

Article

# Prioritized Linguistic Interval-Valued Aggregation Operators and Their Applications in Group Decision-Making Problems

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**Abstract:** The linguistic interval-valued intuitionistic fuzzy (LIVIF) set is an efficient tool to represent data in the form of interval membership degrees in a qualitative rather than a quantitative manner. The LIVIF set combines the features of interval-valued intuitionistic fuzzy sets (IFSs) and the linguistic variables (LV) and hence provides more freedom to decision-makers. Under this environment, the main objective of this manuscript is to propose some new aggregation operators by capturing the prioritized relationship between the objects. For this, different weighted averaging and geometric aggregation operators are proposed in which preferences related to each object are expressed in terms of LIVIF numbers. Desirable properties of the proposed operators are studied. Further, a group decision-making (DM) approach is presented to solve the multi-attribute DM problems, and its efficiency has been verified with an illustrative example.

**Keywords:** interval-valued numbers; aggregation operators; group decision-making (DM); linguistic numbers; prioritized operators; multi-attribute group decision-making (MAGDM)

## 1. Introduction

Decision-making (DM) problems are the imperative part of modern decision theory, where a set of alternatives has to be assessed against multiple influential attributes before the best alternative is selected. In decision theory, an important problem is how to express the preference value. Due to the increasing complexity of the socioeconomic environment and the insufficiency of awareness or the data of the DM problems, it is very difficult for a single decision-maker to consider all the relevant aspects of the problem. Thus, there is a need to incorporate the multiple decision-makers into the decision-makers and hence construct a multi-attribute group decision-making (MAGDM) problem. In MAGDM problems, we achieve the target of the problems based on several decision-makers' preferences. However, it is hard for the decision-maker(s) to provide the exact decision as there is always imprecise, vague or uncertain information [1–3]. Thus, to handle the uncertainties in the data, Zadeh [4] introduced the concept of the fuzzy set (FS), and after that, its extensions such as intuitionistic fuzzy sets (IFSs) [5] and interval-valued IFSs (IVIFSs) [6] have become more powerful tools to describe the uncertainties. During the last five decades, researchers have been paying more attention to these theories and have effectively applied them to the different situations in the DM process such as information measures [7–9] and aggregation operators [10–16]. Among these, an aggregation operator (AO) is an important part of the DM process, which usually takes the form of a mathematical function to aggregate all the input values into a single one. Due to its successful wide application, several researchers have made efforts in the research on aggregation operators. For instance, Xu and Yager [14] developed some weighted, ordered weighted and hybrid geometric aggregation operators

for the different intuitionistic fuzzy numbers (IFNs). Later on, Xu and Chen [10] proposed some of these aggregation operators for interval-valued intuitionistic numbers. Garg [17] presented improved aggregation operators based on the interactive operation laws between the membership degrees.

In all the above existing approaches, the information used for accessing the objects is expressed in a quantitative manner. However, in real-life problems, there are many attribute values that are qualitative in nature and cannot be expressed by a numeric value. For instance, in order to measure the performance of a person, we usually prefer terms such as “excellent”, “better”, “good”, “bad”, etc, which cannot be expressed by a numeric value. In such cases, it is easy to describe the preference values as a linguistic variable (LV). For this, Zadeh [18] proposed the concept of a LV. In the field of AOs, Xu [19] presented a DM approach based on aggregating operators and the possibility degree method under uncertain linguistic information. Zhou et al. [20] presented prioritized operators for aggregating the uncertain linguistic fuzzy information. Garg and Arora [21] presented scaled prioritized AOs under the intuitionistic fuzzy soft environment. Garg and Arora [22] proposed some generalized power AOs based on t-norm operations for intuitionistic fuzzy soft set environment. Later on, Zhang [23] defined the linguistic intuitionistic fuzzy set (LIFS) in which membership and non-membership degree is represented by the linguistic term, and they presented aggregating operators based on it. Chen et al. [24] presented an MAGDM approach in the LIFS environment. Liu and Wang [25] defined some improved operational laws for LIFNs and aggregation operators based on it. Xian et al. [26] presented a new hybrid aggregation operator and DM approach based on it. Garg and Kumar [27] presented some aggregation operators for LIFSs by using the connection number of the set pair analysis theory. Liu and Qin [28] presented the power averaging operator for aggregating the LIFNs and multi-attribute group DM method. Liu and Liu [29] proposed power Bonferroni mean operators for aggregating the LIFN information. Garg and Kumar [30] proposed the possibility degree measures for the LIFS and the linguistic intuitionistic fuzzy aggregation operators using Einstein t-norm operations. Garg [31] presented the linguistic Pythagorean FSs.

The above theories have been successfully applied in many different fields, but their aggregating operators and the DM approaches for IFS/LIFS strictly considered that the different attributes are at the same priority level. Furthermore, in many DM situations, there is always an unequal importance given to each attribute based on their priority level; for example, if we are buying a car based on the attributes such as cost and safety features. In this case, we should assign a higher priority to safety features than cost. To handle such types of problems, Yager [32] proposed the prioritized averaging operator, which highlights the support of input values during the aggregation process. Xu and Yager [15] and Yu [33] investigated the prioritized averaging and geometric aggregation operators under the IFS environment. Arora and Garg [34] presented prioritized averaging and geometric aggregation operators under intuitionistic fuzzy soft set information. Rani and Garg [35] presented power aggregation operators for the complex IFS.

As it is seen from the above study that LIFS theory is widely used by the researchers, due to the complexity of the DM problems, sometimes, decision-makers are not capable of providing their judgment in the form of crisp membership degrees (MDs) and non-membership degrees (NMDs). Consequently, an extension of the existing theories might be extremely valuable to depict the uncertainties because of his/her reluctant judgment in complex DM problems. Thus, to provide more freedom to the decision-makers, Garg and Kumar [36] presented the concept of the linguistic interval-valued intuitionistic fuzzy (LIVIFS) set in which membership and non-membership degrees are represented by interval-valued linguistic terms. Henceforth, an LIVIFS is a more generalized extension of the existing theories such as IFSs, IVIFSs and LIFSs. Later on, Garg and Kumar [37] presented some distance measure-based extended technique for order preference by similarity to ideal solution (TOPSIS) approaches for solving the MAGDM problem under the LIVIFS environment. Thus, by considering the fact that the LIVIFS is a combination of both LV and IVIFS, it can easily express the qualitative, as well as the quantitative aspects. As far as the authors are aware, there is no investigation on how to aggregate the different preferences of decision-makers under the LIVIF

information. Therefore, an interesting and important issue is how to utilize the collective decision matrix and the unknown preferences information to find the most desirable alternative(s) during the DM process.

Keeping the advantage of the LIVIFS and the prioritized relation during the process, we introduced some new prioritized weighted averaging and geometric AOs to aggregate the different linguistic interval-valued intuitionistic fuzzy numbers (LIVIFNs). The main characteristic of these operators is that they consider the prioritized relationship between the input arguments. Thus, these considerations have led us to consider the following main objectives for this paper:

1. to represent the preference of decision-makers in terms of linguistic features;
2. to present some new aggregation operators to aggregate the preferences of the decision-makers;
3. to propose an algorithm to solve the group DM problems;
4. to present an illustrative example to describe an algorithm.

To achieve Objective 1, in this article, we have utilized the features of the interval-valued terms and the LVs to describe the information of the various experts as the LIVIF environment in which MDs and NMDs are represented by linguistic interval-valued terms. Objective 2 is achieved by developing some new prioritized aggregation operators, namely the LIVIF prioritized weighted averaging (LIVIFPWA) operator, LIVIF prioritized ordered weighted averaging (LIVIFPOWA) operator, LIVIF prioritized weighted geometric (LIVIFPWG) operator and LIVIF prioritized ordered weighted geometric (LIVIFPOWG) operator. The various desirable properties of these proposed operators are discussed in detail. To achieve Objective 3, we establish an MAGDM method based on the proposed operators where rating values corresponding to each alternative are expressed as LIVIFNs. Finally, Objective 4 is explained through an illustrative example to demonstrate the approach, and the computed results are compared with some of the existing approaches to show their efficiency.

The rest of the manuscript is organized as follows. Section 2 discusses some basic notion of IVIFS, LIFS and LIVIFS. In Section 3, we develop some prioritized weighted averaging and geometric aggregation operators and investigate their desirable properties. Section 4 describes an MAGDM approach based on the proposed operators for ranking the different alternatives under the LIVIF environment. In Section 5, the presented approach is illustrated with a numerical example, and the computed results are compared with some existing approaches. Section 6 ends with the concluding remarks.

## 2. Preliminaries

In this section, some concepts related to IVIFS, LIFS and LIVIFS are reviewed briefly.

**Definition 1.** [6] An IVIFS ‘A’ in X is defined as:

$$A = \left\{ \left( x, [u_A^L(x), u_A^U(x)], [v_A^L(x), v_A^U(x)] \right) \mid x \in X \right\}, \tag{1}$$

where  $[u_A^L(x), u_A^U(x)], [v_A^L(x), v_A^U(x)] \subseteq [0, 1]$  represents the MDs and NMDs of x to A. For any  $x \in X, 0 \leq u_A^U + v_A^U \leq 1$ , the pair  $([u_A^L, u_A^U], [v_A^L, v_A^U])$  is called an IVIF number (IVIFN).

**Definition 2.** [38] Let  $S = \{s_t \mid t = 0, 1, 2, \dots, h\}$  be a finite odd cardinality linguistic term set (LTS). Each linguistic term  $s_t$  must have the following characteristics.

