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The Extremal Cacti on Multiplicative Degree-Kirchhoff Index

Fangguo He [1] **and Zhongxun Zhu** [2,*]

[1] College of Mathematics and Physics, Huanggang Normal University, Huanggang 438000, China; hfg0118@126.com

[2] College of Mathematics and Statistics, South Central University for Nationalities, Wuhan 430074, China

* Correspondence: zzxun73@mail.scuec.edu.cn

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Abstract: For a graph G, the resistance distance $r_G(x,y)$ is defined to be the effective resistance between vertices x and y, the multiplicative degree-Kirchhoff index $R^*(G) = \sum_{\{x,y\}\subset V(G)} d_G(x)d_G(y)r_G(x,y)$, where $d_G(x)$ is the degree of vertex x, and $V(G)$ denotes the vertex set of G. L. Feng et al. obtained the element in $Cact(n;t)$ with first-minimum multiplicative degree-Kirchhoff index. In this paper, we first give some transformations on $R^*(G)$, and then, by these transformations, the second-minimum multiplicative degree-Kirchhoff index and the corresponding extremal graph are determined, respectively.

Keywords: resistance distance; multiplicative degree-Kirchhoff index; cactus

1. Introduction

Throughout this paper, we consider finite, undirected simple graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ (or V) and edge set $E(G)$. For a graph G, the distance between vertices x and y, denoted by $d_G(x,y)$, is the length of a shortest path between them.

For distance, Harold Wiener in 1947 defined a famous index $W(G)$ [1], named Wiener index, where $W(G) = \sum_{x,y\in V} d_G(x,y)$. It is the earliest and one of the most thoroughly studied distance-based graph invariants. Later, Dobrynin and Kochetova [2] gave a modified version of the Wiener index $D^+(G) = \sum_{x,y\in V}(d_G(x)+d_G(y))d_G(x,y)$. It is called degree distance and has attracted much attention (see [3–6]). For a graph G, the degree distance $D^+(G)$ is the essential part of the molecular topological index $MTI(G)$ introduced by Schultz [7], which is defined as $MTI(G) = \sum_{x\in V} d_G^2(x) + D^+(G)$, where $\sum_{x\in V} d_G^2(x)$ is the well-known first Zagreb index [8]. Klein et al. [9] discovered the relation between degree distance and Wiener index for a tree G on n vertices:

$$D^+(G) = 4W(G) - n(n-1).$$

The Gutman index of a connected graph G is defined as $D^*(G) = \sum_{x,y\in V} d_G(x)d_G(y)d_G(x,y)$. It was introduced in [10] and has been studied extensively (see, e.g., [11,12]). For a tree G on n vertices, Gutman [10] showed that

$$D^*(G) = 4W(G) - (n-1)(2n-1).$$

In 1993, Klein and Randić [13] introduced a distance function named resistance distance on a graph. They viewed a graph G as an electrical network such that each edge of G is assumed to be a unit resistor, and the resistance distance between the vertices x and y of the graph G, denoted by $r_G(x,y)$, is then defined to be the effective resistance between the vertices x and y in G. The Kirchhoff index $Kf(G)$ of G is defined as

$$Kf(G) = \sum_{x,y \in V} r_G(x,y).$$

The index has been widely studied in mathematical, physical and chemical aspects; for details on the Kirchhoff index, the readers are referred to [14–18]. In 1996, Gutman and Mohar [19] obtained the result by which a relationship is established between the Kirchhoff index and the Laplacian spectrum: $Kf(G) = n\sum_{i=1}^{n-1} \frac{1}{\mu_i}$, where $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$ are the eigenvalues of the Laplacian matrix of a connected graph G with n vertices.

Similarly, if the distance is replaced by resistance distance in the expression for the degree distance and Gutman index, respectively, then one arrives at the following indices

$$\begin{aligned} R^+(G) &= \sum_{x,y \in V} (d_G(x) + d_G(y))r_G(x,y), \\ R^*(G) &= \sum_{x,y \in V} d_G(x)d_G(y)r_G(x,y). \end{aligned}$$

$R^+(G)$ and $R^*(G)$ are called the additive degree-Kirchhoff index and multiplicative degree-Kirchhoff index, respectively, and were introduced by Gutman et al. [20] and Chen et al. [21], respectively. The indices have been well studied in both mathematical and chemical literature. In [22] some properties of $R^+(G)$ are determined and the extremal graph of cacti with minimum R^+-value characterized. Bianchi et al. [23] studied some upper and lower bounds for $G^+(G)$ whose expressions do not depend on the resistance distances. Feng et al. [24] characterized n-vertex unicyclic graphs having maximum, second maximum, minimum, and second minimum multiplicative degree-Kirchhoff index. Palacios [25] studied some interplay of the three Kirchhoff indices and found lower and upper bounds for the additive degree-Kirchhoff index. Yang and Klein [26] derived a formula for $R^*(G)$ of subdivisions and triangulations of graphs. To simplify the calculation of $R^*(G)$, the present authors [27] also obtained a formula for $R^*(G)$ with respect to the subgraph of G. For more work on the topological indices, we refer the reader to [13,21,22,28–31].

