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A Novel Interval Three-Way Concept Lattice Model with Its Application in Medical Diagnosis

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Abstract: Medical diagnosis has been recognized as one of the key processes in clinical medicine, which determines diseases from some given symptoms. Nonetheless, previous works about medical diagnosis have some drawbacks because medical data are usually fuzzy, uncertain, incomplete and imprecise. To achieve the optimal medical diagnosis decision by reducing cost and enhancing accuracy, this paper develops a new method named interval three-way concept lattice model. Firstly, we redefine the decision rules and metric function of three-way decision based on interval concept lattice. Secondly, we build a visualized hierarchical structure of relationship between concepts through interval concept construction algorithm which helps us to make decision preferably and clearly. Finally, we establish a dynamic strategy optimization model for medical diagnosis decision making. In addition, a medical case demonstrates the effectiveness and feasibility of this proposed model.

Keywords: medical diagnosis; formal concept analysis; three-way decision; interval fuzzy concept lattice

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1. Introduction

Recently, the government's new medical reform program has boosted the construction of digital hospital and regional medical network construction. With the standardization and widely used of Electronic Health Records (EHR) and Hospital Information System (HIS), the Health care industry has accumulated a lot of medical data, including the patient's medical records, patient diagnosis records, medical image data and so on [1]. It will provide a great convenience for the hospital diagnosis and management of the process when we use medical data effectively. However, now most hospitals, in order to process the data, are mostly confined to adding, deleting, changing, and checking operations, which belongs to the low level of application of medical data without deep analysis.

Medical diagnosis has been recognized as one of the key processes in clinical medicine, which determines diseases from some given symptoms. Many researchers centralized focus on enhancing the accuracy of disease diagnosis by using computerized techniques. Nowadays, Computerized techniques and theories, for example, fuzzy set theories, statistical tools, machine learning algorithms, and recommender systems, have been being widely applied for medical diagnosis [2].

As early as 1959, Ledley's [3] study aroused widespread concern by using of the Bayesian formula to estimate the probability of medical disease diagnosis. Subsequently, Warner [4] used the naïve Bayes theory to establish a probabilistic model to diagnose congenital heart disease and promoted the development of medical decision support information system. However, the hypothesis of Naive Bayes theory is too simple which assumes that symptoms and diseases are independent of each other. From this point of view, scholars explored other, better methods of a medical decision-making model.

Medical data are usually fuzzy, uncertain, incomplete and imprecise, such as symptoms data, the relation between patient and symptoms, and the relation between symptoms, diseases, etc. [5]. Zadeh [6], the pioneer of fuzzy sets theory, firstly applied the fuzzy sets to analyze the relationship between symptoms and diseases in 1969. In addition, fuzzy set theory with extensions was developed, such as intuitionistic fuzzy sets [7,8], 2-tuple fuzzy set [9], fuzzy soft sets [10], probabilistic linguistic term sets [11], interval-valued fuzzy sets [12–14], neutrosophic fuzzy sets [15–19] and Z-numbers [20]. Although the research results of fuzzy sets theories have been very rich [21], due to the complexity and uncertainty of disease diagnosis, it is still a hot topic of the medical diagnosis with fuzzy set theory is used to deal with the problem of medical diagnosis decision.

Three-way decision model (TWDM), which consists of positive, uncertain, and negative regions, is an useful decision-making approach and it has been widely applied to many domains [22]. The application of TWDM in medical diagnosis should be noted. The three-way decision model has firstly introduced a generic framework and applied to the medical diagnosis domain by Yao [23]. In the medical field, Hu [24] developed a novel model for making treatment decisions. Besides, Yao [25] propose a web-based medical decision support system in TWDM on the basis of game theoretic rough sets, which can significantly decrease the uncertainty of medical diagnosis.

Nowadays, lattice theory has been richly used in analyzing biomedical data [26,27]. In 1982, Wille earliest promoted formal concept analysis (FCA) and applied it to knowledge processing tasks [28]. FCA builds a mathematical framework for scholars to analyze data and process tasks [29]. In addition, one of the key benefits of FCA is its capability to predict alternative or potential cases that are similar to those examined (but not present) in each case study [30], therefore endowing it with also generalizing predictive character and a further outreach. Now FCA has been used in various domains successfully especially in medical decision field [31]. Recently, to represent precisely the vagueness and uncertainty of information, FCA has been extended to fuzzy concept lattice analysis based on formal context which combined FCA with fuzzy set theory [32]. In addition, many studies augmented fuzzy formal concept analysis with intuitionistic fuzzy set [33], Bipolar fuzzy sets [34], neutrosophic set [35], rough set [36] and so on. Furthermore, the extension of fuzzy concept lattice and three-way decision theory has attracted the attention of some scholars [37]. There are future trends to extend FCA in a defined fuzzy universe due to their suitable applicability and it is significant for us to use fuzzy concept lattice effectively [31].

However, there are still some drawbacks to previous works in medical diagnosis decision, Firstly, it is not properly appropriate for the missing data situation, which is almost universal. Original information of medical data is composed of different types of data obtained from different times and give predictive values of diseases. Secondly, the previous works will lead to the loss or distortion of information about the development tendency of the values by the time. Last but certainly not least, the diagnosis process is dependent on the defuzzification method [7]. All limitations above of previous methods reduce the accuracy of medical diagnosis. Therefore, a novel method which combined three-way decision and interval-valued fuzzy concept lattice is a great option to eliminate the defects of the previous researches [38]. The IVIFS can describe properly the uncertain and incomplete medical data which represents the membership and non-membership degree of attributes in the interval [0,1]. Meanwhile, the object of universe can be divided into acceptation, uncertain, and rejection regions according to the conditional attributes. In addition, the generation of interval-valued concept lattices is a clustering process. Thus, the parent-child relationship in lattice structure can deal with boundary domain samples and reduce decision losses.

The intention of this article is to develop the interval three-way concept lattice model in medical diagnosis decisions, which is based on the three-way decision and interval fuzzy concept lattice. The proposed model forms a clear and visualized hierarchical structure of the relationship between concepts through formal concept analysis. In addition, a medical diagnosis case validates that we can make decision quickly and preferably by the model.

The rest of this article is constituted as follows: Section 2 introduces medical diagnosis problem related works including interval-valued intuitionistic fuzzy sets, three-way decision and formal concept analysis. Section 3 presents the proposed approach which is called the interval three-way concept lattice model. Section 4 validates the model through an illustrative case based on medical diagnosis information. Finally, Section 5 includes the conclusions and the future research directions.