1.  $s_k \leq s_t \Leftrightarrow k \leq t$ ;
2. Negation operator:  $Neg(s_k) = s_t$  where  $t = h - k$ ;
3. Max operator:  $\max(s_k, s_t) = s_k \Leftrightarrow s_k \geq s_t$ ;
4. Min operator:  $\min(s_k, s_t) = s_k \Leftrightarrow s_k \leq s_t$

In DM problems [23,24], most of the times, the information is expressed as qualitative terms rather than numerical values. In such situations, it is necessary to consider them as LVs. Let  $S = \{s_i \mid i = 0, 1, \dots, h\}$  be a linguistic term set with odd cardinality, where  $s_i$ 's represent a possible value for a LV. Further, Xu [19] extended this set to a continuous set  $S_{[0,h]} = \{s_z \mid s_0 \leq s_z \leq s_h\}$ , where, if  $s_z \in S$ , then  $s_z$  is called the original, otherwise, virtual.

**Definition 3.** [23] For a finite universal set  $X$  and a continuous linguistic term set  $S_{[0,h]}$ , an LIFS  $A$  is stated as:

$$A = \left\{ \left( x, s_{\tau(x)}, s_{\theta(x)} \right) \mid x \in X \right\}, \tag{2}$$

where  $s_{\tau}, s_{\theta} \in S_{[0,h]}$  represent the linguistic membership and non-membership degrees, respectively, and  $0 \leq \tau + \theta \leq h$ . The linguistic indeterminacy is defined as  $s_{\pi} = s_{h-\tau-\theta}$ . Usually, the pair  $(s_{\tau}, s_{\theta})$  is called the LIF number (LIFN), and it is often written as  $\gamma = \langle s_{\tau}, s_{\theta} \rangle$  where  $s_{\tau}, s_{\theta} \in S_{[0,h]}$  and  $\tau + \theta \leq h$ . If  $s_{\tau}, s_{\theta} \in S$ , then the LIFN is called original, otherwise virtual.

**Definition 4.** [32] For real numbers  $\gamma_t (t = 1, 2, \dots, n)$ , the prioritized weighted aggregation (PWA) operator is given as:

$$PWA(\gamma_1, \gamma_2, \dots, \gamma_t) = \sum_{t=1}^n w_t \gamma_t,$$

where  $w_t = \frac{T_t}{\sum_{j=1}^n T_j}$ ,  $T_1 = 1$  and  $T_t = \prod_{k=1}^{t-1} \gamma_k$ ,  $t = 2, 3, \dots, n$ .

**Definition 5.** [36] Let  $S_{[0,h]}$  be a continuous linguistic term set. A LIVIFS is defined in the finite universe of discourse  $X$  mathematically with the form:

$$A = \left\{ \left( x, [s_{\tau(x)}, s_{\eta(x)}], [s_{\theta(x)}, s_{\upsilon(x)}] \right) \mid x \in X \right\}, \tag{3}$$

where  $[s_{\tau}, s_{\eta}]$  and  $[s_{\theta}, s_{\upsilon}]$  are all subsets of  $[s_0, s_h]$  and represent the linguistic membership and non-membership degrees of  $x$  to  $A$ , respectively. For any  $x \in X$ ,  $s_{\eta(x)} + s_{\upsilon(x)} \leq s_h$  (i.e.,  $\eta + \upsilon \leq h$ ) is always satisfied, and in turn, the linguistic intuitionistic index of  $x$  to  $A$  is defined as  $s_{\pi(x)} = [s_{h-\eta(x)-\upsilon(x)}, s_{h-\tau(x)-\theta(x)}]$ . Usually, the pair  $\left( [s_{\tau(x)}, s_{\eta(x)}], [s_{\theta(x)}, s_{\upsilon(x)}] \right)$  is called the linguistic interval-valued intuitionistic fuzzy number (LIVIFN).

For convenience, we denote the LIVIFN as  $\gamma = ([s_{\tau}, s_{\eta}], [s_{\theta}, s_{\upsilon}])$ , where  $[s_{\tau}, s_{\eta}] \subseteq [s_0, s_h]$ ,  $[s_{\theta}, s_{\upsilon}] \subseteq [s_0, s_h]$ ,  $\eta + \upsilon \leq h$  and also  $s_{\tau}, s_{\eta}, s_{\theta}, s_{\upsilon} \in S_{[0,h]}$  holds.

**Definition 6.** [36] Let  $\gamma_1 = ([s_{\tau_1}, s_{\eta_1}], [s_{\theta_1}, s_{\upsilon_1}])$  and  $\gamma_2 = ([s_{\tau_2}, s_{\eta_2}], [s_{\theta_2}, s_{\upsilon_2}])$  be two LIVIFNs, then:

1.  $\gamma_1 \preceq \gamma_2$  if  $\tau_1 \leq \tau_2, \eta_1 \leq \eta_2, \theta_1 \geq \theta_2$ , and  $\upsilon_1 \geq \upsilon_2$ ;
2.  $\gamma_1 = \gamma_2$  if and only if  $\gamma_1 \preceq \gamma_2$  and  $\gamma_2 \preceq \gamma_1$ ;
3.  $\gamma_1^c = ([s_{\theta_1}, s_{\upsilon_1}], [s_{\tau_1}, s_{\eta_1}])$  is the complement of  $\gamma_1$ ;
4.  $\gamma_1 \cup \gamma_2 = \left( \left[ \max\{s_{\tau_1}, s_{\tau_2}\}, \left[ \min\{s_{\theta_1}, s_{\theta_2}\}, \right] \right], \left[ \max\{s_{\eta_1}, s_{\eta_2}\}, \left[ \min\{s_{\upsilon_1}, s_{\upsilon_2}\}, \right] \right] \right)$ ;
5.  $\gamma_1 \cap \gamma_2 = \left( \left[ \min\{s_{\tau_1}, s_{\tau_2}\}, \left[ \max\{s_{\theta_1}, s_{\theta_2}\}, \right] \right], \left[ \min\{s_{\eta_1}, s_{\eta_2}\}, \left[ \max\{s_{\upsilon_1}, s_{\upsilon_2}\}, \right] \right] \right)$ .

**Definition 7.** [36] For LIVIFN  $\gamma = ([s_\tau, s_\eta], [s_\theta, s_\nu])$ , a score function is defined as:

$$S(\gamma) = s_{(2h+\tau-\theta+\eta-\nu)/4} \tag{4}$$

and the accuracy function is:

$$H(\gamma) = s_{(\tau+\eta+\theta+\nu)/2} \tag{5}$$

Based on it, we define an order relation between two LIVIFNs  $\gamma_1$  and  $\gamma_2$ , as:

1. If  $S(\gamma_1) > S(\gamma_2)$ , then  $\gamma_1 > \gamma_2$ ;
2. If  $S(\gamma_1) = S(\gamma_2)$ , then  $\begin{cases} H(\gamma_1) > H(\gamma_2) & \Rightarrow \gamma_1 > \gamma_2; \\ H(\gamma_1) = H(\gamma_2) & \Rightarrow \gamma_1 = \gamma_2 \end{cases}$

**Definition 8.** [36] Let  $\gamma_1 = ([s_{\tau_1}, s_{\eta_1}], [s_{\theta_1}, s_{\nu_1}])$  and  $\gamma_2 = ([s_{\tau_2}, s_{\eta_2}], [s_{\theta_2}, s_{\nu_2}])$  be two LIVIFNs and  $\lambda > 0$  be real, then some basic operational laws are defined as follows:

1.  $\gamma_1 \oplus \gamma_2 = \left( \left[ s_{\tau_1+\tau_2-\frac{\tau_1\tau_2}{h}}, s_{\eta_1+\eta_2-\frac{\eta_1\eta_2}{h}} \right], \left[ s_{\frac{\theta_1\theta_2}{h}}, s_{\frac{\nu_1\nu_2}{h}} \right] \right)$ ;
2.  $\gamma_1 \otimes \gamma_2 = \left( \left[ s_{\frac{\tau_1\tau_2}{h}}, s_{\frac{\eta_1\eta_2}{h}} \right], \left[ s_{\theta_1+\theta_2-\frac{\theta_1\theta_2}{h}}, s_{\nu_1+\nu_2-\frac{\nu_1\nu_2}{h}} \right] \right)$ ;
3.  $\lambda\gamma_1 = \left( \left[ s_{h\left(1-\left(1-\frac{\tau_1}{h}\right)^\lambda\right)}, s_{h\left(1-\left(1-\frac{\eta_1}{h}\right)^\lambda\right)} \right], \left[ s_{h\left(\frac{\theta_1}{h}\right)^\lambda}, s_{h\left(\frac{\nu_1}{h}\right)^\lambda} \right] \right)$ ;
4.  $\gamma_1^\lambda = \left( \left[ s_{h\left(\frac{\tau_1}{h}\right)^\lambda}, s_{h\left(\frac{\eta_1}{h}\right)^\lambda} \right], \left[ s_{h\left(1-\left(1-\frac{\theta_1}{h}\right)^\lambda\right)}, s_{h\left(1-\left(1-\frac{\nu_1}{h}\right)^\lambda\right)} \right] \right)$ .

### 3. New Prioritized Aggregation Operator for LIVIFNs

In this section, we have defined the prioritized aggregation operators for a collection of LIVIFNs defined over the finite universal set  $X = \{x_1, x_2, \dots, x_n\}$ , and  $S_{[0,h]}$  is a continuous LTS.

#### 3.1. Properties of LIVIFNs

**Theorem 1.** If  $\gamma_1$  and  $\gamma_2$  are two LIVIFNs, then operations defined in Definition 8 are also LIVIFNs.

