] introduced the concept of the complete residuated lattice, which is an algebraic structure for many-valued logic. It is an important mathematical tool for studying algebraic structure. By using lower and upper approximation operators, information systems and decision rules were investigated in complete residuated lattices [4–19]. Bělohlávek [4] developed the notion of fuzzy contexts using Galois connections with $R \in L^{X\times Y}$ on a complete residuated lattice. El-Dardery [6] introduced L-fuzzy pre-proximity in view points of Sostak's fuzzy topology [9] and Kim's L-fuzzy proximities [13] on strictly two-sided, commutative quantales. Kim [10–15] investigated the properties of Alexandrov L-fuzzy topologies, Alexandrov L-fuzzy quasi-uniformities, and L-fuzzy approximate operators in complete residuated lattices.

In this paper, we introduce the concepts of Alexandrov L-fuzzy pre-proximities on complete residuated lattices, which are a unified approach to the three spaces: Alexandrov L-fuzzy topologies, L-fuzzy lower approximate operators, and L-fuzzy lower approximate operators as an extension of Pawlak's rough sets. Moreover, we investigate their relations among Alexandrov L-fuzzy pre-proximities, Alexandrov L-fuzzy topologies, L-fuzzy lower approximate operators, and L-fuzzy lower approximate operators. We give their examples.

2. Preliminaries

Definition 1 ([4,8–10])**.** *An algebra* $(L, \wedge, \vee, \odot, \to, \bot, \top)$ *is a complete residuated lattice if:*

(L1) $(L, \leq, \vee, \wedge, \bot, \top)$ *is a complete lattice with the greatest element* $\top$ *and the least element* $\bot$*;*
(L2) $(L, \odot, \top)$ *is a commutative monoid;*
(L3) $x \odot y \leq z$ *if and only if* $x \leq y \to z$ *for all* $x, y, z \in L$*.*

In this paper, we always assume that $(L, \leq, \odot, \to, \oplus, ^*)$ is a complete residuated lattice with an order-reversing involution *, which is defined by:

$$x \oplus y = (x^* \odot y^*)^*, \ \ x^* = x \to \bot$$

unless otherwise specified. For all $\alpha \in L$,

$$(\alpha \to f)(x) = \alpha \to f(x),\ (\alpha \odot f)(x) = \alpha \odot f(x),\ \alpha_X(x) = \alpha,$$

$$\top_x(y) = \begin{cases} \top & \text{if } y = x, \\ \bot, & \text{otherwise} \end{cases} \quad \text{and} \quad \top_x^*(y) = \begin{cases} \bot & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

Lemma 1 ([4,7,8])**.** *Let $x, y, z, x_i, y_i, w \in L$. Then, the following hold.*

(1) $\top \to x = x$ *and* $\bot \odot x = \bot$.
(2) *If* $y \le z$, *then* $x \odot y \le x \odot z$, $x \oplus y \le x \oplus z$, $x \to y \le x \to z$, *and* $z \to x \le y \to x$.
(3) $x \le y$ *if and only if* $x \to y = \top$.
(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$ *and* $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$.
(5) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$.
(6) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$.
(7) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$.
(8) $(\bigwedge_i x_i) \oplus y = \bigwedge_i (x_i \oplus y)$.
(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.
(10) $x \odot y = (x \to y^*)^*$ *and* $x \oplus y = x^* \to y$.
(11) $(x \to y) \odot (z \to w) \le (x \odot z) \to (y \odot w)$.
(12) $x \to y \le (x \odot z) \to (y \odot z)$ *and* $(x \to y) \odot (y \to z) \le x \to z$.
(13) $(x \to y) \odot (z \to w) \le (x \oplus z) \to (y \oplus w)$.
(14) $x \to y = y^* \to x^*$.
(15) $(x \vee y) \odot (z \vee w) \le (x \vee z) \vee (y \odot w) \le (x \oplus z) \vee (y \odot w)$.
(16) $\bigvee_i x_i \to \bigvee_i y_i \ge \bigwedge_i (x_i \to y_i)$ *and* $\bigwedge_i x_i \to \bigwedge_i y_i \ge \bigwedge_i (x_i \to y_i)$.
(17) $(x \odot y) \odot (z \oplus w) \le (x \odot z) \oplus (y \odot w)$.
(18) $x \to y \le (y \to z) \to (x \to z)$ *and* $x \to y \le (z \to x) \to (z \to y)$.

Definition 2 ([4])**.** *Let X be a set. A mapping $R : X \times X \to L$ is an L-partial order if:*

(E1) $R(x, x) = \top$ *for all* $x \in X$ *(reflexive);*
(E2) $R(x, y) \odot R(y, z) \le R(x, z)$ *for all* $x, y, z \in X$ *(transitive);*
(E3) *if* $R(x, y) = R(y, x) = \top$, *then* $x = y$ *(antisymmetric).*

Definition 3 ([4])**.** *Let X be a set. Define a mapping $S : L^X \times L^X \to L$ by:*

$$S(f, g) = \bigwedge_{x \in X} (f(x) \to g(x)) \quad \text{for all } f, g \in L^X.$$

Lemma 2 ([4])**.** *Let $f, g, h, k \in L^X$, and $\alpha \in L$. Then, the following hold.*

(1) S *is an L-partial order on* L^X.
(2) $f \le g$ *if and only if* $S(f, g) \ge \top$.
(3) *If* $f \le g$, *then* $S(h, f) \le S(h, g)$ *and* $S(f, h) \ge S(g, h)$.
(4) $S(f, g) \odot S(k, h) \le S(f \oplus k, g \oplus h)$ *and* $S(f, g) \odot S(k, h) \le S(f \odot k, g \odot h)$.
(5) $S(g, h) \le S(f, g) \to S(f, h)$.
(6) $S(f, h) = \bigvee_{g \in L^X} (S(f, g) \odot S(g, h))$.

Definition 4 ([10])**.** *A mapping $\mathcal{J} : L^X \to L^X$ is an L-lower approximation operator on X if:*

(J1) $\mathcal{J}(\top_X) = \top_X$ *where* $\top_X(x) = \top$ *for all* $x \in X$;
(J2) $\mathcal{J}(f) \le f$ *for all* $f \in L^X$;
(J3) $\mathcal{J}(\bigwedge_{i \in \Gamma} f_i) = \bigwedge_{i \in \Gamma} \mathcal{J}(f_i)$ *for all* $\{f_i\}_{i \in \Gamma} \subseteq L^X$;
(J4) $\mathcal{J}(\alpha \to f) = \alpha \to \mathcal{J}(f)$.

The pair $(X, \mathcal{J})$ is called an L-lower approximation space. An L-lower approximation space is called topological if:

(T) $\mathcal{J}(\mathcal{J}(f)) = \mathcal{J}(f)$ *for all* $f \in L^X$.

Definition 5 ([10]). *A mapping* $\mathcal{H} : L^X \to L^X$ *is an L-upper approximation operator on X if:*

(H1) $\mathcal{H}(\bot_X) = \bot_X$ *where* $\bot_X(x) = \bot$ *for all* $x \in X$;
(H2) $\mathcal{H}(f) \geq f$ *for all* $f \in L^X$;
(H3) $\mathcal{H}(\bigvee_{i\in\Gamma} f_i) = \bigvee_{i\in\Gamma} \mathcal{H}(f_i)$ *for all* $\{f_i\}_{i\in\Gamma} \subseteq L^X$;
(H4) $\mathcal{H}(\alpha \odot f) = \alpha \odot \mathcal{H}(f)$.