2. Preliminaries

In this section, we first introduce the medical diagnosis problem, and then briefly present some basic concepts, properties and notations including the three-way decision, formal concept analyze.

2.1. The Medical Diagnosis Problem

Medical diagnosis has been widely recognized as one of the key processes in clinical medicine, which determines diseases from some given symptoms [8]. Nonetheless, medical data are usually fuzzy, uncertain, incomplete and imprecise, such as symptoms data, the relation between patients and symptoms, and the relation between symptoms and diseases etc. Under the circumstances, we state its definition as follows.

Definition 1 [39]. *The number of Patients, symptoms, and diseases are represented respectively as* $P = \{P_1, P_2, \cdots, P_m\}$, $S = \{S_1, S_2, \cdots, S_k\}$, $D = \{D_1, D_2, \cdots, D_n\}$, *where* $m, k, n \in N^+$. *Meanwhile, the set* $R_{PS} = \{R(P_i, S_j)\,|\forall i, j \in N^+\}$ *denotes the relationship between patients and symptoms which indicates the possibility of the patient* P_i *has the symptom* S_j. *The relationship between symptoms and diseases is characterized by the set* $R_{SD} = \{\,R(S_j, D_h)|\forall j, h \in N^+\,\}$ *where* $R(S_j, D_h)$ *shows that the probability of the symptom* S_j *would cause the disease* D_h. *Last but certainly not least, the relation set* $R_{PD} = \{\,R(P_i, D_h)|\forall i, h \in N^+\,\}$ *which reflects the patient* P_i *acquires the disease* D_h *or not. Finally, we can simplify the medical diagnosis problem as* $\{\,R_{PS}, R_{SD}\} \to R_{PD}$.

Nowadays, computerized techniques and theories, for example, fuzzy set theories, statistical tools, machine learning algorithms, and recommender systems, have been being widely applied to medical diagnosis [40]. Sanchez [41] put forward the earliest fuzzy set method in medical diagnosis decision-making. From then on, IFS with its extension fuzzy sets has been applied into medical decision-making filed. Meanwhile, interval-valued IFS (IVIFS) theory is one of an effective way to increase the accuracy of medical diagnosis among the vague incomplete and missing medical data. The IVIFS is denoted as follows.

Definition 2 [39]. *An interval-valued intuitionistic fuzzy set (IVIFS) in a non-empty set X* is,

$$\widetilde{A} = \{\,\langle x, \widetilde{\mu}_{\widetilde{A}}, \widetilde{\nu}_{\widetilde{A}}, \widetilde{\pi}_{\widetilde{A}}\rangle | x \in X\,\},$$

where the functions $\widetilde{\mu}_{\widetilde{A}} : X \to \mathrm{int}([0,1])$ *and* $\widetilde{\nu}_{\widetilde{A}} : X \to \mathrm{int}([0,1])$ *are the interval membership and interval non-membership degrees of each element* $x \in X$ *to the IVIFS* $\widetilde{A}$, $\widetilde{\pi}_{\widetilde{A}} : X \to \mathrm{int}([0,1])$ *indicates the interval hesitation degree of* $x \in X$ *respectively,*

$$\sup(\widetilde{\mu}_{\widetilde{A}}(x)) + \sup(\widetilde{\nu}_{\widetilde{A}}(x)) + \inf(\widetilde{\pi}_{\widetilde{A}}(x)) = 1,$$

$$\inf(\widetilde{\mu}_{\widetilde{A}}(x)) + \inf(\widetilde{\nu}_{\widetilde{A}}(x)) + \sup(\widetilde{\pi}_{\widetilde{A}}(x)) = 1,$$

$$\widetilde{\pi}_{\widetilde{A}}(x) = 1 - \widetilde{\mu}_{\widetilde{A}}(x) - \widetilde{\nu}_{\widetilde{A}}(x).$$

For convenience, the IVIFS $\widetilde{A}$ can be represented as follows.

$$\widetilde{A} = \left(\left[a^l(x), a^u(x)\right], \left[b^l(x), b^u(x)\right], \left[c^l(x), c^u(x)\right]\right).$$

Some notions and properties of IVIFS such as computing distance or similarities are extended to model uncertainty in the medical diagnosis. However, there are still some limitations to the previous method. Firstly, it is not properly appropriate for the missing data situation, which is almost universal. Secondly, historic diagnosis data of patients could not be used effectively [8]. Last but certainly not least, the diagnosis process is dependent on the defuzzification method [7]. All limitations above of previous methods decrease the accuracy of decision-making in medical diagnosis filed. Therefore, a proper approach should be introduced.

2.2. Three-Way Decision Model

TWDM has been generally applied to lots of domains, such as environmental precaution, information filtering, and investment decision, etc. [22]. By the way, the application of TWDM in medical diagnosis should be noted. The three-way decision model has firstly introduced a generic framework and applied to the medical diagnosis domain by Yao [23]. TWDM is constituted by two states and three actions. In addition, it classifies objects into disjoint positive, negative, and uncertain regions.

Definition 3 [23]. *Let $U = \{C, \neg C\}$ denotes weather element X is in C or not. The decision-making state values are expressed as $R(X)$. Meanwhile, the set of actions is given by $B = \left\{B_t, B_i, B_f\right\}$, where B_t, B_i, and B_f respectively represent acceptance, delay, and refusal actions. As well time, α and β are the suitable threshold to classify X into different regions. Finally, the decision rules are shown below.*

(B_t) If $R(X) \geq \alpha$, decide $X \in$ The III region and take acceptance actions.
(B_i) If $\beta \leq R(X) \leq \alpha$, decide $X \in$ The II region and take delay actions.
(B_f) If $R(X) \leq \beta$, decide $X \in$ The I region and take refusal actions.

2.3. Formal Concept Analysis

Formal concept analysis (FCA) consists of concept lattices and formal concepts, and is a powerful tool to analyse information with fuzzy attributes. A formal concept lies in a pair of elements. In addition, concept lattice is an ordered hierarchy derived by formal context with object space and attribute space [42]. Here are some notions and properties of FCA as follows.

Definition 4 [43]. *A formal context is a triple $A = (U, V, R)$, consisted by the object set U, attributes set V, relationship R between U and V. If $x_i \in U$ and $y_j \in V$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$, $\left(x_i, y_j\right) \in R$ is denoted as xRy which means the object x_i owned attribute. Let V_j be the subset of attributes, then $\left|V_j\right| = 2^n$ and a three-way quotient set is denoted as $\Omega(V) = \left\{V_j | j \in V\right\}$. For short, we denote the power set of $\Omega(V)$, y_j by $2^{\Omega(V)}$.*

And every $B \in 2^{\Omega(V)}$ can be considered as a knowledge group jointly inducing three-way decisions.