**Proof.** Let  $\gamma_1 = ([s_{\tau_1}, s_{\eta_1}], [s_{\theta_1}, s_{\nu_1}])$  and  $\gamma_2 = ([s_{\tau_2}, s_{\eta_2}], [s_{\theta_2}, s_{\nu_2}])$  be two LIVIFNs, so we have  $0 \leq \eta_1, \eta_2, \nu_1, \nu_2 \leq h$ ,  $\eta_1 + \nu_1 \leq h$  and  $\eta_2 + \nu_2 \leq h$ . Therefore, we have  $0 \leq \left(1 - \frac{\eta_1}{h}\right) \left(1 - \frac{\eta_2}{h}\right) \leq 1 \Leftrightarrow 0 \leq 1 - \left(1 - \frac{\eta_1}{h}\right) \left(1 - \frac{\eta_2}{h}\right) \leq 1 \Leftrightarrow 0 \leq h \left(1 - \left(1 - \frac{\eta_1}{h}\right) \left(1 - \frac{\eta_2}{h}\right)\right) \leq h$ . Similarly,  $0 \leq \frac{\nu_1\nu_2}{h} \leq h$ . On the other hand, we have  $\eta_1 + \eta_2 - \frac{\eta_1\eta_2}{h} + \frac{\nu_1\nu_2}{h} \leq \eta_1 + \eta_2 - \frac{\eta_1\eta_2}{h} + \frac{(h-\eta_1)(h-\eta_2)}{h} = h$ . Hence,  $\gamma_1 \oplus \gamma_2$  is an LIVIFN. This is similar for the other cases.  $\square$

**Theorem 2.** Let  $\gamma = ([s_\tau, s_\eta], [s_\theta, s_\nu])$ ,  $\gamma_1 = ([s_{\tau_1}, s_{\eta_1}], [s_{\theta_1}, s_{\nu_1}])$ ,  $\gamma_2 = ([s_{\tau_2}, s_{\eta_2}], [s_{\theta_2}, s_{\nu_2}])$  be three LIVIFNs and  $\lambda, \lambda_1, \lambda_2 > 0$  be three real numbers, then:

1.  $\gamma_1 \oplus \gamma_2 = \gamma_2 \oplus \gamma_1$ ;
2.  $\lambda(\gamma_1 \oplus \gamma_2) = \lambda\gamma_1 \oplus \lambda\gamma_2$ ;
3.  $\lambda_1\gamma \oplus \lambda_2\gamma = (\lambda_1 + \lambda_2)\gamma$ ;
4.  $\gamma_1 \otimes \gamma_2 = \gamma_2 \otimes \gamma_1$ ;
5.  $\gamma^{\lambda_1} \otimes \gamma^{\lambda_2} = \gamma^{\lambda_1+\lambda_2}$ ;
6.  $\gamma_1^\lambda \otimes \gamma_2^\lambda = (\gamma_1 \otimes \gamma_2)^\lambda$ ;
7.  $\gamma_1^c \oplus \gamma_2^c = (\gamma_1 \otimes \gamma_2)^c$ ;
8.  $\gamma_1^c \otimes \gamma_2^c = (\gamma_1 \oplus \gamma_2)^c$ ;

9.  $\gamma_1^c \cup \gamma_2^c = (\gamma_1 \cap \gamma_2)^c$ ;
10.  $\gamma_1^c \cap \gamma_2^c = (\gamma_1 \cup \gamma_2)^c$ ;
11.  $(\gamma_1 \cup \gamma_2) \cap \gamma_2 = \gamma_2$ ;
12.  $(\gamma_1 \cap \gamma_2) \cup \gamma_2 = \gamma_2$ ;
13.  $(\gamma^c)^\lambda = (\lambda\gamma)^c$ ;
14.  $\lambda(\gamma^c) = (\gamma^\lambda)^c$ ;
15.  $\gamma_1 \cup \gamma_2 = \gamma_2 \cup \gamma_1$ ;
16.  $\gamma_1 \cap \gamma_2 = \gamma_2 \cap \gamma_1$ ;
17.  $\lambda(\gamma_1 \cup \gamma_2) = \lambda\gamma_1 \cup \lambda\gamma_2$ ;
18.  $\gamma_1^\lambda \cup \gamma_2^\lambda = (\gamma_1 \cup \gamma_2)^\lambda$ .

**Proof.** Here, we shall prove only Parts (1)–(3), while rest can be proven similarly.

1. According to Definition 8, we have:

$$\begin{aligned} \gamma_1 \oplus \gamma_2 &= \left( \begin{bmatrix} s_{\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{h}} \\ s_{\eta_1 + \eta_2 - \frac{\eta_1 \eta_2}{h}} \end{bmatrix}, \begin{bmatrix} s_{\frac{\theta_1 \theta_2}{h}} \\ s_{\frac{v_1 v_2}{h}} \end{bmatrix} \right) \\ &= \left( \begin{bmatrix} s_{\tau_2 + \tau_1 - \frac{\tau_2 \tau_1}{h}} \\ s_{\eta_2 + \eta_1 - \frac{\eta_2 \eta_1}{h}} \end{bmatrix}, \begin{bmatrix} s_{\frac{\theta_2 \theta_1}{h}} \\ s_{\frac{v_2 v_1}{h}} \end{bmatrix} \right) \\ &= \gamma_2 \oplus \gamma_1 \end{aligned}$$

2. For  $\lambda > 0$ , we have:

$$\lambda\gamma_1 = \left( \begin{bmatrix} s_{h(1-(1-\frac{\tau_1}{h})^\lambda)} \\ s_{h(1-(1-\frac{\eta_1}{h})^\lambda)} \end{bmatrix}, \begin{bmatrix} s_{h(\frac{\theta_1}{h})^\lambda} \\ s_{h(\frac{v_1}{h})^\lambda} \end{bmatrix} \right)$$

and:

$$\lambda\gamma_2 = \left( \begin{bmatrix} s_{h(1-(1-\frac{\tau_2}{h})^\lambda)} \\ s_{h(1-(1-\frac{\eta_2}{h})^\lambda)} \end{bmatrix}, \begin{bmatrix} s_{h(\frac{\theta_2}{h})^\lambda} \\ s_{h(\frac{v_2}{h})^\lambda} \end{bmatrix} \right)$$

Therefore,

$$\begin{aligned} \lambda\gamma_1 \oplus \lambda\gamma_2 &= \left( \begin{bmatrix} s_{h(1-(1-\frac{\tau_1}{h})^\lambda)} \\ s_{h(1-(1-\frac{\eta_1}{h})^\lambda)} \end{bmatrix}, \begin{bmatrix} s_{h(\frac{\theta_1}{h})^\lambda} \\ s_{h(\frac{v_1}{h})^\lambda} \end{bmatrix} \right) \oplus \left( \begin{bmatrix} s_{h(1-(1-\frac{\tau_2}{h})^\lambda)} \\ s_{h(1-(1-\frac{\eta_2}{h})^\lambda)} \end{bmatrix}, \begin{bmatrix} s_{h(\frac{\theta_2}{h})^\lambda} \\ s_{h(\frac{v_2}{h})^\lambda} \end{bmatrix} \right) \\ &= \left( \begin{bmatrix} s_{h(1-(1-\frac{\tau_1}{h})^\lambda(1-\frac{\tau_2}{h})^\lambda)} \\ s_{h(1-(1-\frac{\eta_1}{h})^\lambda(1-\frac{\eta_2}{h})^\lambda)} \end{bmatrix}, \begin{bmatrix} s_{h(\frac{\theta_1 \theta_2}{h^2})^\lambda} \\ s_{h(\frac{v_1 v_2}{h^2})^\lambda} \end{bmatrix} \right) \\ &= \lambda(\gamma_1 \oplus \gamma_2) \end{aligned}$$

3. For  $\lambda_1, \lambda_2 > 0$ , we have:

$$\begin{aligned} \lambda_1 \gamma_1 \oplus \lambda_2 \gamma_1 &= \left( \left[ \begin{matrix} s_{h(1-(1-\frac{\tau_1}{h})^{\lambda_1})} \\ s_{h(1-(1-\frac{\eta_1}{h})^{\lambda_1})} \end{matrix} \right], \left[ \begin{matrix} s_{h(\frac{\theta_1}{h})^{\lambda_1}} \\ s_{h(\frac{v_1}{h})^{\lambda_1}} \end{matrix} \right] \right) \oplus \left( \left[ \begin{matrix} s_{h(1-(1-\frac{\tau_1}{h})^{\lambda_2})} \\ s_{h(1-(1-\frac{\eta_1}{h})^{\lambda_2})} \end{matrix} \right], \left[ \begin{matrix} s_{h(\frac{\theta_1}{h})^{\lambda_2}} \\ s_{h(\frac{v_1}{h})^{\lambda_2}} \end{matrix} \right] \right) \\ &= \left( \left[ \begin{matrix} s_{h(1-(1-\frac{\tau_1}{h})^{\lambda_1+\lambda_2})} \\ s_{h(1-(1-\frac{\eta_1}{h})^{\lambda_1+\lambda_2})} \end{matrix} \right], \left[ \begin{matrix} s_{h(\frac{\theta_1}{h})^{\lambda_1+\lambda_2}} \\ s_{h(\frac{v_1}{h})^{\lambda_1+\lambda_2}} \end{matrix} \right] \right) \\ &= (\lambda_1 + \lambda_2) \gamma_1 \end{aligned}$$

□

**Theorem 3.** For two LIVIFNs  $\gamma_1 = ([s_{\tau_1}, s_{\eta_1}], [s_{\theta_1}, s_{v_1}])$  and  $\gamma_2 = ([s_{\tau_2}, s_{\eta_2}], [s_{\theta_2}, s_{v_2}])$ , we have:

1.  $(\gamma_1 \cup \gamma_2) \oplus (\gamma_1 \cap \gamma_2) = \gamma_1 \oplus \gamma_2$ ;
2.  $(\gamma_1 \cup \gamma_2) \otimes (\gamma_1 \cap \gamma_2) = \gamma_1 \otimes \gamma_2$ .