In this paper, we study the multiplicative degree-Kirchhoff index of cacti. To state our results, we introduce some notation and terminology. For graph-theoretical terms that are not defined here, we refer to Bollobás' book [32]. Let P_n, C_n and S_n be the path, the cycle and the star on n vertices, respectively. We denote by $G \cong H$ if graph G is isomorphic to graph H. Let $N_G(x) = \{y|yx \in E\}$. Denote by $d_G(x) = |N_G(x)|$ the degree of the vertex x of G. If $E_0 \subset E$, we denote by $G - E_0$ the subgraph of G obtained by deleting the edges in E_0. If E_1 is the subset of the edge set of the complement of G, $G + E_1$ denotes the graph obtained from G by adding the edges in E_1. Similarly, if $W \subset V(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them and $G[W]$ the subgraph of G induced by W. If $E = \{xy\}$ and $W = \{x\}$, we write $G - xy$ and $G - x$ instead of $G - \{xy\}$ and $G - \{x\}$, respectively.

A graph G is called a cactus if each block of G is either an edge or a cycle. Denote by $Cact(n;t)$ the set of cacti possessing n vertices and t cycles. Let $G \in Cact(n;t), t \geq 2$, a cycle $C = v_1v_2\cdots v_kv_1$ of G is said to be an end cycle if all vertices $v_1, \cdots, v_{k-1}$ are of degree two, and the degree of vertex v_k is greater than two. The vertex $v_k \in V(C)$ is called the anchor of C. Let $G^0(n;t)$ be the graph shown in Figure 1. In this paper, we first give some transformations on $R^*(G)$, and then, by these transformations, we determine the first-minimum and second-minimum multiplicative degree-Kirchhoff index in $Cact(n;t)$ and characterize the corresponding extremal graphs, respectively.

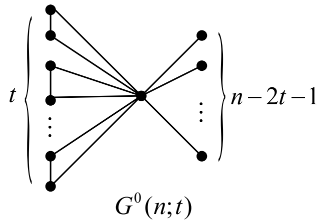

Figure 1. The graph $G^0(n;t)$.

Now, we give some lemmas that are used in the proof of our main results.

Lemma 1. *Ref. [13] Let u be a cut vertex of a connected graph G and x and y be vertices occurring in different components which arise upon deletion of u, then $r_G(x,y) = r_G(x,u) + r_G(u,y)$.*

Lemma 2. *Ref. [27] Let G_1 and G_2 be connected graphs with disjoint vertex sets, with m_1 and m_2 edges, respectively. Let $u_1 \in V(G_1), u_2 \in V(G_2)$. Constructing the graph G by identifying the vertices u_1 and u_2, and denote the so obtained vertex by u. Then,*

$$R^*(G) = R^*(G_1) + R^*(G_2) + 2m_2 \sum_{x \in V(G_1)} d_{G_1}(x) r_{G_1}(u,x) + 2m_1 \sum_{y \in V(G_2)} d_{G_2}(y) r_{G_2}(u,y).$$

For completeness, we also give the proof in this paper.

Proof. Let $V_1 = V(G_1) - u$, $V_2 = V(G_2) - u$. Note that $\forall x \in V_i, d_G(x) = d_{G_i}(x)$ for $i = 1,2$, and $d_G(u) = d_{G_1}(u) + d_{G_2}(u)$. By the definition of $R^*(G)$ and Lemma 1, we have

$$\begin{aligned}
R^*(G) &= \frac{1}{2} \sum_{x,y \in V_1} d_G(x) d_G(y) r_G(x,y) + \frac{1}{2} \sum_{x,y \in V_2} d_G(x) d_G(y) r_G(x,y) + \sum_{x \in V_1} d_G(x) d_G(u) r_G(x,u) + \\
&\quad \sum_{y \in V_2} d_G(y) d_G(u) r_G(u,y) + \sum_{x \in V_1, y \in V_2} d_G(x) d_G(y) r_G(x,y) \\
&= [R^*(G_1) - \sum_{x \in V_1} d_{G_1}(x) d_{G_1}(u) r_{G_1}(x,u)] + [R^*(G_2) - \sum_{y \in V_2} d_{G_2}(y) d_{G_2}(u) r_{G_2}(u,y)] + \\
&\quad \sum_{x \in V_1} d_{G_1}(x) d_G(u) r_G(x,u) + \sum_{y \in V_2} d_{G_2}(y) d_G(u) r_G(u,y) + \\
&\quad \sum_{x \in V_1, y \in V_2} d_{G_1}(x) d_{G_2}(y) [r_{G_1}(x,u) + r_{G_2}(u,y)]
\end{aligned}$$