The pair $(X, \mathcal{H})$ *is called an L-upper approximation space. An L-upper approximation space is called topological if:*

(T) $\mathcal{H}(\mathcal{H}(f)) = \mathcal{H}(f)$ *for all* $f \in L^X$.

Definition 6 ([10–12]). *Let* τ *be a subset of* L^X. τ *is an Alexandrov L-topology on X if:*

(O1) $\bot_X, \top_X \in \tau$;
(O2) *If* $A_i \in \tau$ *for all* $i \in I$, *then* $\bigwedge_{i\in I} A_i, \bigvee_{i\in I} \in \tau$;
(O3) *If* $A \in \tau$ *and* $\alpha \in L$, *then* $\alpha \odot A, \alpha \to A \in \tau$.

Definition 7 ([10]). *A mapping* $\mathcal{T} : L^X \to L$ *is an Alexandrov L-fuzzy topology on X if:*

(AT1) $\mathcal{T}(\bot_X) = \mathcal{T}(\top_X) = \top$;
(AT2) $\mathcal{T}(\bigwedge_i f_i) \geq \bigwedge_i \mathcal{T}(f_i)$ *and* $\mathcal{T}(\bigvee_i f_i) \geq \bigwedge_i \mathcal{T}(f_i)$ *for all* $\{f_i\}_{i\in\Gamma} \subseteq L^X$;
(AT3) $\mathcal{T}(\alpha \odot f) \geq \mathcal{T}(f)$ *and* $\mathcal{T}(\alpha \to f) \geq \mathcal{T}(f)$ *for all* $\alpha \in L$ *and* $f \in L^X$.

The pair $(X, \mathcal{T})$ *is called an L-fuzzy topological space.*

Theorem 1 ([10–12]). (1) *Let* $\mathcal{J} : L^X \to L^X$ *be an L-lower approximation operator. Define* $\mathcal{H}_{\mathcal{J}} : L^X \to L^X$ *by* $\mathcal{H}_{\mathcal{J}}(f) = \mathcal{J}^*(f^*)$. *Then,* $\mathcal{H}_{\mathcal{J}}$ *is an L-upper approximation operator.*

(2) *Let* $\mathcal{H} : L^X \to L^X$ *be an L-upper approximation operator. Define* $\mathcal{J}_{\mathcal{H}} : L^X \to L^X$ *by* $\mathcal{J}_{\mathcal{H}}(f) = \mathcal{H}^*(f^*)$. *Then,* $\mathcal{J}_{\mathcal{H}}$ *is an L-lower approximation operator.*

(3) *Let* $\mathcal{T} : L^X \to L$ *be an Alexandrov L-fuzzy topology. Define* $\mathcal{T}^* : L^X \to L$ *by* $\mathcal{T}^*(f) = \mathcal{T}(f^*)$. *Then,* $\mathcal{T}^*$ *is an Alexandrov L-fuzzy topology.*

(4) *Let* $\tau \subset L^X$ *be an Alexandrov L-topology. Define* $\tau^* = \{f \mid f^* \in \tau\}$. *Then,* τ^* *is an Alexandrov L-topology.*

Theorem 2 ([10]). *Let* $(X, \mathcal{H})$ *be an L-upper approximation space. Define a mapping* $\mathcal{T}_{\mathcal{H}} : L^X \to L$ *by* $\mathcal{T}_{\mathcal{H}}(f) = S(\mathcal{H}(f), f)$. *Then,* $\mathcal{T}_{\mathcal{H}}$ *is an Alexandrov L-fuzzy topology on X with* $\mathcal{T}^*_{\mathcal{H}}(f) = S(f, \mathcal{J}_{\mathcal{H}}(f))$ *where* $\mathcal{J}_{\mathcal{H}}(f) = \mathcal{H}^*(f^*)$ *for all* $f \in L^X$.

Theorem 3 ([10]). *Let* $(X, \mathcal{J})$ *be an L-lower approximation space. Define a map* $\mathcal{T}_{\mathcal{J}} : L^X \to L$ *by* $\mathcal{T}_{\mathcal{J}}(f) = S(f, \mathcal{J}(f))$. *Then,* $\mathcal{T}_{\mathcal{J}}$ *is an Alexandrov L-fuzzy topology on X.*

3. The Relationships between Alexandrov *L*-Fuzzy Pre-Proximities and Alexandrov Topological Structures

Definition 8. *A mapping* $\delta : L^X \times L^X \to L$ *is an Alexandrov L-fuzzy pre-proximity on X if:*

(P1) $\delta(\bot_X, \top_X) = \delta(\top_X, \bot_X) = \bot$;
(P2) $\delta(f, g) \geq \bigvee_{x\in X}(f(x) \odot g(x))$;
(P3) *If* $f \leq f_1$ *and* $g \leq g_1$, *then* $\delta(f, g) \leq \delta(f_1, g_1)$;
(P4) *For all* $f_i, f, g_i, g \in L^X$, $\delta(\bigvee_{i\in\Gamma} f_i, g) \leq \bigvee_{i\in\Gamma} \delta(f_i, g)$ *and* $\delta(f, \bigvee_{i\in\Gamma} g_i) \leq \bigvee_{i\in\Gamma} \delta(f, g_i)$;
(P5) *For all* $\alpha \in L$ *and* $f, g \in L^X$, $\delta(\alpha \odot f, g) = \alpha \odot \delta(f, g) = \delta(f, \alpha \odot g)$.

An Alexandrov L-fuzzy pre-proximity δ *on X is called an Alexandrov L-fuzzy quasi-proximity if:*

(P) $\delta(f,g) \geq \bigwedge_{h\in L^X} \delta(f,h) \oplus \delta(h^*,g)$.

Let δ_1 and δ_2 be two Alexandrov L-fuzzy pre-proximities on X. δ_1 is finer than δ_2 if $\delta_2(f,g) \geq \delta_1(f,g)$ for all $f,g \in L^X$.

Example 1. *Let $R \in L^{X\times X}$. Define a mapping $\delta : L^X \times L^X \to L$ by $\delta(f,g) = \bigvee_{x,y\in X}(R(x,y) \odot f(x) \odot g(y))$.*

(1) *Assume that R is reflexive. Then:*

(P1) $\delta(\perp_X, \top_X) = \delta(\top_X, \perp_X) = \perp$;
(P2) $\delta(f,g) \geq \bigvee_{x\in X}(R(x,x) \odot f(x) \odot g(x)) = \bigvee_{x\in X}(f(x) \odot g(x))$;
(P3) *If $f \leq f_1$ and $g \leq g_1$, then $\delta(f,g) \leq \delta(f_1,g_1)$;*
(P4) *For all $f_i, f, g_i, g \in L^X$, $\delta(\bigvee_{i\in\Gamma} f_i, g) = \bigvee_{i\in\Gamma} \delta(f_i,g)$ and $\delta(f, \bigvee_{i\in\Gamma} g_i) = \bigvee_{i\in\Gamma} \delta(f,g_i)$.*
(P5) *For all $\alpha \in L$ and $f,g \in L^X$,*

$$\begin{aligned}\delta(\alpha \odot f, g) &= \bigvee_{x,y\in X} (R(x,y) \odot (\alpha \odot f(x) \odot g(y))) \\ &= \alpha \odot \bigvee_{x,y\in X} (R(x,y) \odot (f(x) \odot g(y))) \\ &= \alpha \odot \delta(f,g).\end{aligned}$$

Hence, δ is an Alexandrov L-fuzzy pre-proximity on X.