**Proof.** Here, we shall show that only (1) and (2) can be proven similarly.

$$\begin{aligned} &(\gamma_1 \cup \gamma_2) \oplus (\gamma_1 \cap \gamma_2) \\ &= \left( \left[ \begin{matrix} \max\{s_{\tau_1}, s_{\tau_2}\} \\ \max\{s_{\eta_1}, s_{\eta_2}\} \end{matrix} \right], \left[ \begin{matrix} \min\{s_{\theta_1}, s_{\theta_2}\} \\ \min\{s_{v_1}, s_{v_2}\} \end{matrix} \right] \right) \oplus \left( \left[ \begin{matrix} \min\{s_{\tau_1}, s_{\tau_2}\} \\ \min\{s_{\eta_1}, s_{\eta_2}\} \end{matrix} \right], \left[ \begin{matrix} \max\{s_{\theta_1}, s_{\theta_2}\} \\ \max\{s_{v_1}, s_{v_2}\} \end{matrix} \right] \right) \\ &= \left( \left[ \begin{matrix} s_{\max\{\tau_1, \tau_2\} + \min\{\tau_1, \tau_2\} - \frac{\max\{\tau_1, \tau_2\} \cdot \min\{\tau_1, \tau_2\}}{h}} \\ s_{\max\{\eta_1, \eta_2\} + \min\{\eta_1, \eta_2\} - \frac{\max\{\eta_1, \eta_2\} \cdot \min\{\eta_1, \eta_2\}}{h}} \end{matrix} \right], \left[ \begin{matrix} s_{\frac{\min\{\theta_1, \theta_2\} \cdot \max\{\theta_1, \theta_2\}}{h}} \\ s_{\frac{\min\{v_1, v_2\} \cdot \max\{v_1, v_2\}}{h}} \end{matrix} \right] \right) \\ &= \left( \left[ \begin{matrix} s_{\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{h}} \\ s_{\eta_1 + \eta_2 - \frac{\eta_1 \eta_2}{h}} \end{matrix} \right], \left[ \begin{matrix} s_{\frac{\theta_1 \theta_2}{h}} \\ s_{\frac{v_1 v_2}{h}} \end{matrix} \right] \right) \\ &= \gamma_1 \oplus \gamma_2 \end{aligned}$$

□

**Theorem 4.** Let  $\gamma_1 = ([s_{\tau_1}, s_{\eta_1}], [s_{\theta_1}, s_{v_1}])$ ,  $\gamma_2 = ([s_{\tau_2}, s_{\eta_2}], [s_{\theta_2}, s_{v_2}])$  and  $\gamma_3 = ([s_{\tau_3}, s_{\eta_3}], [s_{\theta_3}, s_{v_3}])$  be three LIVIFNs, then

1.  $(\gamma_1 \cup \gamma_2) \cap \gamma_3 = (\gamma_1 \cap \gamma_3) \cup (\gamma_2 \cap \gamma_3)$ ;
2.  $(\gamma_1 \cap \gamma_2) \cup \gamma_3 = (\gamma_1 \cup \gamma_3) \cap (\gamma_2 \cup \gamma_3)$ ;
3.  $(\gamma_1 \cup \gamma_2) \oplus \gamma_3 = (\gamma_1 \oplus \gamma_3) \cup (\gamma_2 \oplus \gamma_3)$ ;
4.  $(\gamma_1 \cap \gamma_2) \oplus \gamma_3 = (\gamma_1 \oplus \gamma_3) \cap (\gamma_2 \oplus \gamma_3)$ ;
5.  $(\gamma_1 \cup \gamma_2) \otimes \gamma_3 = (\gamma_1 \otimes \gamma_3) \cup (\gamma_2 \otimes \gamma_3)$ ;
6.  $(\gamma_1 \cap \gamma_2) \otimes \gamma_3 = (\gamma_1 \otimes \gamma_3) \cap (\gamma_2 \otimes \gamma_3)$ .

**Proof.** It is a trivial. □

**Theorem 5.** Let  $\gamma_1, \gamma_2$  and  $\gamma_3$  be three LIVIFNs, then:

1.  $\gamma_1 \cup \gamma_2 \cup \gamma_3 = \gamma_1 \cup \gamma_3 \cup \gamma_2$ ;
2.  $\gamma_1 \cap \gamma_2 \cap \gamma_3 = \gamma_1 \cap \gamma_3 \cap \gamma_2$ ;
3.  $\gamma_1 \oplus \gamma_2 \oplus \gamma_3 = \gamma_1 \oplus \gamma_3 \oplus \gamma_2$ ;
4.  $\gamma_1 \otimes \gamma_2 \otimes \gamma_3 = \gamma_1 \otimes \gamma_3 \otimes \gamma_2$ .

**Proof.** It is a trivial. □

### 3.2. Prioritized Averaging Aggregation Operators

In the next section, we define averaging and geometric AOs to aggregate the different LIVIF information over the continuous LTS  $S_{[0,h]}$ . For this, let  $\Omega$  be the set of all LIVIFNs.

**Definition 9.** For LIVIFNs  $\gamma_t (t = 1, 2, \dots, n)$  and an operator LIVIFPWA :  $\Omega^n \rightarrow \Omega$  defined as:

$$LIVIFPWA(\gamma_1, \gamma_2, \dots, \gamma_n) = \bigoplus_{t=1}^n \frac{T_t}{\sum_{j=1}^n T_j} \gamma_t, \tag{6}$$

where  $T_1 = 1$  and  $T_t = \prod_{k=1}^{t-1} \frac{I(S(\gamma_k))}{h}, t = 2, 3, \dots, n$ . Furthermore,  $I(S(\gamma_k))$  is the subscript of score function  $S(\gamma_k)$ .

**Theorem 6.** The aggregated value by using LIVIFPWA operator, as defined in Definition 9, for  $n$  LIVIFNs  $\gamma_t = ([s_{\tau_t}, s_{\eta_t}], [s_{\theta_t}, s_{\nu_t}]) (t = 1, 2, \dots, n)$  is again LIVIFN and given by:

$$LIVIFPWA(\gamma_1, \gamma_2, \dots, \gamma_n) = \left( \left[ \begin{matrix} s \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\tau_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]', \left[ \begin{matrix} s \\ h \left( \prod_{t=1}^n \left( \frac{\theta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]' \right), \left( \left[ \begin{matrix} s \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\eta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]', \left[ \begin{matrix} s \\ h \left( \prod_{t=1}^n \left( \frac{\nu_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]' \right) \tag{7}$$

where  $T_1 = 1$  and  $T_t = \prod_{k=1}^{t-1} \frac{I(S(\gamma_k))}{h}, t = 2, 3, \dots, n, I(S(\gamma_k))$  is the subscript of score function  $S(\gamma_k)$ .

**Proof.** To prove Equation (7), we use mathematical induction on  $n$ , and the following steps are executed.

Step 1: For  $n = 2$  and by Definition 8, we have:

$$\frac{T_1}{\sum_{j=1}^n T_j} \gamma_1 = \left( \left[ \begin{matrix} s \\ h \left( 1 - \left( 1 - \frac{\tau_1}{h} \right)^{\frac{T_1}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]', \left[ \begin{matrix} s \\ h \left( \left( \frac{\theta_1}{h} \right)^{\frac{T_1}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]' \right), \left( \left[ \begin{matrix} s \\ h \left( 1 - \left( 1 - \frac{\eta_1}{h} \right)^{\frac{T_1}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]', \left[ \begin{matrix} s \\ h \left( \left( \frac{\nu_1}{h} \right)^{\frac{T_1}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]' \right)$$

and

$$\frac{T_2}{\sum_{j=1}^n T_j} \gamma_2 = \left( \left[ \begin{matrix} s \\ h \left( 1 - \left( 1 - \frac{\tau_2}{h} \right)^{\frac{T_2}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]', \left[ \begin{matrix} s \\ h \left( \left( \frac{\theta_2}{h} \right)^{\frac{T_2}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]' \right), \left( \left[ \begin{matrix} s \\ h \left( 1 - \left( 1 - \frac{\eta_2}{h} \right)^{\frac{T_2}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]', \left[ \begin{matrix} s \\ h \left( \left( \frac{\nu_2}{h} \right)^{\frac{T_2}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right]' \right)$$

Therefore,

$$\begin{aligned} \text{LIVIFPWA}(\gamma_1, \gamma_2) &= \frac{T_1}{\sum_{j=1}^n T_j} \gamma_1 \oplus \frac{T_2}{\sum_{j=1}^n T_j} \gamma_2 \\ &= \left( \left[ \begin{array}{c} \left[ \begin{array}{c} s \\ h \left( 1 - \left( 1 - \frac{\tau_1}{h} \right)^{\frac{T_1}{\sum_{j=1}^n T_j}} \left( 1 - \frac{\tau_2}{h} \right)^{\frac{T_2}{\sum_{j=1}^n T_j}} \right) \end{array} \right] \right] \oplus \left[ \begin{array}{c} \left[ \begin{array}{c} s \\ h \left( \left( \frac{\theta_1}{h} \right)^{\frac{T_1}{\sum_{j=1}^n T_j}} \left( \frac{\theta_2}{h} \right)^{\frac{T_2}{\sum_{j=1}^n T_j}} \right) \end{array} \right] \right] \end{array} \right] \\ \left[ \begin{array}{c} s \\ h \left( 1 - \left( 1 - \frac{\eta_1}{h} \right)^{\frac{T_1}{\sum_{j=1}^n T_j}} \left( 1 - \frac{\eta_2}{h} \right)^{\frac{T_2}{\sum_{j=1}^n T_j}} \right) \end{array} \right] \oplus \left[ \begin{array}{c} s \\ h \left( \left( \frac{\nu_1}{h} \right)^{\frac{T_1}{\sum_{j=1}^n T_j}} \left( \frac{\nu_2}{h} \right)^{\frac{T_2}{\sum_{j=1}^n T_j}} \right) \end{array} \right] \end{array} \right) \end{aligned}$$

which is true for  $n = 2$ .