Because $d_G(u) = d_{G_1}(u) + d_{G_2}(u)$ and $r_G(x,u) = r_{G_1}(x,u)$ for $x \in V_1$, $r_G(u,y) = r_{G_2}(u,y)$ for $y \in V_2$, we have

$$\begin{aligned}
R^*(G) &= [R^*(G_1) + d_{G_2}(u)\sum_{x\in V_1} d_{G_1}(x)r_{G_1}(x,u)] + [R^*(G_2) + d_{G_1}(u)\sum_{y\in V_2} d_{G_2}(y)r_{G_2}(u,y)] + \\
&\quad \sum_{y\in V_2}\sum_{x\in V_1} d_{G_1}(x)d_{G_2}(y)r_{G_1}(x,u) + \sum_{x\in V_1}\sum_{y\in V_2} d_{G_1}(x)d_{G_2}(y)r_{G_2}(u,y) \\
&= [R^*(G_1) + d_{G_2}(u)\sum_{x\in V_1} d_{G_1}(x)r_{G_1}(x,u)] + [R^*(G_2) + d_{G_1}(u)\sum_{y\in V_2} d_{G_2}(y)r_{G_2}(u,y)] + \\
&\quad \sum_{y\in V_2} d_{G_2}(y)\cdot\sum_{x\in V_1} d_{G_1}(x)r_{G_1}(x,u) + \sum_{x\in V_1} d_{G_1}(x)\cdot\sum_{y\in V_2} d_{G_2}(y)r_{G_2}(u,y) \\
&= [R^*(G_1) + d_{G_2}(u)\sum_{x\in V_1} d_{G_1}(x)r_{G_1}(x,u)] + [R^*(G_2) + d_{G_1}(u)\sum_{y\in V_2} d_{G_2}(y)r_{G_2}(u,y)] + \\
&\quad [2m_2 - d_{G_2}(u)]\sum_{x\in V_1} d_{G_1}(x)r_{G_1}(x,u) + [2m_1 - d_{G_1}(u)]\sum_{y\in V_2} d_{G_2}(y)r_{G_2}(u,y) \\
&= R^*(G_1) + R^*(G_2) + 2m_2\sum_{x\in V(G_1)} d_{G_1}(x)r_{G_1}(u,x) + 2m_1\sum_{y\in V(G_2)} d_{G_2}(y)r_{G_2}(u,y),
\end{aligned}$$

since $r_{G_i}(u,u) = 0$ for $i = 1,2$. $\square$

Lemma 3. *Ref. [27] Let $G \in \mathscr{U}_n$, then $R^*(G) \geq R^*(G^0(n;1))$, where $\mathscr{U}_n$ is the class of unicyclic graphs. The equality holds if and only if $T \cong G^0(n;1)$.*

2. Transformations

In this section, we give some transformations that decrease $R^*(G)$.

Transformation 1. *Let u_1u_2 be a cut-edge of G, but not an pendent edge, G_1, G_2 be the connected components of $G - u_1u_2$, where $u_1 \in V(G_1), u_2 \in V(G_2)$. Constructing the graph G' from G by deleting u_1u_2 and identifying the vertices u_1, u_2, denote the so obtained vertex by u, adding an pendent edge uv (as shown in Figure 2).*

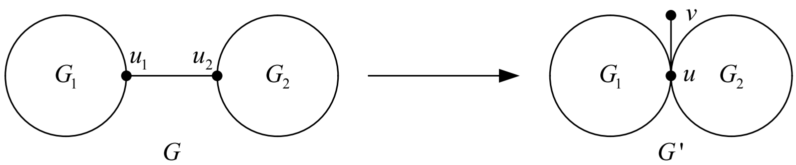

Figure 2. The graphs G and G' in Transformation 1.

Lemma 4. *Let G, G' be the graphs described in Transformation 1, then $R^*(G) > R^*(G')$.*

Proof. Let $|V(G_1)| = n_1, |V(G_2)| = n_2$ and $|E(G_1)| = m_1, |E(G_2)| = m_2$, where $m_1, m_2 \geq 1$. Let H be the graph obtained by attaching to the vertex u_1 of G_1 the pendent vertex u_2, then $|V(H)| = n_1 + 1, |E(H)| = m_1 + 1$. By Lemma 2, we have

$$\begin{aligned}
R^*(G) &= R^*(H) + R^*(G_2) + 2m_2\sum_{x\in V(H)} d_H(x)r_H(u_2,x) + 2(m_1+1)\sum_{y\in V(G_2)} d_{G_2}(y)r_{G_2}(u_2,y), \\
R^*(G') &= R^*(H) + R^*(G_2) + 2m_2\sum_{x\in V(H)} d_H(x)r_H(u,x) + 2(m_1+1)\sum_{y\in V(G_2)} d_{G_2}(y)r_{G_2}(u,y).
\end{aligned}$$

Note that $r_{G_2}(u_2, y) = r_{G_2}(u, y)$, then

$$\begin{aligned} R^*(G) - R^*(G') &= 2m_2[\sum_{x \in V(H)} d_H(x) r_H(u_2, x) - \sum_{x \in V(H)} d_H(x) r_H(u, x)] \\ &= 2m_2\{\sum_{x \in V(G_1)} d_H(x)[r_H(u_1, x) + 1] - [\sum_{x \in V(G_1)} d_H(x) r_H(u, x) + 1]\} \\ &= 2m_2[2(m_1 + 1) - 1 - 1] = 4m_1 m_2 > 0, \end{aligned}$$

so $R^*(G) > R^*(G')$. □

Let $\mathscr{G}_n$ be the class of connected graphs on n vertices. By Transformation 1 and Lemma 4, we have the following result.