(2) *Assume that R is reflexive and transitive. Then, $\bigvee_{y\in X}(R(y,z) \odot R(x,y)) = R(x,z)$. For all $f,g,h \in L^X$, we have by Lemma 1 (17) that:*

$$\begin{aligned}\delta(f,h) \oplus \delta(h^*,g) &= \Big(\bigvee_{x,y\in X} (R(x,y) \odot f(x) \odot h(y))\Big) \oplus \Big(\bigvee_{y,z\in X} (R(y,z) \odot h^*(y) \odot g(z))\Big) \\ &\geq \Big(\bigvee_{x,y,z\in X} (R(x,y) \odot f(x) \odot h(y)) \oplus (R(y,z) \odot h^*(y) \odot g(z))\Big) \\ &\geq \Big(\bigvee_{x,y,z\in X} (R(x,y) \odot R(y,z) \odot f(x) \odot g(z)) \odot (h(y) \oplus h^*(y))\Big) \\ &= \bigvee_{x,y,z\in X} (R(x,y) \odot R(y,z) \odot f(x) \odot g(z)) \\ &= \bigvee_{x,z\in X} (R(x,z) \odot f(x) \odot g(z)) = \delta(f,g).\end{aligned}$$

Thus, $\delta(f,g) \leq \bigwedge_{h\in L^X}(\delta(f,h) \oplus \delta(h^*,g))$.

Let $h(y) = \Big(\bigvee_{x\in X}(R(x,y) \odot f(x))\Big)^*$. Then:

$$\begin{aligned}&\bigwedge_{h\in L^X} (\delta(f,h) \oplus \delta(h^*,g)) \\ &= \bigwedge_{h\in L^X} ((\bigvee_{x,y\in X} (R(x,y) \odot f(x) \odot h(y))) \oplus (\bigvee_{y,z\in X} (R(y,z) \odot h^*(y) \odot g(z)))) \\ &\leq (\bigvee_{y\in X} (h^*(y) \odot h(y))) \oplus (\bigvee_{y,z\in X} (R(y,z) \odot \bigvee_{x\in X} (R(x,y) \odot f(x) \odot g(z)))) \\ &= \perp \oplus (\bigvee_{x,z\in X} (\bigvee_{y\in X} (R(y,z) \odot R(x,y)) \odot f(x) \odot g(z))) \\ &= \bigvee_{x,z\in X} (R(x,z) \odot f(x) \odot g(z)) = \delta(f,g).\end{aligned}$$

Hence, δ is an Alexandrov L-fuzzy quasi-proximity on X.

By taking $R(x,y) = \top_{X\times X}$*, let:*

$$\delta_1(f,g) = \bigvee_{x,y\in X} (\top_{X\times X}(x,y) \odot f(x) \odot g(y)) = \bigvee_{x,y\in X} (f(x) \odot g(y)).$$

Define $\triangle_{X\times X} \in L^{X\times X}$ *by:*

$$\triangle_{X\times X}(x,y) = \begin{cases} \top & \text{if } x = y, \\ \bot & \text{otherwise.} \end{cases}$$

By taking $R(x,y) = \triangle_{X\times X}$*, let:*

$$\delta_2(f,g) = \bigvee_{x,y\in X} (\triangle_{X\times X}(x,y) \odot (f(x) \odot g(y))) = \bigvee_{x\in X} (f(x) \odot g(x)).$$

Then, $\delta_2(f,g) \le \delta(f,g) \le \delta_1(f,g)$ *for all* $f,g \in L^X$.

Lemma 3. *Let* δ *be an Alexandrov L-fuzzy pre-proximity on X. For all* $\alpha \in L$ *and* $f,g,f_i,g_i \in L^X$*, the following hold.*

(1) $\delta(\bigvee_{i\in\Gamma} f_i, g) = \bigvee_{i\in\Gamma} \delta(f_i,g)$ *and* $\delta(f, \bigvee_{i\in\Gamma} g_i) = \bigvee_{i\in\Gamma} \delta(f,g_i)$.
(2) $\delta(\alpha \odot f, \alpha \to g) \le \delta(f,g)$ *and* $\delta(\alpha \to f, \alpha \odot g) \le \delta(f,g)$.

Proof. (1) It follows from (P3) and (P4).
(2) It follows from $\delta(\alpha \odot f, \alpha \to g) = \alpha \odot \delta(f, \alpha \to g) = \delta(f, \alpha \odot (\alpha \to g)) \le \delta(f,g)$.
□

Theorem 4. *Let* δ *be an Alexandrov L-fuzzy pre-proximity on X. Define a mapping* $\delta^s : L^X \times L^X \to L$ *by* $\delta^s(f,g) = \delta(g,f)$*. Then, the following hold.*

(1) δ^s *is an Alexandrov L-fuzzy pre-proximity on X.*
(2) $\delta(f,g) = \bigvee_{x,y\in X}(\delta(\top_x, \top_y) \odot (f(x) \odot g(y))$.
(3) *There exists a reflexive L-fuzzy relation* $R_\delta \in L^{X\times X}$ *such that:*

$$\delta(f,g) = \bigvee_{x,y\in X} (R_\delta(x,y) \odot (f(x) \odot g(y))).$$

(4) *There exists a reflexive L-fuzzy relation* $R_{\delta^s} = R_\delta^{-1} \in L^{X\times X}$ *such that:*

$$\delta^s(f,g) = \bigvee_{x,y\in X} (R_\delta^{-1}(x,y) \odot (f(x) \odot g(y))).$$

Proof. (1) It is easily proven.
(2) Since $f = \bigvee_{x\in X}(f(x) \odot \top_x)$ and $g = \bigvee_{y\in X}(g(y) \odot \top_y)$, we have:

$$\begin{aligned} \delta(f,g) &= \delta(\bigvee_{x\in X} (f(x) \odot \top_x), \bigvee_{y\in X} (g(y) \odot \top_y)) \\ &= \bigvee_{x\in X} \Big(f(x) \odot \delta(\top_x, \bigvee_{y\in X} (g(y) \odot \top_y))\Big) \\ &= \bigvee_{x,y\in X} \Big(f(x) \odot g(y) \odot \delta(\top_x, \top_y)\Big). \end{aligned}$$

(3) Let $R_\delta(x,y) = \delta(\top_x, \top_y)$ in the equation in (2). By (P2),

$$R_\delta(x,x) = \delta(\top_x, \top_x) \geq \bigvee_{x\in X} (\top_x(x) \odot \top_x(x)) = \top.$$