Residual lattice with some simple properties is a principal component of truth degrees. We denote $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ as the residual lattice where 0 and 1 respectively indicate minimum and maximum objects. The operators $\otimes$ and $\rightarrow$ are adjoint operators which satisfied $x \otimes y = \min(x, y)$ and $x \rightarrow y = 1$ if $a \leq b$, otherwise b [44].

Under the background of formal context $A = (U, V, R)$, let the formal fuzzy concept denotes as an ordered pair of (X, Y) from A. We can define following a pair of operators.

If $P(U^{\uparrow}) \rightarrow P(V^{\uparrow})$, then $X^{\uparrow} = \{y \in V | \forall x \in X, (x, y) \in R\}$,
If $P(V^{\downarrow}) \rightarrow P(U^{\downarrow})$, then $Y^{\downarrow} = \{x \in U | \forall y \in Y, (x, y) \in R\}$.

Once $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$, we call X and Y as the extent and the intent of concept (X, Y) respectively. $S(U, V, R)$, generated from the formal context $A = (U, V, R)$, is denoted as universal set of all formal fuzzy concepts [45]. For each formal concept $(X_1, Y_1), (X_2, Y_2) \in S(U, V, R)$, we define it satisfied the partial ordering relation:

$$(X_1, Y_1) \leq (X_2, Y_2) \Leftrightarrow X_1 \subset X_2 \Leftrightarrow Y_1 \supset Y_2.$$

At the same time, the infimum and the supremum for compete lattice are respectively denoted as follows.

$$(X_1, Y_1) \wedge (X_2, Y_2) = \left(X_1 \cap X_2, (Y_1 \cup Y_2)^{\downarrow\uparrow}\right),$$

$$(X_1, Y_1) \vee (X_2, Y_2) = \left((X_1 \cup X_2)^{\uparrow\downarrow}, Y_1 \cap Y_2\right).$$

Expanding it to the general form is expressed as:

$$\wedge_{i \in I} (X_i, Y_i) = \left(\cap_{i \in I} X_i, (\cup_{i \in I} Y_i)^{\downarrow\uparrow}\right),$$

$$\vee_{i \in I} (X_i, Y_i) = \left((\cup_{i \in I} X_i)^{\uparrow\downarrow}, \cap_{i \in I} Y_i\right).$$

3. Interval Three-Way Concept Lattice Model

In this section, the proposed interval three-way concept lattice model was presented. Let P, S and D denotes the set of patients, symptoms, and diseases as Definition 1. The patient $P_i(\forall i \in 1, 2, \cdots, m)$ has some characteristics of symptoms $S_j(\forall j \in 1, 2, \cdots, k)$. In addition, the diseases $D_h(\forall h \in 1, 2, \cdots, n)$ also contain intuitionistic linguistic labels. It is significant to translate medical information into the interval-valued intuitionistic fuzzy set.

3.1. Algorithm for Three-Way Concepts Construction Using IVIFS

Let us denote a fuzzy formal context as $A = (U, V, R)$ where $|U| = m$, $|V| = n$ and, R shows three-way relation among them using IVIFS for expressing much clearly. Then interval three-way concept lattice is derived as follows.

Definition 5. *Let suppose an IVIFS for attributes of symptoms i.e.,*

$$V(x_j) = \left\{x_j, \left(\left[a_v^l(x_j), a_v^u(x_j)\right], \left[b_v^l(x_j), b_v^u(x_j)\right], \left[c_v^l(x_j), c_v^u(x_j)\right]\right) \in [0,1]^3 | \forall x_j \in V\right\}$$

Membership function, non-membership function and hesitation degree is respectively denoted as interval-valued $\left[a_v^l(x_j), a_v^u(x_j)\right]$, $\left[b_v^l(x_j), b_v^u(x_j)\right]$ and $\left[c_v^l(x_j), c_v^u(x_j)\right]$. The same procedure may be easily adapted to obtain IVIFS for U.

The obtained pair (U, V) is taken for the formal fuzzy concept iff: $U^{\uparrow} = V$ and $V^{\downarrow} = U$. We can interpret it as IVIFS of an object which has maximal membership, minimum non-membership and hesitation value in the interval space $[0, 1]^3$ using residual lattice. The integrating of (U, V) can be defined as.

$$U \cup V = \left\langle \begin{array}{c} x_j, \left[\max\left(a_u^l(x_j), a_v^l(x_j)\right), \max(a_u^u(x_j), a_v^u(x_j))\right], \\ \left[\min\left(b_u^l(x_j), b_v^l(x_j)\right), \min(b_u^u(x_j), b_v^u(x_j))\right], \\ \left[\min\left(c_u^l(x_j), c_v^l(x_j)\right), \min(c_u^u(x_j), c_v^u(x_j))\right] \end{array} \right\rangle, \tag{1}$$

where $x_j \in V$.

$$U \cap V = \left\langle \begin{array}{c} x_j, \left[\min\left(a_u^l(x_j), a_v^l(x_j)\right), \min\left(a_u^u(x_j), a_v^u(x_j)\right)\right], \\ \left[\max\left(b_u^l(x_j), b_v^l(x_j)\right), \max\left(b_u^u(x_j), b_v^u(x_j)\right)\right], \\ \left[\max\left(c_u^l(x_j), c_v^l(x_j)\right), \max\left(c_u^u(x_j), c_v^u(x_j)\right)\right] \end{array} \right\rangle, \tag{2}$$

where $x_j \in V$.

Until the membership value getting no longer bigger, then the pair of IVIFS (U, V) is referred to a formal concept, where U is known as extent and V is known as intent. Above these, we propose an algorithm for interval three-way concepts construction as Algorithm 1 shows.

Algorithm 1. Three-way concept lattice using IVIFS.