Corollary 1. *Let G_0 be a graph with the smallest multiplicative degree-Kirchhoff index in $\mathscr{G}_n$, then all cut-edges are pendent edges.*

Transformation 2. *For $G \in Cact(n; t)$, let C_k be a cycle with $k(\geq 4)$ vertices, contained in G. Let there be a unique vertex $u \in C_h$ which is adjacent to a vertex in $V(G) - V(C)$. Assuming that $uv, vw \in E(C)$, construct a new graph $G^* = G - vw + uw$ (as shown in Figure 3).*

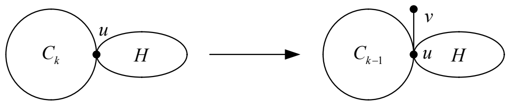

Figure 3. The graphs G and G^* in Transformation 2.

For $u \in V(C_k)$, by direct calculation, we have

$$R^*(C_k) = \frac{k^3 - k}{3} \qquad (1)$$

$$\sum_{y \in V(C_k)} r_{C_k}(u, y) = \frac{k^2 - 1}{6} \qquad (2)$$

Lemma 5. *Let G, G^* be the graphs described in Transformation 2, then $R^*(G) > R^*(G^*)$.*

Proof. Let S be the graph obtained by attaching to the vertex u of C_{k-1} the pendent vertex v. By Lemma 1, we have

$$\begin{aligned} R^*(G) &= R^*(H) + R^*(C_k) + 2k \sum_{x \in V(H)} d_H(x) r_H(u, x) + 2|E(H)| \sum_{y \in V(C_k)} d_{C_k}(y) r_{C_k}(u, y), \\ R^*(G^*) &= R^*(H) + R^*(S) + 2k \sum_{x \in V(H)} d_H(x) r_H(u, x) + 2|E(H)| \sum_{y \in V(S)} d_S(y) r_S(u, y). \end{aligned}$$

Further by Equations (1) and (2), then

$$\begin{aligned} R^*(G) - R^*(G^*) &= [R^*(C_k) - R^*(S)] + 2|E(H)|[\sum_{y \in V(C_k)} d_{C_k}(y) r_{C_k}(u, y) - \sum_{y \in V(S)} d_S(y) r_S(u, y)] \\ &= [\frac{k^3 - k}{3} - (\frac{(k-1)(k^2+2)}{3} + 2k - 1)] + 2|E(H)| \cdot \frac{2k - 4}{3} \\ &\geq \frac{k^2 - 5k + 3}{3} + \frac{4k - 8}{3} = \frac{k^2 - k - 5}{3} > 0, \end{aligned}$$

since $k \geq 4, |E(H)| \geq 1$. □

Transformation 3. *Let $G \in Cact(n;t), t \geq 2$, be a cactus without cut edges. Let C be an end cycle of G and u be its anchor. Let v be a vertex of C different from u. The graphs G_1 and G_2 are constructed by adding r pendent edges to the vertices u and v of G respectively (as shown in Figure 4).*

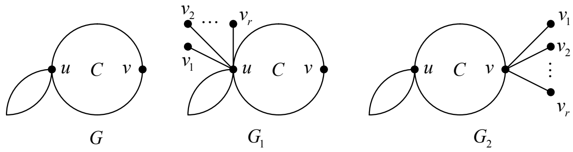

Figure 4. The graphs G, G_1 and G_3 in Transformation 3.

Lemma 6. *Let G, G_1, G_2 be the graphs described in Transformation 3, then $R^*(G_2) > R^*(G_1)$.*

Proof. Let $H = G[V(G) - V(C)], H_1 = G[V(G_1) \setminus (V(G) - u)]$ and $H_2 = G[V(G_2) \setminus (V(G) - v)]$, then $H_1 \cong H_2 \cong K_{1,r}$. By Lemma 2, we have

$$\begin{aligned} R^*(G_1) &= R^*(G) + R^*(K_{1,r}) + 2r \sum_{x \in V(G)} d_G(x) r_G(u,x) + 2|E(G)| \sum_{y \in V(K_{1,r})} d_{K_{1,r}}(y) r_{K_{1,r}}(u,y), \\ R^*(G_2) &= R^*(G) + R^*(K_{1,r}) + 2r \sum_{x \in V(G)} d_G(x) r_G(v,x) + 2|E(G)| \sum_{y \in V(K_{1,r})} d_{K_{1,r}}(y) r_{K_{1,r}}(v,y). \end{aligned}$$

Note that $r_G(u,u) = r_G(v,v) = 0$, $V(G) = V(H) \cup \{V(C) - u\} \cup \{u\}$ and $r_G(v,x) = r_G(x,u) + r_G(v,u)$ for $x \in V(H)$, then

$$\begin{aligned} R^*(G_2) - R^*(G_1) &= 2r[\sum_{x \in V(G)} d_G(x) r_G(v,x) - \sum_{x \in V(G)} d_G(x) r_G(u,x)] \\ &= 2r\{[\sum_{x \in V(H)} d_G(x)(r_G(x,u) + r_G(v,u)) + \sum_{x \in V(C)-u} d_G(x) r_G(v,x) + d_G(u) r_G(v,u)] \\ &\quad -[\sum_{x \in V(H)} d_G(x) r_G(u,x) + \sum_{x \in V(C)-u} d_G(x) r_G(u,x)]\} \end{aligned}$$