Moreover, $\delta(f,g) = \bigvee_{x,y\in X}(R_\delta(x,y) \odot f(x) \odot g(y))$.
(4) Since $R_{\delta^s}(x,y) = \delta^s(\top_x, \top_y) = \delta(\top_y, \top_x) = R_\delta^{-1}(x,y)$ by (2), we have:

$$\begin{aligned}\delta^s(f,g) &= \delta(g,f)\\ &= \bigvee_{x,y\in X} (R_\delta(x,y) \odot (g(x) \odot f(y)))\\ &= \bigvee_{x,y\in X} \Big(R_\delta^{-1}(y,x) \odot (f(y) \odot g(x))\Big).\end{aligned}$$

□

Theorem 5. *Let δ be an Alexandrov L-fuzzy pre-proximity on X. Define a mapping $\mathcal{T}_\delta : L^X \to L$ by $\mathcal{T}_\delta(f) = \delta^*(f, f^*)$. Then, $\mathcal{T}_\delta$ is an Alexandrov L-fuzzy topology on X such that $\mathcal{T}_\delta^* = \mathcal{T}_{\delta^s}$. If $\delta_1 \leq \delta_2$, then $\mathcal{T}_{\delta_1} \geq \mathcal{T}_{\delta_2}$.*

Proof. (AT1) $\mathcal{T}_\delta(\top_X) = \delta^*(\top_X, \top_X^*) = \top$ and $\mathcal{T}_\delta(\bot_X) = \delta^*(\bot_X, \bot_X^*) = \top$.
(AT2) By (P3) and (P4), we have:

$$\mathcal{T}_\delta(\bigwedge_i f_i) = \delta^*(\bigwedge_i f_i, \bigvee_i f_i^*) \geq \delta^*(f_i, \bigvee_i f_i^*) = \bigwedge_i \delta^*(f_i, f_i^*) = \bigwedge_i \mathcal{T}_\delta(f_i)$$

and:

$$\mathcal{T}_\delta(\bigvee_i f_i) = \delta^*(\bigvee_i f_i, \bigwedge_i f_i^*) \geq \delta^*(\bigvee_i f_i, f_i^*) = \bigwedge_i \delta^*(f_i, f_i^*) = \bigwedge_i \mathcal{T}_\delta(f_i).$$

(AT3) By Lemma 3 (2), we have:

$$\begin{aligned}&\mathcal{T}_\delta(\alpha \odot f) = \delta^*(\alpha \odot f, \alpha \to f^*) = \alpha \to \delta^*(f, \alpha \to f^*) = \delta^*(f, \alpha \odot (\alpha \to f^*))\\ &\geq \delta^*(f, f^*) = \mathcal{T}_\delta(f), \mathcal{T}_\delta(\alpha \to f) = \delta^*(\alpha \to f, \alpha \odot f^*) \geq \delta^*(f, f^*) = \mathcal{T}_\delta(f).\end{aligned}$$

Then, $\mathcal{T}_\delta$ is an Alexandrov L-fuzzy topology on X. Moreover,

$$\mathcal{T}_\delta^*(f) = \mathcal{T}_\delta(f^*) = \delta^*(f^*, f) = \delta^{s*}(f, f^*) = \mathcal{T}_{\delta^s}(f).$$

□

Example 2. *Let $R \in L^{X\times X}$ be a reflexive fuzzy relation. Define a mapping $\delta : L^X \times L^X \to L$ by $\delta(f,g) = \bigvee_{x,y\in X}(R(x,y) \odot f(x) \odot g(y))$. Then:*

$$\begin{aligned}\mathcal{T}_\delta(f) &= \delta^*(f, f^*) = \Big(\bigvee_{x,y\in X} (R(x,y) \odot f(x) \odot f^*(y))\Big)^*\\ &= \bigwedge_{x,y\in X} (R(x,y) \to (f(x) \to f(y)).\end{aligned}$$

If $R = \top_{X\times X}$, then $\mathcal{T}_\delta(f) = \bigwedge_{x,y\in X}(f(x) \to f(y))$.
If $R = \triangle_{X\times X}$, then $\mathcal{T}_\delta(f) = \bigwedge_{x\in X}(f(x) \to f(x)) = \top$.

From the following two theorems, we obtain the L-lower approximation operator and the L-lower approximation operator induced by an Alexandrov L-fuzzy pre-proximity.

Theorem 6. *Let δ be an Alexandrov L-fuzzy pre-proximity on X. Define a mapping $\mathcal{H}_\delta : L^X \to L^X$ by $\mathcal{H}_\delta(f)(x) = \delta(f, \top_x)$. Then, the following hold.*

(1) *$\mathcal{H}_\delta$ is an L-upper approximation operator on X.*
(2) $\delta(\top_x, \top_x) = \top$.
(3) *There exists a reflexive L-fuzzy relation $R_\delta \in L^{X\times X}$ such that:*

$$\mathcal{H}_\delta(f)(x) = \bigvee_{y\in X} (R_\delta(y,x) \odot f(y)).$$

Moreover, there exists a reflexive L-fuzzy relation $R_{\delta^s} = R_\delta^{-1} \in L^{X\times X}$ such that:

$$\mathcal{H}_{\delta^s}(f)(x) = \bigvee_{y\in X} (R_\delta(x,y) \odot f(y)).$$

(4) $\bigvee_{y\in X}(\delta(\top_x, \top_y) \odot \delta(\top_y, \top_z)) \le \delta(\top_x, \top_z)$ *if and only if $\mathcal{H}_\delta$ is a topological L-upper approximation operator on X.*
(5) $\mathcal{T}_{\mathcal{H}_\delta}(f) = \delta^*(f, f^*) = \mathcal{T}_\delta(f)$ *for all $f \in L^X$.*
(6) $\delta(f,g) = \bigvee_{x\in X}(\mathcal{H}_\delta(f)(x) \odot g(x))$ *for all $f, g \in L^X$.*

Proof. (1) (H1) Since $\delta(\bot_X, \top_x) \le \delta(\bot_X, \top_X) = \bot$, we have $\mathcal{H}_\delta(\bot_X)(x) = \delta(\bot_X, \top_x) = \bot$.
(H2) $\mathcal{H}_\delta(f)(x) = \delta(f, \top_x) \ge \bigvee_{x\in X}(f(x) \odot \top_x(x)) = f(x)$.
(H3) From Lemma 3, we obtain:

$$\begin{aligned}\mathcal{H}_\delta(\bigvee_{i\in\Gamma} f_i)(x) = \delta(\bigvee_{i\in\Gamma} f_i, \top_x) &= \bigvee_{i\in\Gamma} \delta(f_i, \top_x) \\ &= \bigvee_{i\in\Gamma} \mathcal{H}_\delta(f_i)(x).\end{aligned}$$

(H4) By (P4), $\mathcal{H}_\delta(\alpha \odot f)(x) = \delta(\alpha \odot f, \top_x) = \alpha \odot \delta(f, \top_x) = \alpha \odot \mathcal{H}_\delta(f)$. Hence, $\mathcal{H}_\delta$ is an L-upper approximation operator on X.