Step 2: Assume that Equation (7) is true for  $n = k$ , that is:

$$\text{LIVIFPWA}(\gamma_1, \gamma_2, \dots, \gamma_k) = \left( \left[ \begin{array}{c} \left[ \begin{array}{c} s \\ h \left( 1 - \prod_{t=1}^k \left( 1 - \frac{\tau_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^k T_j}} \right) \end{array} \right] \right] \oplus \left[ \begin{array}{c} \left[ \begin{array}{c} s \\ h \left( \prod_{t=1}^k \left( \frac{\theta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^k T_j}} \right) \end{array} \right] \right] \end{array} \right] \\ \left[ \begin{array}{c} s \\ h \left( 1 - \prod_{t=1}^k \left( 1 - \frac{\eta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^k T_j}} \right) \end{array} \right] \oplus \left[ \begin{array}{c} s \\ h \left( \prod_{t=1}^k \left( \frac{\nu_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^k T_j}} \right) \end{array} \right] \end{array} \right)$$

then for  $n = k + 1$ , we have:

$$\begin{aligned} &\text{LIVIFPWA}(\gamma_1, \gamma_2, \dots, \gamma_{k+1}) \\ &= \text{LIVIFPWA}(\gamma_1, \gamma_2, \dots, \gamma_k) \oplus \frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j} \gamma_{k+1} \\ &= \left( \left[ \begin{array}{c} \left[ \begin{array}{c} s \\ h \left( 1 - \prod_{t=1}^k \left( 1 - \frac{\tau_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^k T_j}} \right) \end{array} \right] \right] \oplus \left[ \begin{array}{c} \left[ \begin{array}{c} s \\ h \left( \prod_{t=1}^k \left( \frac{\theta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^k T_j}} \right) \end{array} \right] \right] \end{array} \right] \\ \left[ \begin{array}{c} s \\ h \left( 1 - \prod_{t=1}^k \left( 1 - \frac{\eta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^k T_j}} \right) \end{array} \right] \oplus \left[ \begin{array}{c} s \\ h \left( \prod_{t=1}^k \left( \frac{\nu_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^k T_j}} \right) \end{array} \right] \end{array} \right) \\ &\oplus \left( \left[ \begin{array}{c} \left[ \begin{array}{c} s \\ h \left( 1 - \left( 1 - \frac{\tau_{k+1}}{h} \right)^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}} \right) \end{array} \right] \right] \oplus \left[ \begin{array}{c} \left[ \begin{array}{c} s \\ h \left( \left( \frac{\theta_{k+1}}{h} \right)^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}} \right) \end{array} \right] \right] \end{array} \right] \\ \left[ \begin{array}{c} s \\ h \left( 1 - \left( 1 - \frac{\eta_{k+1}}{h} \right)^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}} \right) \end{array} \right] \oplus \left[ \begin{array}{c} s \\ h \left( \left( \frac{\nu_{k+1}}{h} \right)^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}} \right) \end{array} \right] \end{array} \right) \end{aligned}$$

$$\begin{aligned}
 &= \left( \left[ \begin{matrix} s \\ h \left( 1 - \prod_{t=1}^k \left( 1 - \frac{\tau_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \left( 1 - \frac{\tau_{k+1}}{h} \right)^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}} \right) \right], \left[ \begin{matrix} s \\ h \left( \prod_{t=1}^k \left( \frac{\theta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \left( \frac{\theta_{k+1}}{h} \right)^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}} \right) \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ h \left( 1 - \prod_{t=1}^k \left( 1 - \frac{\eta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \left( 1 - \frac{\eta_{k+1}}{h} \right)^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}} \right) \right], \left[ \begin{matrix} s \\ h \left( \prod_{t=1}^k \left( \frac{v_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \left( \frac{v_{k+1}}{h} \right)^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}} \right) \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ h \left( 1 - \prod_{t=1}^{k+1} \left( 1 - \frac{\tau_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \right], \left[ \begin{matrix} s \\ h \left( \prod_{t=1}^{k+1} \left( \frac{\theta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ h \left( 1 - \prod_{t=1}^{k+1} \left( 1 - \frac{\eta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \right], \left[ \begin{matrix} s \\ h \left( \prod_{t=1}^{k+1} \left( \frac{v_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \right] \right)
 \end{aligned}$$

Thus, we get that Equation (7) holds for  $n = k + 1$  also, which completes the proof of the theorem.

□

**Example 1.** Let  $S_{[0,8]} = \{s_z \mid s_0 \leq s_z \leq s_8\}$  be a continuous LTS, and let  $\gamma_1 = ([s_2, s_4], [s_1, s_2])$ ,  $\gamma_2 = ([s_2, s_3], [s_1, s_4])$  and  $\gamma_3 = ([s_3, s_5], [s_2, s_3])$  be three LIVIFNs, and based on these numbers, we obtain  $T_1 = 1.0000$ ,  $T_2 = 0.5938$  and  $T_3 = 0.2969$ . Hence, the aggregated value of these numbers is calculated as:

$$\begin{aligned}
 LIVIFPWA(\gamma_1, \gamma_2, \gamma_3) &= \left( \left[ \begin{matrix} s \\ 8 \left( 1 - \prod_{t=1}^3 \left( 1 - \frac{\tau_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^3 T_j}} \right) \right], \left[ \begin{matrix} s \\ 8 \left( \prod_{t=1}^3 \left( \frac{\theta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^3 T_j}} \right) \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ 8 \left( 1 - \prod_{t=1}^3 \left( 1 - \frac{\eta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^3 T_j}} \right) \right], \left[ \begin{matrix} s \\ 8 \left( \prod_{t=1}^3 \left( \frac{v_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^3 T_j}} \right) \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ 8 \left( 1 - \left( 1 - \frac{2}{8} \right)^{0.5289} \times \left( 1 - \frac{2}{8} \right)^{0.3141} \times \left( 1 - \frac{3}{8} \right)^{0.1570} \right) \right], \left[ \begin{matrix} s \\ 8 \left( \left( \frac{1}{8} \right)^{0.5289} \times \left( \frac{1}{8} \right)^{0.3141} \times \left( \frac{2}{8} \right)^{0.1570} \right) \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ 8 \left( 1 - \left( 1 - \frac{4}{8} \right)^{0.5289} \times \left( 1 - \frac{3}{8} \right)^{0.3141} \times \left( 1 - \frac{5}{8} \right)^{0.1570} \right) \right], \left[ \begin{matrix} s \\ 8 \left( \left( \frac{2}{8} \right)^{0.5289} \times \left( \frac{4}{8} \right)^{0.3141} \times \left( \frac{3}{8} \right)^{0.1570} \right) \right] \right) \\
 &= \left( \left[ \begin{matrix} s_{2.1693} \\ s_{3.8991} \end{matrix} \right], \left[ \begin{matrix} s_{1.1150} \\ s_{2.6498} \end{matrix} \right] \right)
 \end{aligned}$$

It is observed that the proposed LIVIFPWA operator satisfies the certain properties, which are stated as follows.

**Theorem 7. (Idempotency)** Let  $\gamma_t (t = 1, 2, \dots, n)$  be a collection of LIVIFNs. If  $\gamma$  is another LIVIFN such that  $\gamma_i = \gamma$  for all  $t$ , then:

$$LIVIFPWA(\gamma_1, \gamma_2, \dots, \gamma_n) = \gamma$$

**Proof.** As  $\gamma_t = \gamma$  for all  $t$  and by the definition of the LIVFWA operator, we have:

$$\begin{aligned} \text{LIVIFPWA}(\gamma_1, \gamma_2, \dots, \gamma_n) &= \frac{T_1}{\sum_{j=1}^n T_j} \gamma_1 \oplus \frac{T_2}{\sum_{j=1}^n T_j} \gamma_2 \oplus \dots \oplus \frac{T_n}{\sum_{j=1}^n T_j} \gamma_n \\ &= \frac{T_1}{\sum_{j=1}^n T_j} \gamma \oplus \frac{T_2}{\sum_{j=1}^n T_j} \gamma \oplus \dots \oplus \frac{T_n}{\sum_{j=1}^n T_j} \gamma \\ &= \sum_{t=1}^n \frac{T_t}{\sum_{j=1}^n T_j} \gamma \\ &= \gamma. \end{aligned}$$

□

**Theorem 8.** (Boundedness) Let  $\gamma^- = ([s_{\tau^-}, s_{\eta^-}], [s_{\theta^-}, s_{v^-}])$  and  $\gamma^+ = ([s_{\tau^+}, s_{\eta^+}], [s_{\theta^+}, s_{v^+}])$ , where  $\tau^- = \min_t \{\tau_t\}$ ,  $\eta^- = \min_t \{\eta_t\}$ ,  $\theta^- = \max_t \{\theta_t\}$ ,  $v^- = \max_t \{v_t\}$ ,  $\tau^+ = \max_t \{\tau_t\}$ ,  $\eta^+ = \max_t \{\eta_t\}$ ,  $\theta^+ = \min_t \{\theta_t\}$ ,  $v^+ = \min_t \{v_t\}$ , then we have:

$$\gamma^- \preceq \text{LIVIFPWA}(\gamma_1, \gamma_2, \dots, \gamma_n) \preceq \gamma^+.$$

**Proof.** For the membership part of the LIVIFPWA operator on a collection of LIVIFNs  $\gamma_t = ([s_{\tau_t}, s_{\eta_t}], [s_{\theta_t}, s_{v_t}])$ ,  $t = 1, 2, \dots, n$ , we have:

$$\begin{aligned} \tau^- = \min_t \{\tau_t\} &\leq \tau_t \leq \max_t \{\tau_t\} = \tau^+ \\ \Rightarrow 1 - \frac{\tau^-}{h} &\geq 1 - \frac{\tau_t}{h} \geq 1 - \frac{\tau^+}{h} \\ \Rightarrow \prod_{t=1}^n \left(1 - \frac{\tau^-}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}} &\geq \prod_{t=1}^n \left(1 - \frac{\tau_t}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \geq \prod_{t=1}^n \left(1 - \frac{\tau^+}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \\ \Rightarrow h \left(1 - \left(1 - \frac{\tau^-}{h}\right)^{\sum_{t=1}^n \frac{T_t}{\sum_{j=1}^n T_j}}\right) &\leq h \left(1 - \prod_{t=1}^n \left(1 - \frac{\tau_t}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}}\right) \leq h \left(1 - \left(1 - \frac{\tau^+}{h}\right)^{\sum_{t=1}^n \frac{T_t}{\sum_{j=1}^n T_j}}\right) \\ \Rightarrow \tau^- &\leq h \left(1 - \prod_{t=1}^n \left(1 - \frac{\tau_t}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}}\right) \leq \tau^+. \end{aligned}$$

which implies that:

$$s_{\tau^-} \leq s \left( h \left(1 - \prod_{t=1}^n \left(1 - \frac{\tau_t}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}}\right) \right) \leq s_{\tau^+} \tag{8}$$