Considering that $d_G(u) = d_H(u) + d_C(u)$ and $\sum_{x \in V(C)-u} d_G(x) r_G(u,x) = \sum_{x \in V(C)-u} d_G(x) r_G(v,x) + d_C(u) r_G(v,u)]$, we obtain

$$R^*(G_2) - R^*(G_1) = 2r[\sum_{x \in V(H)} d_G(x) r_G(v,u) + d_G(u) r_G(v,u)] > 0.$$

This completes the proof. □

Transformation 4. *Let $C = v_1 v_2 v_3 v_1$ be a cycle of the graph G, C is called a pendant triangle if $d_G(v_1) = d_G(v_2) = 2$ and $d_G(v_3) > 2$. Suppose that G is a cactus graph and $u, v \in V(G)$ are two vertices, such that $u x_i y_i u$ $(i = 1, 2, \ldots, s)$ and $v p_j q_j v$ $(j = 1, 2, \ldots, t)$ are pendant triangles with the anchor u and v, respectively. We form two new graphs A and B according to the following transformation.*

$$\begin{aligned} A &= G - \cup_{i=1}^{s} \{u x_i, u y_i\} + \cup_{i=1}^{s} \{v x_i, v y_i\}, \\ B &= G - \cup_{i=1}^{t} \{v p_i, v q_i\} + \cup_{i=1}^{t} \{u p_i, u q_i\}. \end{aligned}$$

Lemma 7. *Let G, A, B be the graphs described in Transformation 4, then either $R^*(G) > R^*(A)$ or $R^*(G) > R^*(B)$.*

Proof. Let $M = \{x_1, y_1, \ldots, x_s, y_s\}$, $N = \{p_1, q_1, \ldots, p_t, q_t\}$ and $H = V(G) - M - N - \{u, v\}$. Because of $V(G) = M \cup N \cup H \cup \{u, v\}$, by Lemma 2, we have

$$\begin{aligned} R^*(G) &= [\sum_{\{x,y\}\subset M} + \sum_{\{x,y\}\subset N} + \sum_{\{x,y\}\subset H}] d_G(x)d_G(y)r_G(x,y) + \sum_{x\in H, y\in M} d_G(x)d_G(y)r_G(x,y) + \\ & \sum_{x\in H, y\in N} d_G(x)d_G(y)r_G(x,y) + \sum_{x\in M, y\in N} d_G(x)d_G(y)r_G(x,y) + d_G(u)d_G(v)r_G(u,v) \\ & + \sum_{x\in H} d_G(x)d_G(u)r_G(x,u) + \sum_{x\in H} d_G(x)d_G(v)r_G(x,v) + \sum_{x\in M} d_G(x)d_G(u)r_G(x,u) + \\ & \sum_{x\in M} d_G(x)d_G(v)r_G(x,v) + \sum_{x\in N} d_G(x)d_G(u)r_G(x,u) + \sum_{x\in N} d_G(x)d_G(v)r_G(x,v). \end{aligned}$$

and analogously

$$\begin{aligned} R^*(A) &= [\sum_{\{x,y\}\subset M} + \sum_{\{x,y\}\subset N} + \sum_{\{x,y\}\subset H}] d_A(x)d_A(y)r_A(x,y) + \sum_{x\in H, y\in M} d_A(x)d_A(y)r_A(x,y) + \\ & \sum_{x\in H, y\in N} d_A(x)d_A(y)r_A(x,y) + \sum_{x\in M, y\in N} d_A(x)d_A(y)r_A(x,y) + d_A(u)d_A(v)r_A(u,v) \\ & + \sum_{x\in H} d_A(x)d_A(u)r_A(x,u) + \sum_{x\in H} d_A(x)d_A(v)r_A(x,v) + \sum_{x\in M} d_A(x)d_A(u)r_A(x,u) + \\ & \sum_{x\in M} d_G(x)d_A(v)r_A(x,v) + \sum_{x\in N} d_A(x)d_A(u)r_A(x,u) + \sum_{x\in N} d_A(x)d_A(v)r_A(x,v). \end{aligned}$$

Note that $d_G(x) = d_A(x)$ for $x \in V(G) - \{u, v\}$ and $r_G(x, y) = r_A(x, y)$ for $x, y \in M$ or $x, y \in N$ or $x, y \in H$, then

$$\begin{aligned} a_0 &= [\sum_{\{x,y\}\subset M} + \sum_{\{x,y\}\subset N} + \sum_{\{x,y\}\subset H}] d_G(x)d_G(y)r_G(x,y), \\ b_0 &= [\sum_{\{x,y\}\subset M} + \sum_{\{x,y\}\subset N} + \sum_{\{x,y\}\subset H}] d_A(x)d_A(y)r_A(x,y), \\ a_0 - b_0 &= 0 \end{aligned} \tag{3}$$