1. Input a fuzzy formal context $A = (U, V, R)$, where $|U| = m$, $|V| = n$.
2. Find all the subsets of attributes set V and represent as k_i.
3. **for each** $i = 1$ to 2^n **do**
4. Set up the membership to maximum value for subset $(k_i) = \max\{[1,1], [0,0], [0,0]\}$
5. Apply down operator $\downarrow$ to the set V
   $V(Yp_i)^{\downarrow} = \left\{y_i, \left(\left[a_{Yp_i}^l(y_i), a_{Yp_i}^u(y_i)\right], \left[b_{Yp_i}^l(y_i), b_{Yp_i}^u(y_i)\right], \left[c_{Yp_i}^l(y_i), c_{Yp_i}^u(y_i)\right]\right)\right\}^{\downarrow} = U(Xp_i)$
6. Figure the membership, hesitation and non-membership value for the derived objects set:
   $\left[a_{Xp_i}^l, a_{Xp_i}^u\right] = \left[\min_{i \in a_{Xp_i}^l}\left(a_{Xp_i}^l, a_{Yp_i,}^l\right), \min_{i \in a_{Xp_i}^l}\left(a_{Xp_i}^u, a_{Yp_i}^u\right)\right]$
   $\left[b_{Xp_i}^l, b_{Xp_i}^u\right] = \left[\max_{i \in a_{Xp_i}^l}\left(b_{Xp_i}^l, b_{Yp_i,}^l\right), \max_{i \in a_{Xp_i}^l}\left(b_{Xp_i}^u, b_{Yp_i}^u\right)\right]$
   $\left[c_{Xp_i}^l, c_{Xp_i}^u\right] = \left[\max_{i \in a_{Xp_i}^l}\left(c_{Xp_i}^l, c_{Yp_i,}^l\right), \max_{i \in a_{Xp_i}^l}\left(c_{Xp_i}^u, c_{Yp_i}^u\right)\right]$
7. Apply up operator $\uparrow$ to the set U
   $U(Xp_i)^{\uparrow} = \left\{x_i, \left(\left[a_{Xp_i}^l(x_i), a_{Xp_i}^u(x_i)\right], \left[b_{Xp_i}^l(x_i), b_{Xp_i}^u(x_i)\right], \left[c_{Xp_i}^l(x_i), c_{Xp_i}^u(x_i)\right]\right)\right\}^{\uparrow}$
8. **if** $U(Xp_i)^{\uparrow} = V(Xp_i)$. //For intent represent in the set of attributes.
9. Then it is a concept.
10. **else**
11. Other attribute $z \in V$ covers the obtained objects
12. Then $k_i = (z \cup k_i)$. // add the novel attribute as an intent
13. The new concept $\left(X_{k_i}, Y_{k_i}\right)$ is generated.
14. **end if**
15. **end for**
16. Set the membership of last concept set = $\{[0,\ 0], [0,\ 0], [1,1]\}$ for $V\left(Y_j\right)$.
17. Construct the concept lattice via Galois connection.
18. **Return** the set of interval three-way concepts
   $U(x_i) = \left\{x_i, \left(\left[a_u^l(x_i), a_u^u(x_i)\right], \left[b_u^l(x_i), b_u^u(x_i)\right], \left[c_u^l(x_i), c_u^u(x_i)\right]\right)\right\}$
   $V\left(y_j\right) = \left\{y_j, \left(\left[a_v^l\left(y_j\right), a_v^u\left(y_j\right)\right], \left[b_v^l\left(y_j\right), b_v^u\left(y_j\right)\right], \left[c_v^l\left(y_j\right), c_v^u\left(y_j\right)\right]\right)\right\}$.

The Algorithm 1 generates interval-values intuitionistic fuzzy concepts via subset of attributes. In addition, through Galois connection, these interval concepts can be constructed as a visual hierarchical structure which makes the process of decision-making more intuitive.

3.2. A Novel Interval Three-Way Concept Lattice Model

After computing by the Algorithm 1, we can get an interval three-way concept lattice as $A_\alpha^\beta = (U, V, R)$. $G = \left(M^\alpha, M^\beta, Y\right)$ is one of interval concept of A_α^β. Set U was classified into three parts by the upper and lower extent of interval-valued intuitionistic fuzzy concept [38].

$$POS_\alpha^\beta(x) = M^\beta = \left\{ x \in U \middle| \frac{|f(x \cap Y)|}{|Y|} \geq \beta \right\},$$

$$BND_\alpha^\beta(x) = M^\beta - M^\alpha = \left\{ x \in U \middle| \alpha \leq \frac{|f(x \cap Y)|}{|Y|} < \beta \right\},$$

$$NEG_\alpha^\beta(x) = U - M^\alpha = \left\{ x \in U \middle| \frac{|f(x \cap Y)|}{|Y|} < \alpha \right\}.$$

If $x \in POS_\alpha^\beta(x)$, then we take acceptance actions. If $x \in NEG_\alpha^\beta(x)$, then we take refusal actions. Otherwise, we delayed the decision actions. B_t, B_i, and B_f represent respectively the acceptance action, delay action, and refusal action. The score function is denoted as follows.

Definition 6. *Let* $C = \{C, \neg C\}$ *denotes weather element* X *is in* C *or not. Then the proportion of each action are,*

Correct acceptance rate:

$$\frac{\left|C \cap POS_\alpha^\beta(x)\right|}{\left|POS_\alpha^\beta(x)\right|} = \frac{\left|C \cap M^\beta\right|}{\left|M^\beta\right|} \tag{3}$$

False acceptance rate:

$$\frac{\left|C^c \cap POS_\alpha^\beta(x)\right|}{\left|POS_\alpha^\beta(x)\right|} = \frac{\left|C^c \cap M^\beta\right|}{\left|M^\beta\right|} \tag{4}$$

Correct delay rate:

$$\frac{\left|C \cap BND_\alpha^\beta(x)\right|}{\left|BND_\alpha^\beta(x)\right|} = \frac{\left|C \cap \left(M^\alpha - M^\beta\right)\right|}{\left|M^\alpha - M^\beta\right|} \tag{5}$$

False delay rate:

$$\frac{\left|C^c \cap BND_\alpha^\beta(x)\right|}{\left|BND_\alpha^\beta(x)\right|} = \frac{\left|C^c \cap \left(M^\alpha - M^\beta\right)\right|}{\left|M^\alpha - M^\beta\right|} \tag{6}$$

Correct refusal rate:

$$\frac{\left|C \cap NEG_\alpha^\beta(x)\right|}{\left|NEG_\alpha^\beta(x)\right|} = \frac{\left|C \cap (U - M^\alpha)\right|}{\left|U - M^\alpha\right|} \tag{7}$$

False refusal rate:

$$\frac{\left|C^C \cap NEG_\alpha^\beta(x)\right|}{\left|NEG_\alpha^\beta(x)\right|} = \frac{\left|C^C \cap (U - M^\alpha)\right|}{\left|U - M^\alpha\right|} \tag{8}$$

Definition 7. *Let* $C = \{C, \neg C\}$ *denotes weather element* X *is in* C *or not. We can divide the receiving rate, delaying rate, and refusing rate into* C *or* $\neg C$ *respectively. From the point of view, the score functions can be considered as,* *Acceptance rate in C*:

$$\frac{\left|C \cap POS_{\alpha}^{\beta}(x)\right|}{|C|} = \frac{\left|C \cap M^{\beta}\right|}{|C|} \tag{9}$$