Similarly for the upper limit of the membership part, we have:

$$s_{\eta^-} \leq s \left( h \left(1 - \prod_{t=1}^n \left(1 - \frac{\eta_t}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}}\right) \right) \leq s_{\eta^+}$$

On the other hand, for the non-membership part of the LIVIFPWA operator, we have:

$$\begin{aligned} \theta^+ &= \min_t \{\theta_t\} \leq \theta_t \leq \max_t \{\theta_t\} = \theta^- \\ \Rightarrow \frac{\theta^+}{h} &\leq \frac{\theta_t}{h} \leq \frac{\theta^-}{h} \\ \Rightarrow \prod_{t=1}^n \left(\frac{\theta^+}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}} &\leq \prod_{t=1}^n \left(\frac{\theta_t}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \leq \prod_{t=1}^n \left(\frac{\theta^-}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \\ \Rightarrow \left(\frac{\theta^+}{h}\right)^{\sum_{t=1}^n \frac{T_t}{\sum_{j=1}^n T_j}} &\leq \prod_{t=1}^n \left(\frac{\theta_t}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \leq \left(\frac{\theta^-}{h}\right)^{\sum_{t=1}^n \frac{T_t}{\sum_{j=1}^n T_j}} \\ \Rightarrow \theta^+ &\leq h \left( \prod_{t=1}^n \left(\frac{\theta_t}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \leq \theta^- \end{aligned}$$

which implies that

$$s_{\theta^+} \leq s \left( \prod_{t=1}^n \left(\frac{\theta_t}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \leq s_{\theta^-}$$

Similarly, we can obtain:

$$s_{\nu^+} \leq s \left( \prod_{t=1}^n \left(\frac{\nu_t}{h}\right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \leq s_{\nu^-}$$

Hence, according to Definition 6, we obtain:

$$\gamma^- \preceq \text{LIVIFPWA}(\gamma_1, \gamma_2, \dots, \gamma_n) \preceq \gamma^+.$$

□

**Remark 1.** From the proposed LIVIFPWA operator, it is observed that it does not satisfy the monotonicity property, i.e., there exist some collections of LIVIFNs  $\gamma_t$  and  $\beta_t$  where  $t = 1, 2, \dots, n$ , which satisfy the relation  $\gamma_t \leq \beta_t$  for all  $t$ , but  $\text{LIVIFPWA}(\gamma_1, \gamma_2, \dots, \gamma_n) \not\preceq \text{LIVIFPWA}(\beta_1, \beta_2, \dots, \beta_n)$ .

Now, we extend the idea of LIVIFPWA into the LIVIFPOWA operator as follows.

**Definition 10.** For LIVIFNs  $\gamma_t (t = 1, 2, \dots, n)$ , a mapping LIVIFPOWA :  $\Omega^n \rightarrow \Omega$  given by:

$$\text{LIVIFPOWA}(\gamma_1, \gamma_2, \dots, \gamma_n) = \bigoplus_{t=1}^n \frac{T_t}{\sum_{j=1}^n T_j} \gamma_{\sigma(t)}, \tag{9}$$

where  $\gamma_{\sigma(t)} = ([s_{\tau_{\sigma(t)}}, s_{\eta_{\sigma(t)}}], [s_{\theta_{\sigma(t)}}, s_{\nu_{\sigma(t)}}])$  is the  $t^{\text{th}}$  largest value of the  $\gamma_t$ ,  $T_1 = 1$  and  $T_t = \prod_{k=1}^{t-1} \frac{I(s(\gamma_{\sigma(k)}))}{h}$ .

**Theorem 9.** The aggregated value for different LIVIFNs  $\gamma_t = ([s_{\tau_t}, s_{\eta_t}], [s_{\theta_t}, s_{v_t}])$  ( $t = 1, 2, \dots, n$ ) by using the LIVIFPOWA operator is also a LIVIFN and given by:

$$LIVIFPOWA(\gamma_1, \gamma_2, \dots, \gamma_n) = \left( \left[ \begin{array}{c} s \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\tau_{\sigma(t)}}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{array} \right]', \left[ \begin{array}{c} s \\ h \left( \prod_{t=1}^n \left( \frac{\theta_{\sigma(t)}}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{array} \right]' \right)$$

$$\left( \left[ \begin{array}{c} s \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\eta_{\sigma(t)}}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{array} \right]', \left[ \begin{array}{c} s \\ h \left( \prod_{t=1}^n \left( \frac{v_{\sigma(t)}}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{array} \right]' \right)$$

where  $T_1 = 1, T_t = \prod_{k=1}^{t-1} \frac{I(S(\gamma_{\sigma(k)}))}{h}, t = 2, 3, \dots, n, S(\gamma_{\sigma(k)})$  represents the score function of the LIVIFN  $\gamma_{\sigma(k)}$  and  $I(S(\gamma_{\sigma(k)}))$  is the subscript of  $S(\gamma_{\sigma(k)})$ .

The proof follows from Theorem 6.

Further, the LIVIFPOWA operator satisfies the properties of idempotency and boundedness.

### 3.3. Prioritized Geometric Aggregation Operators

In the following, motivated by the geometric AOs [10], we present some LIVIF prioritized weighted geometric (LIVIFPWG) AOs as follows.

**Definition 11.** Let  $\gamma_t = ([s_{\tau_t}, s_{\eta_t}], [s_{\theta_t}, s_{v_t}]), t = 1, 2, \dots, n$ , be a collection of LIVIFNs, then LIVIFPWG :  $\Omega^n \rightarrow \Omega$  is defined as follows:

$$LIVIFPWG(\gamma_1, \gamma_2, \dots, \gamma_n) = \bigotimes_{t=1}^n \gamma_t^{\frac{T_t}{\sum_{j=1}^n T_j}}, \tag{10}$$

where  $T_1 = 1, T_t = \prod_{k=1}^{t-1} \frac{I(S(\gamma_k))}{h}, t = 2, 3, \dots, n$  and  $I(S(\gamma_k))$  is the subscript of score function  $S(\gamma_k)$ .

**Theorem 10.** The aggregated value of  $n$  LIVIFNs  $\gamma_t = ([s_{\tau_t}, s_{\eta_t}], [s_{\theta_t}, s_{v_t}])$  by Definition 11 is still LIVIFN and given by:

$$LIVIFPWG(\gamma_1, \gamma_2, \dots, \gamma_n) = \left( \left[ \begin{array}{c} s \\ h \left( \prod_{t=1}^n \left( \frac{\tau_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{array} \right]', \left[ \begin{array}{c} s \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\theta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{array} \right]' \right)$$

$$\left( \left[ \begin{array}{c} s \\ h \left( \prod_{t=1}^n \left( \frac{\eta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{array} \right]', \left[ \begin{array}{c} s \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{v_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{array} \right]' \right)$$
(11)

**Proof.** This is similar to Theorem 6, so we omit it here. □

The following properties are satisfied by LIVIFPWG for a collection of LIVIFNs  $\gamma_t, t = 1, 2, \dots, n$ .

**Theorem 11. (Idempotency)** If all  $\gamma_t$  are equal, i.e.  $\gamma_t = \gamma = ([s_{\tau}, s_{\eta}], [s_{\theta}, s_{v}]),$  for all  $t$ , then:

$$LIVIFPWG(\gamma_1, \gamma_2, \dots, \gamma_n) = \gamma.$$

**Proof.**

$$\begin{aligned}
 \text{LIVIFPWG}(\gamma_1, \gamma_2, \dots, \gamma_n) &= \left( \left[ \begin{matrix} s \\ h \left( \prod_{t=1}^n \left( \frac{\tau_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right], \left[ \begin{matrix} s \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\theta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ h \left( \prod_{t=1}^n \left( \frac{\eta_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right], \left[ \begin{matrix} s \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{v_t}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ h \left( \prod_{t=1}^n \left( \frac{\tau}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right], \left[ \begin{matrix} s \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\theta}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ h \left( \prod_{t=1}^n \left( \frac{\eta}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right], \left[ \begin{matrix} s \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{v}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ h \left( \left( \frac{\tau}{h} \right)^{\frac{\sum_{t=1}^n T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right], \left[ \begin{matrix} s \\ h \left( 1 - \left( 1 - \frac{\theta}{h} \right)^{\frac{\sum_{t=1}^n T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right] \right) \\
 &= \left( \left[ \begin{matrix} s \\ h \left( \left( \frac{\eta}{h} \right)^{\frac{\sum_{t=1}^n T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right], \left[ \begin{matrix} s \\ h \left( 1 - \left( 1 - \frac{v}{h} \right)^{\frac{\sum_{t=1}^n T_t}{\sum_{j=1}^n T_j}} \right) \end{matrix} \right] \right) \\
 &= ([s_\tau, s_\eta], [s_\theta, s_v])
 \end{aligned}$$

□

**Theorem 12.** Let  $\gamma^- = ([s_{\tau^-}, s_{\eta^-}], [s_{\theta^-}, s_{v^-}])$  and  $\gamma^+ = ([s_{\tau^+}, s_{\eta^+}], [s_{\theta^+}, s_{v^+}])$ , where  $\tau^- = \min_t \{\tau_t\}$ ,  $\eta^- = \min_t \{\eta_t\}$ ,  $\theta^- = \max_t \{\theta_t\}$ ,  $v^- = \max_t \{v_t\}$ ,  $\tau^+ = \max_t \{\tau_t\}$ ,  $\eta^+ = \max_t \{\eta_t\}$ ,  $\theta^+ = \min_t \{\theta_t\}$ ,  $v^+ = \min_t \{v_t\}$ , then we have:

$$\gamma^- \preceq \text{LIVIFPWG}(\gamma_1, \gamma_2, \dots, \gamma_n) \preceq \gamma^+.$$