(2) $\delta(\top_x, \top_x) \ge \bigvee_{x\in X}(\top_x(x) \odot \top_x(x)) = \top$.
(3) We obtain $\mathcal{H}_\delta(f)(x) = \delta(f, \top_x) = \delta(\bigvee_{y\in X}(f(y) \odot \top_y), \top_x) = \bigvee_{y\in X}(f(y) \odot \delta(\top_y, \top_x))$. Put $R_\delta(x,y) = \delta(\top_x, \top_y)$. By (2), R_δ is reflexive. Then, $\mathcal{H}_\delta(f)(x) = \bigvee_{y\in X}(f(y) \odot R_\delta(y,x))$. Moreover, $R_{\delta^s}(x,y) = \delta^s(\top_x, \top_y) = \delta(\top_y, \top_x) = R_\delta(y,x) = R_\delta^{-1}(x,y)$ such that:

$$\begin{aligned}\mathcal{H}_{\delta^s}(f)(x) &= \bigvee_{y\in X} (f(y) \odot \delta^{s*}(\top_y, \top_x)) \\ &= \bigvee_{y\in X} (f(y) \odot \delta(\top_x, \top_y)) = \bigvee_{y\in X} (f(y) \odot R_\delta(x,y)).\end{aligned}$$

(4) Since $\mathcal{H}_\delta(f) = \bigvee_{y\in X}(\mathcal{H}_\delta(f)(y) \odot \top_y)$, we have:

$$\begin{aligned}
\mathcal{H}_\delta(\mathcal{H}_\delta(f))(x) &= \delta(\mathcal{H}_\delta(f), \top_x) = \delta(\bigvee_{y\in X}(\mathcal{H}_\delta(f)(y) \odot \top_y), \top_x) \\
&= \bigvee_{y\in X}(\mathcal{H}_\delta(f)(y) \odot \delta(\top_y, \top_x)) = \bigvee_{y\in X}(\delta(f, \top_y) \odot \delta(\top_y, \top_x)) \\
&= \bigvee_{y\in X}(\delta(\bigvee_{z\in X}(f(z) \odot \top_z), \top_y) \odot \delta(\top_y, \top_x)) \\
&= \bigvee_{y\in X}(\bigvee_{z\in X}(f(z) \odot \delta(\top_z, \top_y)) \odot \delta(\top_y, \top_x)) \\
&= \bigvee_{z\in X}(f(z) \odot \bigvee_{y\in X}(\delta(\top_z, \top_y) \odot \delta(\top_y, \top_x))) \\
&\le \bigvee_{z\in X}(f(z) \odot \delta(\top_z, \top_x)) = \delta(\bigvee_{z\in X}(f(z) \odot \top_z), \top_x) \\
&= \delta(f, \top_x) = \mathcal{H}_\delta(f)(x).
\end{aligned}$$

Conversely, since $\mathcal{H}_\delta(\mathcal{H}_\delta(\top_z))(x) \le \mathcal{H}_\delta(\top_z)(x)$, for $\mathcal{H}_\delta(\top_z) = \bigvee_{y\in X}(\mathcal{H}_\delta(\top_z)(y) \odot \top_y)$, we have:

$$\begin{aligned}
\mathcal{H}_\delta(\mathcal{H}_\delta(\top_z))(x) &= \mathcal{H}_\delta(\bigvee_{y\in X}(\mathcal{H}_\delta(\top_z)(y) \odot \top_y))(x) \\
&= \bigvee_{y\in X}(\mathcal{H}_\delta(\top_z)(y) \odot \mathcal{H}_\delta(\top_y)(x)) \le \mathcal{H}_\delta(\top_z)(x).
\end{aligned}$$

(5) For all $f \in L^X$, we have:

$$\begin{aligned}
\mathcal{T}_{\mathcal{H}_\delta}(f) &= S(\mathcal{H}_\delta(f), f) = \bigwedge_{x\in X}(\mathcal{H}_\delta(f)(x) \to f(x)) \\
&= \bigwedge_{x\in X}(\delta(f, \top_x) \to f(x)) = \bigwedge_{x\in X}(f^*(x) \to \delta^*(f, \top_x)) \\
&= \bigwedge_{x\in X}\delta^*(f, f^*(x) \odot \top_x) = \delta^*(f, \bigvee_{x\in X}(f^*(x) \odot \top_x)) \\
&= \delta^*(f, f^*) = \mathcal{T}_\delta(f).
\end{aligned}$$

(6)

$$\begin{aligned}
\bigvee_{x\in X}(\mathcal{H}_\delta(f)(x) \odot g(x)) &= \bigvee_{x\in X}(\delta(f, \top_x) \odot g(x)) \\
&= \delta(f, \bigvee_{x\in X}(\top_x \odot g(x))) = \delta(f, g).
\end{aligned}$$

□

Theorem 7. *Let δ be an Alexandrov L-fuzzy pre-proximity on X. Define a mapping $\mathcal{J}_\delta : L^X \to L^X$ by $\mathcal{J}_\delta(f)(x) = \delta^*(\top_x, f^*)$. Then, the following hold.*

(1) *$\mathcal{J}_\delta$ is an L-lower approximation operator on X.*
(2) *There exists a reflexive L-fuzzy relation $R_\delta \in L^{X\times X}$ such that:*

$$\mathcal{J}_\delta(f)(x) = \bigwedge_{y\in X}(R_\delta(x, y) \to f(y)).$$

Moreover, there exists a reflexive L-fuzzy relation $R_{\delta^s} = R_\delta^{-1} \in L^{X\times X}$ *such that:*

$$\mathcal{J}_{\delta^s}(f)(x) = \bigwedge_{y\in X}(R_\delta(y,x) \to f(y)).$$

(3) *For all* $f \in L^X$, $\bigvee_{y\in X}(\delta(\top_x,\top_y)\odot\delta(\top_y,\top_z)) \le \delta(\top_x,\top_z)$ *if and only if* $\mathcal{J}_\delta(\mathcal{J}_\delta(f)) \ge \mathcal{J}_\delta(f)$.
(4) $\mathcal{T}_{\mathcal{J}_\delta}(f) = \delta^*(f,f^*) = \mathcal{T}_\delta(f)$ *for all* $f \in L^X$.
(5) $\mathcal{J}_{\delta^s}(f) = \delta(f^*,\top_x^*) = \mathcal{H}_\delta^*(f^*)$ *for all* $f \in L^X$ *and* $\mathcal{T}^*_{\mathcal{J}_\delta} = \mathcal{T}_{\delta^s} = \mathcal{T}_{\mathcal{J}_{\delta^s}}$.
(6) $\delta(f,g) = S(f,\mathcal{J}_\delta(g))$ *for all* $f,g \in L^X$.