Delay rate in C:

$$\frac{\left|C \cap BND_{\alpha}^{\beta}(x)\right|}{|C|} = \frac{\left|C \cap \left(M^{\alpha} - M^{\beta}\right)\right|}{|C|} \tag{10}$$

Refusal rate in C:

$$\frac{\left|C \cap NEG_{\alpha}^{\beta}(x)\right|}{|C|} = \frac{|C \cap (U - M^{\alpha})|}{|C|} \tag{11}$$

Acceptance rate in $\neg C$:

$$\frac{\left|C^{c} \cap POS_{\alpha}^{\beta}(x)\right|}{|C^{c}|} = \frac{\left|C^{c} \cap M^{\beta}\right|}{|C^{c}|} \tag{12}$$

Delay rate in $\neg C$:

$$\frac{\left|C^{c} \cap BND_{\alpha}^{\beta}(x)\right|}{|C^{c}|} = \frac{\left|C^{c} \cap \left(M^{\alpha} - M^{\beta}\right)\right|}{|C^{c}|} \tag{13}$$

Refusal rate in $\neg C$:

$$\frac{\left|C^{C} \cap NEG_{\alpha}^{\beta}(x)\right|}{|C^{c}|} = \frac{|C^{C} \cap (U - M^{\alpha})|}{|C^{c}|} \tag{14}$$

Generally, TWDM focus on cost losses including misclassification costs and delay costs in different states. In Table 1, l_{tp}, l_{ip}, and l_{fp} indicates the costs for each action of B_t, B_i, and B_f respectively represent acceptance, delay, and refusal action when an element pertains to C, and vice versa [22].

As depicted in Table 1, the loss expectation function is respectively denoted as.

$$R(B_t) = l_{tp}\left(\left|X \cap M^{\beta}\right| / \left|M^{\beta}\right|\right) + l_{tn}\left(\left|\overline{X} \cap M^{\beta}\right| / \left|M^{\beta}\right|\right) \tag{15}$$

$$R(B_i) = l_{ip}\left(\left|X \cap \left(M^{\alpha} - M^{\beta}\right)\right| / \left|M^{\alpha} - M^{\beta}\right|\right) + l_{in}\left(\left|\overline{X} \cap \left(M^{\alpha} - M^{\beta}\right)\right| / \left|M^{\alpha} - M^{\beta}\right|\right) \tag{16}$$

$$R\left(B_f\right) = l_{fp}(|X \cap (U - M^{\alpha})| / |U - M^{\alpha}|) + l_{fn}(|\overline{X} \cap (U - M^{\alpha})| / |U - M^{\alpha}|) \tag{17}$$

Table 1. The loss expectation matrix.

	C(P)	$\neg$**C(N)**
B_t	l_{tp}	l_{tn}
B_i	l_{ip}	l_{in}
B_f	l_{fp}	l_{fn}

Definition 8 [46]. *Let the fuzzy formal context denotes as* $A = (U, V, R)$ *with decision interval* $[\alpha, \beta]$. *After computing by the algorithm of Table 1, we can get an interval-valued intuitionistic concept lattice as* $A_{\alpha}^{\beta} = (U, V, R)$. $G = \left(M^{\alpha}, M^{\beta}, Y\right)$ *is one of interval concept of* A_{α}^{β}. *Then* $\widetilde{A} = \left(M^{\alpha}, M^{\beta}, Y; R(B_t), R(B_i), R\left(B_f\right)\right)$ *represents interval three-way fuzzy concept lattice. For each decision element x of action* B_a $(a \in t, i, f)$ *with loss R, it can be repressed as* $J = (B_a, R(B_a))$.

Three-way decision space is constituted by interval concept lattice based on prior formal context. The interval three-way decision domain $\widetilde{A}_{\alpha}^{\beta}(U, V, R)$ which is divided by interval concept lattice can be used as a new object of making the decision. In addition, the decision space relationship can decide the next action decision to reduce losses. For a new object x with attribute set $V' \subseteq V$, we search for the interval three-way fuzzy concept owned intent of V'. Here are the procedures of the proposed interval three-way concept lattice model.

Step 1. Suppose an action strategy $J_k \in JS = (J_1, J_2, \cdots, J_n)$ is generated by interval three-way fuzzy concept $\widetilde{A} = \left(M^{\alpha}, M^{\beta}, Y; R(B_t), R(B_i), R\left(B_f\right)\right)$.

Step 2. Computing the loss R_{tk} of each J_k by loss function above.

Step 3. If there is the only action J_k which is satisfied

$$R_{tk} = \min(P_{t1}, P_{t2}, \cdots, P_{tn})$$

Then accept strategy J_k and refuse other strategies.

Step 4. If there are multiple actions J what are satisfied the formula above, then take delay decision and add newly attributes until $R_{tk} = \min(P_{t1}, P_{t2}, \cdots, P_{tn})$ holds.

Step 5. The action strategy J_n and J_m is decided respectively by interval concept lattice

$$\widetilde{A}_n = \left(M_n^{\alpha}, M_n^{\beta}, Y; R(B_t), R(B_i), R\left(B_f\right)\right)$$

$$\widetilde{A}_m = \left(M_m^{\alpha}, M_m^{\beta}, Y; R(B_t), R(B_i), R\left(B_f\right)\right)$$

At the same time, $R_{tn} = R_{tm} = \min(R_{t1}, R_{t2}, \cdots, R_{tn})$ holds. Then we should search the subset of $\widetilde{A}_n$ and $\widetilde{A}_m$. After adding attribute according to those intents of subsets, we search the minimum loss for taking acceptance action [37].

Once there is none concept, we should search for the concept which possesses the intent Y'. It exists that $Y' \subseteq Y$ and $|Y' - Y| = 1$. Analogously, we can get the decided action by repeating the above steps and same criteria.

4. Application of the Proposed Model in Medical Diagnosis

In this part, we conduct an illustrative experiment in medical diagnosis to validate the performance of interval three-way concept lattice model. As Table 2 shows, there are six patients with different symptoms. The decision is based on the following:

(1) S_1 as fever,
(2) S_2 as cough,
(3) S_3 as headache,
(4) S_4 as fatigue,
(5) S_5 as nausea,
(6) S_6 as inappetence, and
(7) S_7 as chest tightness

Diseases of these patients are repressed as,

(1) D_1 as flu,
(2) D_2 as chicken pox,
(3) D_3 as measles,
(4) D_4 as heatstroke

Suppose loss function value respectively as, $l_{tp} = 0$, $l_{ip} = 9$, $l_{fp} = 15$, $l_{tn} = 17$, $l_{in} = 2$, $l_{nn} = 0$

Table 2. The medical diagnosis formal context.