**Proof.** For each  $t$  and by the definition of  $\gamma^-$ ,  $\gamma_t$  and  $\gamma^+$ , we can easily see that  $\gamma^- \preceq \gamma_t \preceq \gamma^+$ . Thus, from Equation (10), we get:

$$\text{LIVIFPWG}(\gamma_1, \gamma_2, \dots, \gamma_n) = \bigotimes_{t=1}^n \gamma_t^{\frac{T_t}{\sum_{j=1}^n T_j}} \preceq \bigotimes_{t=1}^n (\gamma^+)^{\frac{T_t}{\sum_{j=1}^n T_j}} = (\gamma^+)^{\frac{\sum_{t=1}^n T_t}{\sum_{j=1}^n T_j}} = \gamma^+$$

and:

$$\text{LIVIFPWG}(\gamma_1, \gamma_2, \dots, \gamma_n) = \bigotimes_{t=1}^n \gamma_t^{\frac{T_t}{\sum_{j=1}^n T_j}} \succeq \bigotimes_{t=1}^n (\gamma^-)^{\frac{T_t}{\sum_{j=1}^n T_j}} = (\gamma^-)^{\frac{\sum_{t=1}^n T_t}{\sum_{j=1}^n T_j}} = \gamma^-$$

Hence,

$$\gamma^- \preceq \text{LIVIFPWG}(\gamma_1, \gamma_2, \dots, \gamma_n) \preceq \gamma^+.$$

□

**Definition 12.** Let  $\gamma_t = ([s_{\tau_t}, s_{\eta_t}], [s_{\theta_t}, s_{\nu_t}])$ ,  $t = 1, 2, \dots, n$ , be a collection of LIVIFNs, then the LIVIFPOWG :  $\Omega^n \rightarrow \Omega$  operator is defined as follows:

$$\text{LIVIFPOWG}(\gamma_1, \gamma_2, \dots, \gamma_n) = \bigotimes_{t=1}^n \gamma_{\sigma(t)}^{\frac{T_t}{\sum_{j=1}^n T_j}}, \tag{12}$$

where  $\gamma_{\sigma(t)} = ([s_{\tau_{\sigma(t)}}, s_{\eta_{\sigma(t)}}], [s_{\theta_{\sigma(t)}}, s_{\nu_{\sigma(t)}}])$  is the  $t^{\text{th}}$  largest value of the  $\gamma_t$ ,  $t = 1, 2, \dots, n$ .  $T_1 = 1$  and  $T_t = \prod_{k=1}^{t-1} \frac{I(S(\gamma_{\sigma(k)}))}{h}$ ,  $t = 2, 3, \dots, n$ .

**Theorem 13.** The value for LIVIFNs  $\gamma_t = ([s_{\tau_t}, s_{\eta_t}], [s_{\theta_t}, s_{\nu_t}])$ , ( $t = 1, 2, \dots, n$ ) by using the LIVIFPOWG operator is again LIVIFN and given by:

$$\text{LIVIFPOWG}(\gamma_1, \gamma_2, \dots, \gamma_n) = \left( \left[ \begin{array}{c} s \left( \prod_{t=1}^n \left( \frac{\tau_{\sigma(t)}}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \\ h \left( \prod_{t=1}^n \left( \frac{\eta_{\sigma(t)}}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{array} \right], \left[ \begin{array}{c} s \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\theta_{\sigma(t)}}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \\ h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\nu_{\sigma(t)}}{h} \right)^{\frac{T_t}{\sum_{j=1}^n T_j}} \right) \end{array} \right] \right)$$

where  $T_t = \prod_{k=1}^{t-1} \frac{I(S(\gamma_{\sigma(k)}))}{h}$ ,  $t = 2, 3, \dots, n$ ,  $T_1 = 1$ .

**Proof.** This is similar to the above. □

#### 4. Proposed MAGDM Approach under the LIVIF Environment

In this section, we present an approach for solving MAGDM problems using the proposed operators.

Consider an MAGDM problem consisting of  $A_1, A_2, \dots, A_m$  alternatives and  $G_1, G_2, \dots, G_n$  attributes with their own prioritized relation as  $G_1 \succ G_2 \succ \dots \succ G_n$ . Here,  $\succ$  refers to “preferred to”. To evaluate them, a set of  $l$  decision-makers  $D^{(1)}, D^{(2)}, \dots, D^{(l)}$  is taken with their own prioritized relation as  $D^{(1)} \succ D^{(2)} \succ \dots \succ D^{(l)}$ . Each decision-maker has to evaluate the given alternatives under the LIVIFS environment and represented as  $\tilde{\gamma}_{kt}^{(q)} = ([\tilde{s}_{\tau_{kt}^{(q)}}, \tilde{s}_{\eta_{kt}^{(q)}}], [\tilde{s}_{\theta_{kt}^{(q)}}, \tilde{s}_{\nu_{kt}^{(q)}}])$  ( $q = 1, 2, \dots, l; k = 1, 2, \dots, m; t = 1, 2, \dots, n$ ). Then, the presented approach to get the best alternative(s) is summarized in the following steps.

Step 1: Construct the decision matrices  $\tilde{R}^{(q)} = (\tilde{\gamma}_{kt}^{(q)})_{m \times n}$  for each decision-maker as:

$$\tilde{R}^{(q)} = \begin{matrix} & G_1 & G_2 & \dots & G_n \\ A_1 & \tilde{\gamma}_{11}^{(q)} & \tilde{\gamma}_{12}^{(q)} & \dots & \tilde{\gamma}_{1n}^{(q)} \\ A_2 & \tilde{\gamma}_{21}^{(q)} & \tilde{\gamma}_{22}^{(q)} & \dots & \tilde{\gamma}_{2n}^{(q)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & \tilde{\gamma}_{m1}^{(q)} & \tilde{\gamma}_{m2}^{(q)} & \dots & \tilde{\gamma}_{mn}^{(q)} \end{matrix} \quad (13)$$

Step 2: Normalize  $\tilde{R}^{(q)}$ , if required, into  $R^{(q)} = (\gamma_{kt}^{(q)})_{m \times n}$  where  $\gamma_{kt}^{(q)}$  is given as:

$$\gamma_{kt}^{(q)} = \begin{cases} \left( \left[ \begin{matrix} \tilde{s}_{\tau_{kt}}^{(q)}, \tilde{s}_{\eta_{kt}}^{(q)} \end{matrix} \right], \left[ \begin{matrix} \tilde{s}_{\theta_{kt}}^{(q)}, \tilde{s}_{\nu_{kt}}^{(q)} \end{matrix} \right] \right) & ; \text{if } G_t \text{ is the benefit type attribute} \\ \left( \left[ \begin{matrix} \tilde{s}_{\theta_{kt}}^{(q)}, \tilde{s}_{\nu_{kt}}^{(q)} \end{matrix} \right], \left[ \begin{matrix} \tilde{s}_{\tau_{kt}}^{(q)}, \tilde{s}_{\eta_{kt}}^{(q)} \end{matrix} \right] \right) & ; \text{if } G_t \text{ is the cost type attribute} \end{cases} \quad (14)$$

Step 3: Calculate  $T_{kt}^{(q)} (q = 1, 2, \dots, l)$  as follows:

$$T_{kt}^{(1)} = 1, \quad (15)$$

$$T_{kt}^{(q)} = \prod_{p=1}^{q-1} \frac{I(S(\gamma_{kt}^{(p)}))}{h}, \quad q = 2, 3, \dots, l, \quad (16)$$

Step 4: Aggregate each expert preference into the collective one by using either the LIVIFPOWA or the LIVIFPOWG operator, and get decision matrix  $R = (\gamma_{kt})_{m \times n}, k = 1, 2, \dots, m; t = 1, 2, \dots, n$ , where  $\gamma_{kt} = ([s_{\tau_{kt}}, s_{\eta_{kt}}], [s_{\theta_{kt}}, s_{\nu_{kt}}])$ . For instance, by utilizing the LIVIFPOWA operator, we have:

$$\gamma_{kt} = \text{LIVIFPOWA} (\gamma_{kt}^{(1)}, \gamma_{kt}^{(2)}, \dots, \gamma_{kt}^{(l)}) \quad (17)$$

$$= \left( \left[ \begin{matrix} s \\ h \left( 1 - \prod_{q=1}^l \left( 1 - \frac{\tau_{kt}^{\sigma(q)}}{h} \right)^{w_{kt}^{(q)}} \right) \end{matrix} \right], \left[ \begin{matrix} s \\ h \left( \prod_{q=1}^l \left( \frac{\theta_{kt}^{\sigma(q)}}{h} \right)^{w_{kt}^{(q)}} \right) \end{matrix} \right] \right), \left( \left[ \begin{matrix} s \\ h \left( 1 - \prod_{q=1}^l \left( 1 - \frac{\eta_{kt}^{\sigma(q)}}{h} \right)^{w_{kt}^{(q)}} \right) \right], \left[ \begin{matrix} s \\ h \left( \prod_{q=1}^l \left( \frac{\nu_{kt}^{\sigma(q)}}{h} \right)^{w_{kt}^{(q)}} \right) \end{matrix} \right] \right) \quad (18)$$

while by the LIVIFPOWG operator, we have:

$$\gamma_{kt} = \text{LIVIFPOWG} (\gamma_{kt}^{(1)}, \gamma_{kt}^{(2)}, \dots, \gamma_{kt}^{(l)}) \quad (19)$$

$$= \left( \left[ \begin{matrix} s \\ h \left( \prod_{q=1}^l \left( \frac{\tau_{kt}^{\sigma(q)}}{h} \right)^{w_{kt}^{(q)}} \right) \right], \left[ \begin{matrix} s \\ h \left( 1 - \prod_{q=1}^l \left( 1 - \frac{\theta_{kt}^{\sigma(q)}}{h} \right)^{w_{kt}^{(q)}} \right) \end{matrix} \right] \right), \left( \left[ \begin{matrix} s \\ h \left( \prod_{q=1}^l \left( \frac{\eta_{kt}^{\sigma(q)}}{h} \right)^{w_{kt}^{(q)}} \right) \right], \left[ \begin{matrix} s \\ h \left( 1 - \prod_{q=1}^l \left( 1 - \frac{\nu_{kt}^{\sigma(q)}}{h} \right)^{w_{kt}^{(q)}} \right) \end{matrix} \right] \right) \quad (20)$$

where  $\gamma_{kt}^{\sigma(q)} = ([s_{\tau_{kt}}^{\sigma(q)}, s_{\eta_{kt}}^{\sigma(q)}], [s_{\theta_{kt}}^{\sigma(q)}, s_{\nu_{kt}}^{\sigma(q)}])$  and  $w_{kt}^{(q)} = \frac{T_{kt}^{(q)}}{\sum_{q=1}^l T_{kt}^{(q)}}$ .