Considering that $r(u, y) = \frac{2}{3}$ for $y \in M$ and $r(x, y) = r(x, u) + r(u, y)$, we get

$$\begin{aligned} a_1 = \sum_{x\in H, y\in M} d_G(x)d_G(y)r_G(x,y) &= \sum_{x\in H, y\in M} d_G(x)d_G(y)[r_G(x,u) + \tfrac{2}{3}], \\ b_1 = \sum_{x\in H, y\in M} d_A(x)d_A(y)r_A(x,y) &= \sum_{x\in H, y\in M} d_G(x)d_G(y)[r_G(x,v) + \tfrac{2}{3}], \\ a_1 - b_1 &= \sum_{x\in H, y\in M} d_G(x)d_G(y)[r_G(x,u) - r_G(x,v)] \end{aligned} \tag{4}$$

and analogously,

$$\begin{aligned} a_2 = \sum_{x\in H, y\in N} d_G(x)d_G(y)r_G(x,y) &= \sum_{x\in H, y\in N} d_G(x)d_G(y)[r_G(x,v) + \tfrac{2}{3}], \\ b_2 = \sum_{x\in H, y\in N} d_A(x)d_A(y)r_A(x,y) &= \sum_{x\in H, y\in N} d_G(x)d_G(y)[r_G(x,v) + \tfrac{2}{3}], \\ a_2 - b_2 &= 0 \end{aligned} \tag{5}$$

Because $r_G(x, y) = r_G(x, u) + r_G(u, v) + r_G(v, y) = r_G(u, v) + \frac{4}{3}$ for $x \in M, y \in N$, then

$$\begin{aligned} a_3 = \sum_{x\in M, y\in N} d_G(x)d_G(y)r_G(x,y) &= \sum_{x\in M, y\in N} d_G(x)d_G(y)[r_G(u,v) + \tfrac{4}{3}], \\ b_3 = \sum_{x\in M, y\in N} d_A(x)d_A(y)r_A(x,y) &= \tfrac{4}{3} \sum_{x\in M, y\in N} d_G(x)d_G(y), \\ a_3 - b_3 &= \sum_{x\in M, y\in N} d_G(x)d_G(y)r_G(u,v) \end{aligned} \tag{6}$$

$$\begin{aligned} a_4 &= d_G(u)d_G(v)r_G(u,v), \\ b_4 = d_A(u)d_A(v)r_A(u,v) &= [d_G(u)-2s][d_G(v)+2s]r_G(u,v), \\ a_4-b_4 &= 4s^2r_G(u,v)+2sd_G(v)r_G(u,v)-2sd_G(u)r_G(u,v) \end{aligned} \tag{7}$$

$$\begin{aligned} a_5 &= \sum_{x\in H} d_G(x)d_G(u)r_G(x,u), \\ b_5 = \sum_{x\in H} d_A(x)d_A(u)r_A(x,u) &= \sum_{x\in H} d_G(x)[d_G(u)-2s]r_G(x,u), \\ a_5-b_5 &= \sum_{x\in H} 2sd_G(x)r_G(x,u) \end{aligned} \tag{8}$$

$$\begin{aligned} a_6 &= \sum_{x\in H} d_G(x)d_G(v)r_G(x,v), \\ b_6 = \sum_{x\in H} d_A(x)d_A(v)r_A(x,v) &= \sum_{x\in H} d_G(x)[d_G(v)+2s]r_G(x,v), \\ a_6-b_6 &= -\sum_{x\in H} 2sd_G(x)r_G(x,v) \end{aligned} \tag{9}$$

$$\begin{aligned} a_7 = \sum_{x\in M} d_G(x)d_G(u)r_G(x,u) &= \frac{2}{3}\sum_{x\in M} d_G(x)d_G(u), \\ b_7 = \sum_{x\in M} d_A(x)d_A(u)r_A(x,u) &= \sum_{x\in M} d_G(x)[d_G(u)-2s][r_G(u,v)+\tfrac{2}{3}], \\ a_7-b_7 &= \sum_{x\in M} [(2s-d_G(u))d_G(x)r_G(u,v)+\tfrac{4}{3}sd_G(x)] \end{aligned} \tag{10}$$

Because $r_G(x,v) = r_G(u,v) + \frac{2}{3}$ for $x \in M$. After the transformation, the degree of the vertex v increases by $2s$, and $r_A(x,v) = \frac{2}{3}$ for $x \in M$.

$$\begin{aligned} a_8 = \sum_{x\in M} d_G(x)d_G(v)r_G(x,v) &= \sum_{x\in M} d_G(x)d_G(v)[r_G(u,v)+\tfrac{2}{3}], \\ b_8 = \sum_{x\in M} d_G(x)d_A(v)r_A(x,v) &= \frac{2}{3}\sum_{x\in M} d_G(x)[d_G(v)+2s], \\ a_8-b_8 &= \sum_{x\in M} [d_G(x)d_G(v)r_G(u,v)-\tfrac{4}{3}sd_G(x)] \end{aligned} \tag{11}$$

$$\begin{aligned} a_9 = \sum_{x\in N} d_G(x)d_G(u)r_G(x,u) &= \sum_{x\in N} d_G(x)d_G(u)[r_G(v,u)+\tfrac{2}{3}], \\ b_9 = \sum_{x\in N} d_A(x)d_A(u)r_A(x,u) &= \sum_{x\in N} d_G(x)[d_G(u)-2s][r_G(v,u)+\tfrac{2}{3}], \\ a_9-b_9 &= \sum_{x\in N} 2sd_G(x)[r_G(v,u)+\tfrac{2}{3}] \end{aligned} \tag{12}$$

$$\begin{aligned} a_{10} = \sum_{x\in N} d_G(x)d_G(v)r_G(x,v) &= \frac{2}{3}\sum_{x\in N} d_G(x)d_G(v), \\ b_{10} = \sum_{x\in N} d_A(x)d_A(v)r_A(x,v) &= \frac{2}{3}\sum_{x\in N} d_G(x)[d_G(v)+2s], \\ a_{10}-b_{10} &= -\frac{4}{3}\sum_{x\in N} sd_G(x) \end{aligned} \tag{13}$$