P	S_1	S_2	S_3	S_4	S_5	S_6	S_7	D
1	1	0	1	1	0	0	0	D_1
2	1	1	0	0	1	0	0	D_2
3	1	1	1	0	0	0	0	D_3
4	0	0	1	0	0	1	1	D_4
5	1	0	0	1	0	0	0	D_1
6	0	0	1	1	0	0	0	D_1

Decision interval $[\alpha, \beta]$ which reflects the risk preference of decision-makers, is decided by decision-makers. In this case, we suppose decision interval is $[0.6, 0.8]$. After using the proposed algorithm in Algorithm 1, the interval three-way concept is obtained as shown in Table 3.

Table 3. Interval three-way concept.

Concept	Upper Extent	Lower Extent	Intent	R_t	R_i	R_f
A_1	Φ	Φ	$S_1S_2S_3S_4S_5S_6S_7D_1D_2D_3D_4$	—	—	—
A_2	1235	1235	S_1D_1	8.50	0.00	7.50
A_3	1346	1346	S_3D_1	8.50	0.00	7.50
A_4	156	156	S_4D_1	0.00	0.00	0.00
A_5	13	13	$S_1S_3D_1$	8.50	0.00	7.50
A_6	15	15	$S_1S_4D_1$	0.00	0.00	3.75
A_7	16	16	$S_3S_4D_1$	0.00	0.00	3.75
A_8	1356	1	$S_1S_3S_4D_1$	0.00	6.67	0.00
A_9	1235	1235	S_1D_2	12.75	0.00	0.00
A_{10}	23	23	$S_1S_2D_2$	8.50	0.00	0.00
A_{11}	2	2	$S_1S_5D_2$	0.00	0.00	0.00
A_{12}	2	2	$S_2S_5D_2$	0.00	0.00	0.00
A_{13}	23	2	$S_1S_2S_5D_2$	0.00	2.00	0.00
A_{14}	1235	1235	S_1D_3	12.75	0.00	0.00
A_{15}	1346	1346	S_3D_3	12.75	0.00	0.00
A_{16}	23	23	$S_1S_2D_3$	8.50	0.00	0.00
A_{17}	13	13	$S_1S_3D_3$	8.50	0.00	0.00
A_{18}	3	3	$S_2S_3D_3$	0.00	0.00	0.00
A_{19}	123	3	$S_1S_2S_3D_3$	0.00	2.00	0.00
A_{20}	1346	1346	S_3D_4	12.75	0.00	0.00
A_{21}	4	4	$S_3S_6S_7D_4$	0.00	0.00	0.00
A_0	123456	123456	Φ	0.00	0.00	0.00

According to Table 3 and Algorithm 1, Hasse diagram of interval fuzzy concept lattice is derived through Galois connection as shown in Figure 1. In addition, it also represents three-way decision space $\widetilde{A}_{\alpha}^{\beta}(U, V, R)$. In Figure 1, decision-makers can intuitively realize the meaning of each concept and relationships between concepts. For example, concept $A_0(123456, 123456, \phi, 0.00, 0.00, 0.00)$ expresses the extent is 123456 and intent is ϕ, it indicates all the patients do not have the same symptoms. Sub-concept $A_2(1235, 1235, S_1D_1, 8.50, 0.00, 7.50)$ of A_0 depicts patients 1, 2, 3, and 5 can be classified into the same class which has a fever and get the flu, and the cost of taking acceptation, delay, and rejection action is 8.50, 0.00, and 7.50 respectively. For sub-concept $A_1(\phi, \phi, S_1S_2S_3S_4S_5S_6S_7D_1D_2D_3D_4, -, -, -)$, it signifies there is no disease with all the symptoms of a fever, cough, headache, fatigue, nausea, lack of appetite, and chest tightness.

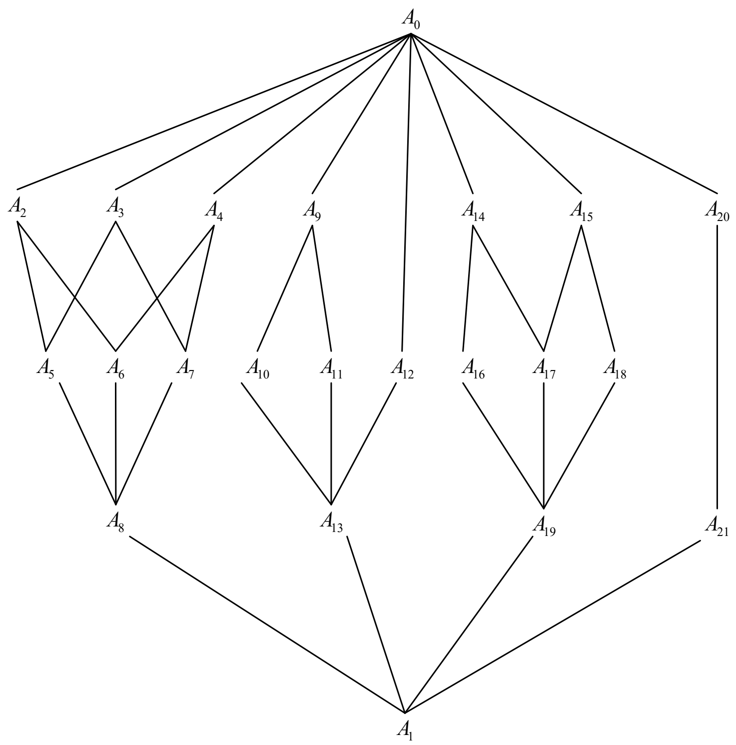

Figure 1. The three-way decision space $\widetilde{A}_{\alpha}^{\beta}(U,V,R)$.

Supposed there comes a new patient x with symptoms of fever S_1 and headache S_3. First of all, we should search the interval three-way concept for attributes as S_1 S_3. Then it can be obtained as:

$$A_5 = (13, 13, S_1S_3D_1; 8.50, 0.00, 0.75),$$

$$A_{17} = (13, 13, S_1S_3D_3; 8.50, 0.00, 0.00).$$

A_5 and A_{17} respectively indicated D_1 taking action $J_1 = \left(a_{t1}, R\left(a_{p1}\right) = 8.50\right)$ and D_3 taking action $J_2 = \left(a_{t2}, R\left(a_{p2}\right) = 8.50\right)$ which constitutes the binary decision stratagem space $JS = (J_1, J_2)$. It equals to $R(a_{t1}) = R(a_{t2}) = 8.50$ that is to say that the loss of acceptance of decision making by D_1 and D_3 for x is the same.