Proof. (1) (J1) Since $\delta^*(\top_x,\top_X^*) \ge \delta^*(\top_X,\perp_X) = \top$, we have $\mathcal{J}_\delta(\top_X)(x) = \delta^*(\top_x,\top_X^*) = \top$.
(J2) Note that:

$$\mathcal{J}_\delta(f)(x) = \delta^*(\top_x,f^*) \le (\bigvee_{x\in X}(\top_x(x)\odot f^*(x)))^* = f(x).$$

(J3) By Lemma 3, we obtain:

$$\mathcal{J}_\delta(\bigwedge_{i\in\Gamma} f_i)(x) = \delta^*(\top_x,\bigvee_{i\in\Gamma} f_i^*) = \bigwedge_{i\in\Gamma}\delta^*(\top_x,f_i^*) = \bigwedge_{i\in\Gamma}\mathcal{J}_\delta(f_i)(x).$$

(J4) By (P4), we have:

$$\mathcal{J}_\delta(\alpha\to f)(x) = \delta^*(\top_x,\alpha\odot f^*) = \alpha\to\delta^*(\top_x,f^*) = \alpha\to\mathcal{J}_\delta(f).$$

(2) For $f^* = \bigvee_{y\in X}(f^*(y)\odot\top_y)$, we have:

$$\begin{aligned}\mathcal{J}_\delta(f)(x) &= \delta^*(\top_x,f^*) = \delta^*(\top_x,\bigvee_{y\in X}(f^*(y)\odot\top_y))\\ &= \bigwedge_{y\in X}(f^*(y)\to\delta^*(\top_x,\top_y)) = \bigwedge_{y\in X}(\delta(\top_x,\top_y)\to f(y)).\end{aligned}$$

Let $R_\delta(x,y) = \delta(\top_x,\top_y)$. By (2), R_δ is reflexive and $\mathcal{J}_\delta(f)(x) = \bigwedge_{y\in X}(R_\delta(x,y)\to f(y))$. Moreover, $R_{\delta^s}(x,y) = \delta^{s*}(\top_x,\top_y) = \delta^*(\top_y,\top_x) = R_\delta(y,x) = R_\delta^{-1}(x,y)$ such that:

$$\mathcal{J}_{\delta^s}(f)(x) = \bigwedge_{y\in X}(\delta^{s*}(\top_x,\top_y)\to f(y)) = \bigwedge_{y\in X}(\delta^*(\top_y,\top_x)\to f(y)) = \bigwedge_{y\in X}(R_\delta(y,x)\to f(y)).$$

(3) Since $\mathcal{J}_\delta(f) = \bigwedge_{y\in X}(\mathcal{J}_\delta^*(f)(y)\to\top_y^*)$, we have:

$$\begin{aligned}\mathcal{J}_\delta(\mathcal{J}_\delta(f))(x) &= \delta^*(\top_x,\mathcal{J}_\delta^*(f))\\ &= \delta^*(\top_x,\bigvee_{y\in X}(\mathcal{J}_\delta^*(f)(y)\odot\top_y)) = \bigwedge_{y\in X}(\mathcal{J}_\delta^*(f)(y)\to\delta^*(\top_x,\top_y))\\ &= \bigwedge_{y\in X}(\delta(\top_y,\bigvee_{z\in X}(f^*(z)\odot\top_z))\to\delta^*(\top_x,\top_y))\\ &= \bigwedge_{y\in X}(\bigvee_{z\in X}(f^*(z)\odot\delta(\top_y,\top_z))\to\delta^*(\top_x,\top_y))\\ &= \Big(\bigvee_{z\in X}(f^*(z)\odot\bigvee_{y\in X}(\delta(\top_y,\top_z)\odot\delta(\top_x,\top_y)))\Big)^*\\ &\ge \Big(\bigvee_{y\in X}(f^*(z)\odot\delta(\top_x,\top_z))\Big)^*\\ &= \Big(\delta(\top_x,\bigvee_{y\in X}(f^*(z)\odot\top_z))\Big)^* = \delta^*(\top_x,f^*) = \mathcal{J}_\delta(f)(x).\end{aligned}$$

Conversely, since $\mathcal{J}_\delta(\top_y^*)(x) = \delta^*(\top_x, \top_y)$ and $\mathcal{J}_\delta(\top_z^*) = \bigwedge_{y\in X}(\mathcal{J}_\delta^*(\top_z^*)(y) \to \top_y^*)$, we have that $\mathcal{J}_\delta(\mathcal{J}_\delta(\top_z^*))(x) = \bigwedge_{y\in X}(\mathcal{J}_\delta^*(\top_z^*)(y) \to \mathcal{J}_\delta(\top_y^*)(x) \geq \mathcal{J}_\delta(\top_z^*)(x)$ if and only if $\bigvee_{y\in X}(\mathcal{J}_\delta^*(\top_z^*)(y) \odot \mathcal{J}_\delta^*(\top_y^*)(x) \leq \mathcal{J}_\delta^*(\top_z^*)(x)$ if and only if $\bigvee_{y\in X}(\delta(\top_y, \top_z) \odot \delta(\top_x, \top_y) \leq \delta(\top_x, \top_z)$.

(4) For $f = \bigvee_{x\in X}(f(x) \odot \top_x)$, we have:

$$\mathcal{T}_{\mathcal{J}_\delta}(f) = S(f, \mathcal{J}_\delta(f)) = \bigwedge_{x\in X}(f(x) \to \delta^*(\top_x, f^*)) = \bigwedge_{x\in X}\delta^*(f(x)\odot\top_x, f^*))$$
$$= \delta^*(\bigvee_{x\in X}(f(x)\odot\top_x), f^*) = \delta^*(f, f^*) = \mathcal{T}_\delta(f).$$

(5) For all $f \in L^X$, we have:

$$\mathcal{J}_{\delta^s}(f)(x) = \delta^{s*}(\top_x, f^*) = \delta^*(f^*, \top_x) = \mathcal{H}_\delta^*(f^*),$$
$$\mathcal{T}_{\mathcal{J}_\delta}^*(f) = \mathcal{T}_{\mathcal{J}_\delta}(f^*) = \delta^*(f^*, f) = \mathcal{T}_{\delta^s}(f) = \mathcal{T}_{\mathcal{J}_{\delta^s}}(f).$$

(6) For all $f, g \in L^X$, we have:

$$S(f, \mathcal{J}_\delta(g)) = \bigwedge_{x\in X}(f(x) \to \mathcal{J}_\delta(g))) = \bigwedge_{x\in X}(f(x) \to \delta^*(\top_x, g^*))$$
$$= \delta^*(\bigvee_{x\in X}(f(x)\odot\top_x), g^*) = \delta^*(f, g^*).$$

□

From the following theorem, we obtain the Alexandrov L-fuzzy pre-proximity induced by an L-upper approximation operator.

Theorem 8. *Let $(X, \mathcal{H})$ be an L-upper approximation space. Define a mapping $\delta_\mathcal{H} : L^X \times L^X \to L$ by:*

$$\delta_\mathcal{H}(f, g) = \bigvee_{y\in X}(\mathcal{H}(f)(y) \odot g(y)).$$

Then, the following hold.