By Equations (3)–(13), we have

$$\begin{aligned} R^*(G)-R^*(A) &= 4s\sum_{x\in H} d_G(x)[r_G(x,u)-r_G(x,v)] + 24str_G(u,v) + 6sr_G(u,v)[d_G(v)-d_G(u)] + \\ &\quad 12s^2r_G(u,v) + \sum_{x\in H} 2sd_G(x)[r_G(x,u)-r_G(x,v)]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} R^*(G)-R^*(B) &= 4t\sum_{x\in H} d_G(x)[r_G(x,v)-r_G(x,u)]+24str_G(u,v)+6tr_G(u,v)[d_G(u)-d_G(v)]+\\ &\quad 12t^2r_G(u,v)+2t\sum_{x\in H} d_G(x)[r_G(x,v)-r_G(x,u)]. \end{aligned}$$

If $R^*(G)-R^*(A)\le 0$, then

$$\begin{aligned} & 4\sum_{x\in H} d_G(x)[r_G(x,u)-r_G(x,v)]+6r_G(u,v)[d_G(v)-d_G(u)]+\sum_{x\in H} 2d_G(x)[r_G(x,u)-r_G(x,v)]\\ \le\ & -(24tr_G(u,v)+12sr_G(u,v)). \end{aligned}$$

Further, we have

$$R^*(G)-R^*(B) \ \ge\ t(24tr_G(u,v)+12sr_G(u,v))+24str_G(u,v)+12t^2r_G(u,v)>0.$$

This completes the proof. □

Transformation 5. *Let u be a vertex of G such that there are s pendent vertices $u_1,u_2,\ldots,u_s$ attached to u, and v be another vertex of G such that there are t pendent vertices $v_1,v_2,\ldots,v_t$ attached to v. Let*

$$\begin{aligned} G_3 &= G-\{uu_1,uu_2,\ldots,uu_s\}+\{vu_1,vu_2,\ldots,vu_s\},\\ G_4 &= G-\{vv_1,vv_2,\ldots,vv_t\}+\{uv_1,uv_2,\ldots,uv_t\}. \end{aligned}$$

Similar to the proof of Lemma 7, we can prove the following result.

Lemma 8. *Let G,G_3 and G_4 be graphs as described in Transformation 5, then either $R^*(G)>R^*(G_3)$ or $R^*(G)>R^*(G_4)$.*

3. Main Results

In this section, we determine the elements in $Cact(n;t)$ with first-minimum and second-minimum multiplicative degree-Kirchhoff index by the transformations that we have obtained. Note that the first-minimum multiplicative degree-Kirchhoff index has been obtained in [33]; for completeness, we also give the following proof.

Theorem 1. *Ref. [33] Let $G\in Cact(n;t)$, then $R^*(G)\ge 16t^2-8t+(n-2t-1)^2+(n-2t-1)(n-2t-2)+\frac{34}{3}t(n-2t-1)$. The equality holds if and only if $G\cong G^0(n,t)$.*

Proof. Let $\tilde{G}$ be the unique graph having the minimum multiplicative degree-Kirchhoff index in $Cact(n;t)$.

Case 1. If $t=1$, $Cact(n;t)$ is the class of unicyclic graphs. By Lemma 3, we know the results hold.

Case 2. If $t=2$, $Cact(n;t)$ is the class of bicyclic graphs. By Lemma 4, we conclude that $\tilde{G}$ contains two cycles attached to a common vertex u, and all cut-edges are all pendent edges (if any). Further, by Lemmas 8 and 6, all pendent edges (if any) are also attached to u. Finally, by Lemma 5, the two cycles must be triangles, that is, $\tilde{G}\cong G^0(n,2)$. This obtains the desirable results.

Case 3. If $t\ge 3$, by Lemma 4, we conclude that all cut-edges are all pendent edges (if any) in $\tilde{G}$. Further, by Lemmas 8 and 6, $\tilde{G}$ has at least two end cycles. Repeated by Lemmas 5–8, we arrive at the conclusion $\tilde{G}\cong G^0(n,t)$

By direct calculation, we have

$$R^*(G^0(n;t)) \ = \ 16t^2-8t+(n-2t-1)^2+(n-2t-1)(n-2t-2)+\frac{34}{3}t(n-2t-1).$$

This completes the proof. □

Theorem 2. *Let $G \in Cact(n;t) \setminus G^0(n;t)$, then $R^*(G) \geq [16t^2 + \frac{34t-22}{3}] + [(n-2t-2)^2 + (n-2t-2)(n-2t-3)] + \frac{(16t+14)(n-2t-2)}{3} + 2(3t+1)(n-2t-2)$. The equality holds if and only if $G \cong G_3^0$.*