For further confirmed what disease x acquires, sub-concepts of A_5 and A_{17} are searched as:

$$A_8 = (1356, 1, S_1S_3S_4D_1; 0.00, 6.67, 0.00),$$

$$A_{19} = (123, 3, S_1S_2S_3D_3; 0.00, 2.00, 0.00).$$

It needs to add decision attributes respectively, which means we should enquire about the patient's symptoms for an accurate decision. If he or she has the symptom of fatigue S_4, then the diagnosis result is D_1 which means the patient acquires flu. Else, the diagnosis result is measles D_3, once he has the symptom of cough S_2.

In this section, we perform the proposed interval three-way concept lattice method on an illustrative medical diagnosis case. When applied the approach described into the example, the medical information was collected in IVIFS which effectively reflected the semantics in actual. $[0.6, 0.8]$ To reflect the risk preference of decision-makers and make the computing briefly, we take cut sets on decision interval $[\alpha, \beta]$. Using the interval three-way concepts construction algorithm, a clear and visualized Hasse diagram of interval fuzzy concept lattice can be generated which helps us to make decision preferably. Through the proposed score function, we computed the losses of acceptation, delay, and rejection strategy respectively. In addition, the decision can be made by comparing the loss of each strategy. While taking the delay strategy, we can wait for further observation until adding new attribute. Then we should search for the sub-concept and take the strategy of action minimum loss as the final decision. Thus, the proposed model can handle the medical diagnosis problem properly, and it can be applied to various fields.

5. Conclusions

In this paper, we put forward an interval three-way concept lattice model, which can provide decision making for the objects in the boundary domain and reduce the loss of decision making. According to the concept extension, we divide the decision domains into three decision regions as the new positive, boundary and negative region grounded on given decision rules. We redefine interval three-way decision measurements and decision loss functions based on IVIFS, TWDM and FCA. In addition, we put forward an algorithm for the interval three-way concepts construction, which can build a clear and visualized hierarchical relationship between concepts. Through the concept lattice hierarchy, we establish a dynamic strategy optimization model for medical diagnosis field. In addition, an example of medical diagnosis validates the effectiveness of the model and the operator. The interval three-way concept lattice model can apply to other fields beside medical diagnosis.

The proposed model provides solutions for medical diagnosis in the fuzzy environment. Firstly, the IVIFS can describe the uncertain and incomplete medical data properly and then improve the accuracy of diagnosis. Secondly, an algorithm is presented for the interval three-way concepts construction. In addition, a visualized hierarchical structure can be derived which helps us to make decisions, preferably such as in Figure 1. Thirdly, loss functions are redefined and a dynamic strategy optimization model is established which can be effectively used in many domains. However, there are still some issues which are not considered in this paper. First is the complexity of construction interval fuzzy concept lattice. Second is the data of illustration example is small.

Thus, further studies can be focused on the following aspects. Firstly, the reduction of interval three-way concept lattice can be presented which reduce the complexity of construction concept lattice. Secondly, it might optimize the proposed model and apply it to handle large-sized data of symptoms and disease in medical diagnosis field. Thirdly, it can be further investigated to combine the three-way concept lattice theory and other fuzzy sets, such as neutrosophic sets, picture fuzzy sets, and Z-numbers.

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References

1. Hu, J.; Zhang, X.; Yang, Y.; Liu, Y.; Chen, X. New doctors ranking system based on VIKOR method. *Int. Trans. Op. Res.* **2018**. [CrossRef]
2. Das, S.; Guha, D.; Dutta, B. Medical diagnosis with the aid of using fuzzy logic and intuitionistic fuzzy logic. *Appl. Intell.* **2016**, *45*, 1–18. [CrossRef]
3. Ledley, R.S.; Lusted, L.B. Reasoning foundations of medical diagnosis; symbolic logic, probability, and value theory aid our understanding of how physicians reason. *Science* **1959**, *130*, 9–21. [CrossRef] [PubMed]
4. Warner, H.R.; Toronto, A.F.; Veasey, L.G.; Stephenson, R. A Mathematical approach to medical diagnosis. *J. Am. Med. Assoc.* **1961**, *177*, 177–183. [CrossRef]
5. Yang, Y.; Hu, J.; Liu, Y.; Chen, X. Doctor recommendation based on an intuitionistic normal cloud model considering patient preferences. *Cognit. Comput.* **2018**. [CrossRef]
6. Zadeh, L.A. Biological Application of the Theory of Fuzzy Sets and Systems. In Proceedings of the International Symposium on Biocybernetics of the Central Nervous System, Washington, DC, USA, 15–19 January 1990; pp. 199–206.