(1) *$\delta_\mathcal{H}$ is an Alexandrov L-fuzzy proximity such that:*

$$\delta_\mathcal{H}(f, g) = \bigvee_{x,y\in X}(\mathcal{H}(\top_y)(x) \odot (f(y) \odot g(x))).$$

(2) *$\delta_\mathcal{H}(f, g) \leq \bigwedge_{h\in L^X}(\delta_\mathcal{H}(f, h) \oplus \delta_\mathcal{H}(h^*, g))$. Moreover, the equality holds if $\mathcal{H}$ is topological.*
(3) *If $\mathcal{H}$ is topological, then $\delta_\mathcal{H}$ is an Alexandrov L-fuzzy quasi-proximity on X.*
(4) *$\mathcal{H} = \mathcal{H}_{\delta_\mathcal{H}}$.*
(5) *$\mathcal{T}_\mathcal{H}(f) = \delta_\mathcal{H}(f, f) = \mathcal{T}_{\delta_\mathcal{H}}(f)$ for all $f \in L^X$.*
(6) *If δ is an Alexandrov L-fuzzy pre-proximity on X, then $\delta_{\mathcal{H}_\delta}(f, g) = \delta(f, g)$ for all $f, g \in L^X$.*

Proof. (1) (P1) Since $\mathcal{H}(\bot_X) = \bot_X$ and $\mathcal{H}(\top_X) = \top_X$, we have:

$$\delta_\mathcal{H}(\top_X, \bot_X) = \bigvee_{y\in X}(\mathcal{H}(\top_X)(y) \odot \bot_X(y)) = \bot,$$
$$\delta_\mathcal{H}(\bot_X, \top_X) = \bigvee_{y\in X}(\mathcal{H}(\bot_X)(y) \odot \top_X(y)) = \top.$$

(P2) Since $\mathcal{H}(f) \geq f$, we have:

$$\delta_\mathcal{H}(f, g) = \bigvee_{y\in X}(\mathcal{H}(f)(y) \odot g(y)) \geq \bigvee_{x\in X}(f(x) \odot g(x)).$$

(P3) If $f \leq f_1$ and $g \leq g_1$, then $\mathcal{H}(f) \leq \mathcal{H}(f_1)$. Thus,

$$\delta_{\mathcal{H}}(f,g) = \bigvee_{y\in X} (\mathcal{H}(f)(y) \odot g(y)) \leq \bigvee_{y\in X} (\mathcal{H}(f_1)(y) \odot g_1(y)) = \delta_{\mathcal{H}}(f_1,g_1).$$

(P4) Note that:

$$\begin{aligned}\delta_{\mathcal{H}}(\bigvee_{i\in\Gamma} f_i, g) &= \bigvee_{x\in X} (\mathcal{H}(\bigvee_{i\in\Gamma} f_i)(x) \odot g(x)) \\ &= \bigvee_{x\in X} (\bigvee_{i\in\Gamma} \mathcal{H}(f_i)(x) \odot g(x)) = \bigvee_{i\in\Gamma} \delta_{\mathcal{H}}(f_i,g),\end{aligned}$$

$$\delta_{\mathcal{H}}(f, \bigvee_{i\in\Gamma} g_i) = \bigvee_{x\in X} (f(x) \odot \bigvee_{i\in\Gamma} g_i(x)) = \bigvee_{i\in\Gamma} \delta_{\mathcal{H}}(f,g_i)$$

and:

$$\begin{aligned}\delta_{\mathcal{H}}(\alpha \odot f, g) &= \bigvee_{x\in X} (\mathcal{H}(\alpha \odot f)(x) \odot g(x)) \\ &= \bigvee_{x\in X} (\alpha \odot \mathcal{H}(f)(x) \odot g(x)) = \alpha \odot \delta_{\mathcal{H}}(f,g).\end{aligned}$$

Hence, $\delta_{\mathcal{H}}$ is an Alexandrov L-fuzzy pre-proximity. For $f = \bigvee(f(y) \odot \top_y)$, we have:

$$\begin{aligned}\delta_{\mathcal{H}}(f,g) &= \bigvee_{x\in X} (\mathcal{H}(f)(x) \odot g(x)) = \bigvee_{x\in X} (\mathcal{H}(\bigvee(f(y) \odot \top_y))(x) \odot g(x)) \\ &= \bigvee_{x\in X} (\bigvee_{y\in X} (f(y) \odot \mathcal{H}(\top_y)(x)) \odot g(x)) \\ &= \bigvee_{x,y\in X} (\mathcal{H}(\top_y)(x) \odot (f(y) \odot g(x))).\end{aligned}$$

(2) For each $f, g, h \in L^X$, we have:

$$\begin{aligned}&\delta_{\mathcal{H}}(f,h) \oplus \delta_{\mathcal{H}}(h^*,g) \\ &= \left(\bigvee_{x\in X} (\mathcal{H}(f)(x) \odot h(x))\right) \oplus \left(\bigvee_{x\in X} (\mathcal{H}(h^*)(x) \odot g(x))\right) \\ &\geq \bigvee_{x\in X} ((\mathcal{H}(f)(x) \odot h(x)) \oplus (\mathcal{H}(h^*)(x) \odot g(x))) \\ &\geq \bigvee_{x\in X} ((\mathcal{H}(f)(x) \odot f(x)) \odot (h(x) \oplus \mathcal{H}(h^*)(x))) \quad \text{by Lemma 1 (17)} \\ &= \bigvee_{x\in X} ((\mathcal{H}(f)(x) \odot f(x)) \odot (h^*(x) \to \mathcal{H}(h^*)(x))) \\ &= \delta_{\mathcal{H}}(f,g).\end{aligned}$$

Hence, $\delta_{\mathcal{H}}(f,g) \leq \bigwedge_{h\in L^X}(\delta_{\mathcal{H}}(f,h) \oplus \delta_{\mathcal{H}}(h^*,g))$.

If $\mathcal{H}$ is topological, then:

$$\begin{aligned}
&\bigwedge_{h\in L^X} (\delta_{\mathcal{H}}(f,h) \oplus \delta_{\mathcal{H}}(h^*,g)) \\
&= \bigwedge_{h\in L^X} ((\bigvee_{x\in X} (\mathcal{H}(f(x)) \odot h(x))) \oplus (\bigvee_{x\in X} (\mathcal{H}(h^*)(x) \odot g(x)))) \\
&(\text{ put } h^* = \mathcal{H}(f),) \\
&\leq (\bigvee_{x\in X} (\mathcal{H}(f(x)) \odot \mathcal{H}^*(f(x))) \oplus (\bigvee_{x\in X} (\mathcal{H}(\mathcal{H}(f))(x) \odot g(x)))) \\
&= (\bigvee_{x\in X} (\mathcal{H}(\mathcal{H}(f))(x) \odot g(x)))) = \delta_{\mathcal{H}}(f,g).
\end{aligned}$$

(3) It follows by (2).
(4) For all $f \in L^X$, we have:

$$\mathcal{H}_{\delta_{\mathcal{H}}}(f) = \delta_{\mathcal{H}}(f, \top_x) = \bigvee_{y\in X} (\mathcal{H}(f)(y) \odot \top_x(y)) = \mathcal{H}(f)(x).$$

(5) For all $f \in L^X$, we have:

$$\mathcal{T}_{\delta_{\mathcal{H}}}(f) = \delta^*_{\mathcal{H}}(f,f^*) = \left(\bigvee_{x\in X} (\mathcal{H}(f)(x) \odot f^*(x))\right)^* = \mathcal{T}_{\mathcal{H}}(f).$$

(6) For all $f, g \in L^X$, we have:

$$\begin{aligned}
\delta_{\mathcal{H}_\delta}(f,g) &= \bigvee_{y\in X} (\mathcal{H}_\delta(f)(y) \odot g(y)) = \bigvee_{y\in X} (\delta(f,\top_y) \odot g(y)) \\
&= \delta(f, \bigvee_{y\in X} (\top_y \odot g(y))) = \delta(f,g).
\end{aligned}$$

□

By the above theorem, we obtain the Alexandrov L-fuzzy pre-proximity induced by an L-lower approximation operator in a sense $\mathcal{H}_{\mathcal{J}}(f) = \mathcal{J}^*(f^*)$ for all $f \in L^X$.