Proof. By Lemmas 4–8 and Theorem 1, one can conclude that G, which has the second-minimum multiplicative degree-Kirchhoff index in $Cact(n;t)$ must be one of the graphs G_1^0, G_2^0, G_3^0, as shown in Figure 5. By Lemma 2, we have

$$\begin{aligned}
R^*(G_1^0) &= [16t^2 - 8t] + [2 + 4(n-2t-2) + 7(n-2t-3) + (n-2t-2)(n-2t-3) + \\
&\quad (n-2t-3)(n-2t-4)] + \frac{16}{3}t(n-2t-1) + 6t(n-2t+1), \\
R^*(G_2^0) &= [16t^2 + \frac{34}{3}t - 7] + [(n-2t-2)^2 + (n-2t-2)(n-2t-3)] + \frac{(16t+14)(n-2t-2)}{3} \\
&\quad +2(3t+1)(n-2t-2), \\
R^*(G_3^0) &= [16t^2 + \frac{34t-22}{3}] + [(n-2t-2)^2 + (n-2t-2)(n-2t-3)] + \frac{(16t+14)(n-2t-2)}{3} \\
&\quad +2(3t+1)(n-2t-2).
\end{aligned}$$

Then,

$$\begin{aligned}
R^*(G_1^0) - R^*(G_3^0) &= \frac{4n+4t+5}{3}, \\
R^*(G_2^0) - R^*(G_3^0) &= \frac{1}{3}.
\end{aligned}$$

This completes the proof. □

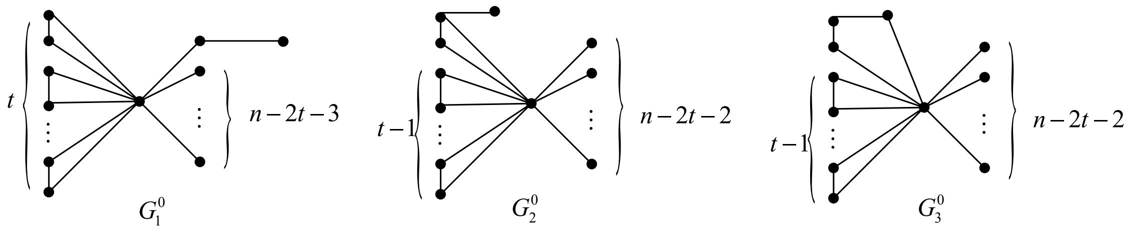

Figure 5. The graphs G_1^0, G_2^0, G_3^0.

By Theorems 1 and 2, we have

Corollary 2. *Among all graphs in $Cact(n;t)$, $G^0(n;t)$ and G_3^0 are the graphs with first-minimum and second-minimum multiplicative degree-Kirchhoff index.*

According to the above discussion, we find that the extremal cacti for the index $R^*(G)$ are the same as the extremal cacti for the Kirchhoff index, the multiplicative degree-Kirchhoff index, the Wiener index and the other indices [22,29,34,35]. Based on the known results for these indices, we guess the element of $Cact(n;t)$ with maximum multiplicative degree-Kirchhoff index is isomorphic to the graph $C_{n,t}$ (as shown in Figure 6).

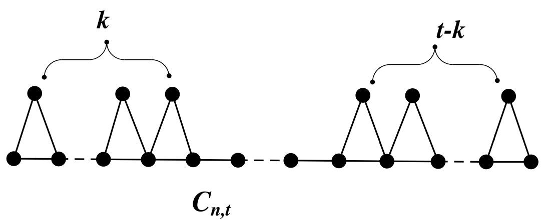

Figure 6. The graphs $C_{n,t}$.

Conjecture 1. *Let $C_{n,t}$ be the graph depicted in Figure 6, where $k = \lfloor \frac{t}{2} \rfloor$. Then, $C_{n,t}$ is the unique element of $Cact(n;t)$ having maximum multiplicative degree-Kirchhoff index.*

In particular, for $Cat(n;t)$, if $t = 1;0$, $Cat(n;1)$ and $Cat(n;0)$ are the set of unicyclic graphs and trees, respectively. For $G \in Cat(n;1)$, the graphs having maximum and minimum multiplicative degree-Kirchhoff index are given in [13], that is

$$R^*(G^0(n,1)) \leq R^*(G) \leq R^*(U_3^n).$$

where U_3^n consists of a cycle of size 3 to which a path with $n-3$ vertices is attached. For $G \in Cat(n;0)$, it is easy to get the result

$$R^*(S_n) \leq R^*(G) \leq R^*(P_n).$$

4. Conclusions

In this paper, we give some transformations on the multiplicative degree-Kirchhoff index. As applications, the second-minimum multiplicative degree-Kirchhoff index on $Cat(n;t)$ and the corresponding extremal graph are determined. We guess $C_{n,t}$ is the graph of $Cat(n;t)$ with maximum $R^*(G)$ value. For solving the problem, our approach would need to be modified; it would be interesting to continue studying the extremal graphs.

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