7. De, S.K.; Biswas, R.; Roy, A.R. An application of intuitionistic fuzzy sets in medical diagnosis. *Fuzzy S. Syst.* **2001**, *117*, 209–213. [CrossRef]
8. Le, H.S.; Thong, N.T. Intuitionistic fuzzy recommender systems: An effective tool for medical diagnosis. *Know. Based Syst.* **2015**, *74*, 133–150.
9. Zhang, X.Y.; Zhang, H.Y.; Wang, J.Q. Discussing incomplete 2-tuple fuzzy linguistic preference relations in multi-granular linguistic MCGDM with unknown weight information. *Soft Comput.* **2017**. [CrossRef]
10. Zhang, H.; Shu, L.; Liao, S. Generalized trapezoidal fuzzy soft set and its application in medical diagnosis. *J. Appl. Math.* **2014**, *2014*, 1–12. [CrossRef]
11. Wang, X.; Wang, J.; Zhang, H. Distance-based multicriteria group decision-making approach with probabilistic linguistic term sets. *Expert Syst.* **2018**. [CrossRef]
12. Wang, J.Q.; Peng, J.J.; Zhang, H.Y.; Chen, X.H. Outranking approach for multi-criteria decision-making problems with hesitant interval-valued fuzzy sets. *Soft Comput.* **2017**. [CrossRef]
13. Li, J.; Wang, J.Q.; Hu, J.H. Interval-valued n-person cooperative games with satisfactory degree constraints. *Cent. Eur. J. Op. Res.* **2018**. [CrossRef]
14. Liang, P.; Hu, J.; Liu, Y.; Chen, X. Public resources allocation using an uncertain cooperative game among vulnerable groups. *Kybernetes* **2018**. [CrossRef]
15. Hu, J.; Pan, L.; Chen, X. An Interval neutrosophic projection-based VIKOR Method for selecting doctors. *Cognit. Comput.* **2017**, *9*, 801–816. [CrossRef]
16. Yang, Y.; Hu, J.; Sun, R.; Chen, X. Medical tourism destinations prioritization using group decision making method with neutrosophic fuzzy preference relations. *Sci. Iran.* **2018**, *25*, 3744–3764. [CrossRef]
17. Li, Y.Y.; Wang, J.Q.; Wang, T.L. A Linguistic neutrosophic multi-criteria group decision-making approach with EDAS method. *Arab. J. Sci. Eng.* **2018**. [CrossRef]
18. Sun, R.; Hu, J.; Chen, X. Novel single-valued neutrosophic decision-making approaches based on prospect theory and their applications in physician selection. *Soft Comput.* **2019**, *23*, 211–225. [CrossRef]
19. Ji, P.; Zhang, H.Y.; Wang, J.Q. A Fuzzy decision support model with sentiment analysis for items comparison in e-commerce: The case study of PConline.com. *IEEE Trans. Syst. Man Cybern. Syst.* **2018**. [CrossRef]
20. Peng, H.-G.; Wang, X.-K.; Wang, T.-L.; Wang, J.-Q. Multi-criteria game model based on the pairwise comparisons of strategies with Z-numbers. *Appl. Soft Comput.* **2019**, *74*, 451–465. [CrossRef]
21. Yang, Y.; Hu, J.; Liu, Y.; Chen, X. Alternative selection of end-of-life vehicle management in China: A group decision-making approach based on picture hesitant fuzzy measurements. *J. Clean. Prod.* **2019**, *206*, 631–645. [CrossRef]
22. Liang, D.; Pedrycz, W.; Liu, D.; Hu, P. Three-way decisions based on decision-theoretic rough sets under linguistic assessment with the aid of group decision making. *Appl. Soft Comput.* **2015**, *29*, 256–269. [CrossRef]
23. Yao, Y.; Deng, X. Sequential three-way decisions with probabilistic rough sets. *Inf. Sci.* **2010**, *180*, 341–353. [CrossRef]
24. Hu, J.; Yang, Y.; Chen, X. A novel TODIM method-based three-way decision model for medical treatment selection. *Int. J. Fuzzy Syst.* **2017**, *33*, 3405–3417. [CrossRef]
25. Yao, J.T.; Azam, N. Web-based medical decision support systems for three-way medical decision making with game-theoretic rough sets. *IEEE Trans. Fuzzy Syst.* **2015**, *23*, 3–15. [CrossRef]
26. Schnabel, M. Representing and processing medical knowledge using formal concept analysis. *Methods Inf. Med.* **2002**, *41*, 160–167. [CrossRef] [PubMed]
27. Fu, H.; Jennings, B.; Malone, P. Analysis and Representation of Biomedical data with Concept Lattice. In Proceedings of the Digital Ecosystems & Technologies Conference, Cairns, Austria, 21–23 February 2007; pp. 577–580.
28. Wille, R. *Restructuring Lattice Theory: An Approach Based on Hierarchies of Concepts*; Springer: Berlin/Heidelberg, Gemany, 2009; pp. 445–470.
29. Krajči, S. A Generalized concept lattice. *Logic J. IGPL* **2010**, *13*, 543–550. [CrossRef]
30. Papadimitriou, F. Modelling landscape complexity for land use management in Rio de Janeiro, Brazil. *Land Use Policy* **2012**, *29*, 855–861. [CrossRef]
31. Singh, P.K.; Kumar, C.A.; Gani, A. A comprehensive survey on formal concept analysis, its research trends and applications. *Int. J. Appl. Math. Comput. Sci.* **2016**, *26*, 495–516. [CrossRef]
32. Xu, F.; Xing, Z.Y.; Yin, H.D. Attribute reductions and concept lattices in interval-valued intuitionistic fuzzy rough set theory: Construction and properties. *J. Intell. Fuzzy Syst.* **2016**, *30*, 1231–1242. [CrossRef]

33. Lei, Z. Formal Concept Analysis in Intuitionistic Fuzzy Formal Context. In Proceedings of the International Conference on Fuzzy Systems & Knowledge Discovery, Yantai, China, 10–12 August 2010; pp. 2012–2015.
34. Singh, P.K.; Kumar, C.A. Bipolar fuzzy graph representation of concept lattice. *Inf. Sci.* **2014**, *288*, 437–448. [CrossRef]
35. Singh, P.K. Complex neutrosophic concept lattice and its applications to air quality analysis. *Chaos Solitons Fractals* **2018**, *109*, 206–213. [CrossRef]
36. Qian, X.; Wang, D.; Yuan, H.; Lu, H.; Xu, J. Analysis of Association Rule Extraction between Rough Set and Concept Lattice. In Proceedings of the International Conference on Computer Science & Education, Melbourne, Australia, 14–17 July 2017; pp. 599–603.
37. Singh, P.K. Three-way fuzzy concept lattice representation using neutrosophic set. *Int. J. Mach. Learn. Cybern.* **2016**, 1–11. [CrossRef]
38. Qi, J.; Wei, L.; Yao, Y. Three-Way Formal Concept Analysis. In Proceedings of the 9th International Conference RSKT 2014, Shanghai, China, 24–26 October 2014; pp. 732–741.
39. Atanassov, K.T. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349. [CrossRef]
40. Kononenko, I. Machine learning for medical diagnosis: History, state of the art and perspective. *Artif. Intell. Med.* **2001**, *23*, 89–109. [CrossRef]
41. Sanchez, E. Resolution of composite fuzzy relation equations. *Inf. Control* **1976**, *30*, 38–48. [CrossRef]
42. Singh, P.K.; Aswani Kumar, C.; Li, J. Knowledge representation using interval-valued fuzzy formal concept lattice. *Soft Comput.* **2016**, *19*, 1–18. [CrossRef]
43. Burusco Juandeaburre, A.; Fuentes González, R. The study of the L-fuzzy concept lattice. *Mathw. Soft Comput.* **1994**, *1*, 209–218.
44. Pollandt, S. *Fuzzy-Begriffe*; Springer: Heidelberg/Berlin, Germany, 1997; pp. 51–87.
45. Ganter, B.; Godin, R. *Formal Concept Analysis*; Springer: Heidelberg/Berlin, Germany, 2005; pp. 170–180.
46. Zhang, C.; Wang, L. An effective interval concept lattice construction algorithm. *ICIC Express Lett. Part B Appl. Int. J. Res. Surv.* **2014**, *5*, 1573–1578.