Corollary 1. *Let $(X, \mathcal{J})$ be an L-lower approximation space. Define a mapping $\delta_{\mathcal{J}} : L^X \times L^X \to L$ by:*

$$\delta_{\mathcal{J}}(f,g) = \bigvee_{y\in X} (\mathcal{J}^*(f^*)(y) \odot g(y)).$$

Then, the following hold.

(1) *$\delta_{\mathcal{J}}$ is an Alexandrov L-fuzzy proximity such that:*

$$\delta_{\mathcal{J}}(f,g) = \bigvee_{x,y\in X} (\mathcal{J}^*(\top^*_y)(x) \odot (f(y) \odot g(x))).$$

(2) *$\delta_{\mathcal{J}}(f,g) \leq \bigwedge_{h\in L^X} (\delta_{\mathcal{J}}(f,h) \oplus \delta_{\mathcal{J}}(h^*,g))$. Moreover, the equality holds if $\mathcal{J}$ is topological.*
(3) *If $\mathcal{J}$ is topological, then $\delta_{\mathcal{J}}$ is an Alexandrov L-fuzzy quasi-proximity on X.*
(4) *$\mathcal{J} = \mathcal{J}_{\delta_{\mathcal{J}}}$.*
(5) *$\mathcal{T}_{\mathcal{J}}(f) = \delta_{\mathcal{J}}(f,f) = \mathcal{T}_{\delta_{\mathcal{J}}}(f)$ for all $f \in L^X$.*
(6) *If δ is an Alexandrov L-fuzzy pre-proximity on X, then $\delta_{\mathcal{J}_\delta}(f,g) = \delta(f,g)$ for all $f, g \in L^X$.*

Example 3. Let $([0,1], \odot, \to, ^*, 0, 1)$ be a complete residuated lattice [4,8–10] where:

$$x \odot y = \max\{0, x+y-1\},\ x \to y = \min\{1-x+y, 1\}$$

$$x \oplus y = \min\{x+y, 1\}, \quad x^* = 1 - x.$$

Let $X = \{x, y, z\}$. Consider the reflexive and transitive L-fuzzy relation $R \in [0,1]^{X \times X}$ defined by:

$$\begin{pmatrix} 1 & 0.7 & 0.8 \\ 0.5 & 1 & 0.4 \\ 0.6 & 0.7 & 1 \end{pmatrix}$$

(1) By Example 1, we obtain two Alexandrov L-fuzzy quasi-proximities $\delta, \delta^s : [0,1]^X \times [0,1]^X \to [0,1]$ where:

$$\delta(f,g) = \bigvee_{x,y \in X} (R(x,y) \odot (f(x) \odot g(y)),$$

$$\delta^s(f,g) = \bigvee_{x,y \in X} (R(y,x) \odot (f(x) \odot g(y)).$$

(2) By Theorem 5, we obtain two Alexandrov L-fuzzy topologies $\mathcal{T}_\delta, \mathcal{T}_{\delta^s} : [0,1]^X \times [0,1]^X \to [0,1]$ where:

$$\begin{aligned} \mathcal{T}_\delta(f) = \delta^*(f, f^*) &= \Big(\bigvee_{x,y \in X} (R(x,y) \odot (f(x) \odot f^*(y)) \Big)^* \\ &= \bigwedge_{x,y \in X} (R(x,y) \to (f(x) \odot f^*(y))^*) \\ &= \bigwedge_{x,y \in X} (R(x,y) \to (f(x) \to f(y)), \end{aligned}$$

$$\mathcal{T}_{\delta^s}(f) = \delta^*(f^*, f) = \bigwedge_{x,y \in X} (R(y,x) \to (f(x) \to f(y)).$$

(3) From Theorem 6 (4), since R is a reflexive and transitive L-fuzzy relation, we obtain two topological L-upper approximation operators $\mathcal{H}_\delta, \mathcal{H}_{\delta^s} : [0,1]^X \to [0,1]^X$ where:

$$\mathcal{H}_\delta(f)(x) = \delta(f, \top_x) = \bigvee_{y \in X} (R(y,x) \odot f(y)),$$

$$\mathcal{H}_{\delta^s}(f)(x) = \bigvee_{y \in X} (R(x,y) \odot f(y)).$$

(4) By Theorem 6 (4), we obtain two topological L-lower approximation operators $\mathcal{J}_\delta, \mathcal{J}_{\delta^s} : [0,1]^X \to [0,1]^X$ where:

$$\mathcal{J}_\delta(f)(x) = \delta^*(\top_x, f^*) = \bigwedge_{y \in X} (R(x,y) \to f(y)),$$

$$\mathcal{J}_{\delta^s}(f)(x) = \delta^*(f^*, \top_x) = \bigwedge_{y \in X} (R(y,x) \to f(y)).$$

(5) From Theorem 8, since $\mathcal{H}_\delta$ and $\mathcal{H}_{\delta^s}$ are topological L-upper approximation operators, we obtain two Alexandrov L-fuzzy quasi-proximities $\delta_{\mathcal{H}_\delta}, \delta_{\mathcal{H}_{\delta^s}} : [0,1]^X \times [0,1]^X \to [0,1]$ where:

$$\delta_{\mathcal{H}_\delta}(f,g) = \bigvee_{y \in X} (\mathcal{H}_\delta(f)(y) \odot (y)) = \bigvee_{x,y \in X} (R(x,y) \odot f(x)) \odot g(y)) = \delta(f,g).$$

$$\delta_{\mathcal{H}_{\delta^s}}(f,g) = \bigvee_{x,y \in X} (R(y,x) \odot (f(x) \odot g(y))) = \delta^s(f,g).$$

(6) By Corollary 1, since $\mathcal{J}_\delta$ and $\mathcal{J}_{\delta^s}$ are topological L-lower approximation operators, we obtain Alexandrov L-fuzzy quasi-proximities $\delta_{\mathcal{J}_\delta}, \delta_{\mathcal{J}_{\delta^s}} : [0,1]^X \times [0,1]^X \to [0,1]$ as:

$$\begin{aligned}\delta_{\mathcal{J}_\delta}(f,g) &= \bigvee_{y\in X} (\mathcal{J}_\delta^*(f^*)(y) \odot (y)) \\ &= \bigvee_{y\in X} ((\bigwedge_{x\in X} (R(y,x) \to f^*(x)))^* \odot g(y)) \\ &= \bigvee_{x,y\in X} (R(y,x) \odot f(x) \odot g(y)) = \delta^s(f,g).\end{aligned}$$

$$\begin{aligned}\delta_{\mathcal{J}_{\delta^s}}(f,g) &= \bigvee_{y\in X} (\mathcal{J}_{\delta^s}^*(f^*)(y) \odot (y)) \\ &= \bigvee_{y\in X} ((\bigwedge_{x\in X} (R(x,y) \to f^*(x)))^* \odot g(y)) \\ &= \bigvee_{x,y\in X} (R(x,y) \odot f(x) \odot g(y)) = \delta(f,g).\end{aligned}$$

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