Step 5: Calculate the value of  $T_{kt}$  as follows:

$$T_{k1} = 1; \quad k = 1, 2, \dots, m \tag{21}$$

$$T_{kt} = \prod_{v=1}^{t-1} \frac{I(S(\gamma_{kv}))}{h}; \quad t = 2, 3, \dots, n, \tag{22}$$

Step 6: Utilize either the LIVIFPWA or the LIVIFPWG operator to obtain the overall values  $\gamma_k$  of each alternative  $A_k (k = 1, 2, \dots, m)$  as:

$$\gamma_k = ([s_{\tau_k}, s_{\eta_k}], [s_{\theta_k}, s_{v_k}]) \tag{23}$$

$$= \text{LIVIFPWA}(\gamma_{k1}, \gamma_{k2}, \dots, \gamma_{kn}) \tag{24}$$

$$= \left( \left[ \begin{matrix} s_h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\tau_{kt}}{h} \right)^{\omega_{kt}} \right)' \\ s_h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\eta_{kt}}{h} \right)^{\omega_{kt}} \right) \end{matrix} \right], \left[ \begin{matrix} s_h \left( \prod_{t=1}^n \left( \frac{\theta_{kt}}{h} \right)^{\omega_{kt}} \right)' \\ s_h \left( \prod_{t=1}^n \left( \frac{v_{kt}}{h} \right)^{\omega_{kt}} \right) \end{matrix} \right] \right) \tag{25}$$

or:

$$\gamma_k = ([s_{\tau_k}, s_{\eta_k}], [s_{\theta_k}, s_{v_k}]) \tag{26}$$

$$= \text{LIVIFPWG}(\gamma_{k1}, \gamma_{k2}, \dots, \gamma_{kn}) \tag{27}$$

$$= \left( \left[ \begin{matrix} s_h \left( \prod_{t=1}^n \left( \frac{\tau_{kt}}{h} \right)^{\omega_{kt}} \right)' \\ s_h \left( \prod_{t=1}^n \left( \frac{\eta_{kt}}{h} \right)^{\omega_{kt}} \right) \end{matrix} \right], \left[ \begin{matrix} s_h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{\theta_{kt}}{h} \right)^{\omega_{kt}} \right)' \\ s_h \left( 1 - \prod_{t=1}^n \left( 1 - \frac{v_{kt}}{h} \right)^{\omega_{kt}} \right) \end{matrix} \right] \right) \tag{28}$$

where  $\omega_{kt} = \frac{T_{kt}}{\sum_{t=1}^n T_{kt}}$ .

Step 7: Compute the score values of  $\gamma_k$  as:

$$S(\gamma_k) = s_{(2h + \tau_k - \theta_k + \eta_k - v_k)/4} \tag{29}$$

If there is no difference between two score values  $S(\gamma_{k_1})$  and  $S(\gamma_{k_2})$  for any two positive  $k_1, k_2$ , then compute the accuracy value of the alternative as:

$$H(\gamma_k) = s_{(\tau_k + \theta_k + \eta_k + v_k)/2} \tag{30}$$

Step 8: Rank all the given alternatives  $A_k (k = 1, 2, \dots, m)$  according to Definition 7 and, hence, select the desirable alternative(s).

### 5. Illustrative Example

In order to demonstrate the above-mentioned approach, an illustrative example has been taken as below under the LIVIFS environment.

#### 5.1. A Case Study

Jharkhand is the eastern state of the India, which has the 40 percent mineral resources of the country and second leading state of the mineral wealth after Chhatisgarh state. It is also known for its vast forest resources. Jamshedpur, Bokaro and Dhanbad cities of the Jharkhand are famous for industries in all over the world. After that, it is the widespread poverty state of the India because it is the primarily a rural state as 76 percent of the population live in the villages which depend on the agriculture and wages. Only 30 percent villages are connected by roads while only 55 percent villages have access to electricity and other facilities. But in the today's life, every one is changing

fast to himself for a better life, therefore, every one moves to the urban cities for a better job. To stop this emigration, Jharkhand government wants to setup the industries based on the agriculture in the rural areas. For this, government have been organized “MOMENTUM JHARKHAND” global investor submit 2017 in Ranchi to invite the companies for investment in the rural areas. The government announced the various facilities for setup the food processing parks in the rural areas and consider the four attributes required for company selection to setup them, namely, project quality ( $G_1$ ), technical capability ( $G_2$ ), company background ( $G_3$ ), reference from previous project ( $G_4$ ) and assign the weights of relative importance of each attributes. The four companies/candidates taken as in the form of the alternatives, namely, Surya Food and Agro Pvt. Ltd. ( $A_1$ ), Mother Dairy Fruit and Vegetable Pvt. Ltd. ( $A_2$ ), Parle Products Ltd. ( $A_3$ ), Heritage Food Ltd. ( $A_4$ ) interested for these projects. Then the main objective of the government is to choose the best company among them for the required task. In order to fulfill this, three senior experts  $D^{(1)}$ ,  $D^{(2)}$  and  $D^{(3)}$  are invited to give their preferences on each attribute in terms of LIVIFNs according to the linguistic term set  $S = \{s_0 = \text{“extremely poor”}, s_1 = \text{“very poor”}, s_2 = \text{“poor”}, s_3 = \text{“slightly poor”}, s_4 = \text{“fair”}, s_5 = \text{“slightly good”}, s_6 = \text{“good”}, s_7 = \text{“very good”}, s_8 = \text{“extremely good”}\}$ . Then, the steps of the proposed approach are executed as follows to find the best alternative(s).

Step 1: Rating values of the three decision-makers are noted in the form of a decision matrix summarized in Tables 1–3, respectively.

**Table 1.** Linguistic interval-valued intuitionistic fuzzy decision matrix  $\tilde{R}^{(1)}$  of decision-maker  $D^{(1)}$ .

	$G_1$	$G_2$	$G_3$	$G_4$
$A_1$	$([s_3, s_5], [s_2, s_3])$	$([s_4, s_5], [s_1, s_2])$	$([s_4, s_5], [s_2, s_3])$	$([s_3, s_4], [s_1, s_2])$
$A_2$	$([s_3, s_5], [s_2, s_3])$	$([s_2, s_4], [s_1, s_2])$	$([s_2, s_4], [s_3, s_4])$	$([s_1, s_3], [s_2, s_3])$
$A_3$	$([s_4, s_6], [s_1, s_2])$	$([s_5, s_6], [s_1, s_1])$	$([s_3, s_4], [s_2, s_3])$	$([s_4, s_5], [s_1, s_3])$
$A_4$	$([s_4, s_5], [s_2, s_3])$	$([s_1, s_3], [s_3, s_4])$	$([s_3, s_5], [s_1, s_3])$	$([s_6, s_7], [s_1, s_1])$

**Table 2.** Linguistic interval-valued intuitionistic fuzzy decision matrix  $\tilde{R}^{(2)}$  of decision-maker  $D^{(2)}$ .

	$G_1$	$G_2$	$G_3$	$G_4$
$A_1$	$([s_2, s_4], [s_1, s_3])$	$([s_4, s_5], [s_1, s_2])$	$([s_4, s_5], [s_1, s_3])$	$([s_3, s_6], [s_1, s_2])$
$A_2$	$([s_3, s_5], [s_1, s_3])$	$([s_1, s_2], [s_1, s_4])$	$([s_2, s_3], [s_3, s_4])$	$([s_3, s_5], [s_1, s_3])$
$A_3$	$([s_3, s_4], [s_2, s_3])$	$([s_5, s_6], [s_1, s_2])$	$([s_3, s_5], [s_2, s_3])$	$([s_3, s_4], [s_2, s_3])$
$A_4$	$([s_4, s_5], [s_1, s_2])$	$([s_1, s_2], [s_3, s_5])$	$([s_3, s_3], [s_2, s_3])$	$([s_2, s_3], [s_1, s_2])$

**Table 3.** Linguistic interval-valued intuitionistic fuzzy decision matrix  $\tilde{R}^{(3)}$  of decision-maker  $D^{(3)}$ .

	$G_1$	$G_2$	$G_3$	$G_4$
$A_1$	$([s_2, s_4], [s_1, s_2])$	$([s_2, s_3], [s_2, s_4])$	$([s_3, s_5], [s_2, s_3])$	$([s_5, s_6], [s_1, s_2])$
$A_2$	$([s_1, s_4], [s_2, s_3])$	$([s_4, s_5], [s_1, s_2])$	$([s_2, s_4], [s_1, s_3])$	$([s_3, s_4], [s_2, s_4])$
$A_3$	$([s_2, s_3], [s_1, s_3])$	$([s_3, s_5], [s_2, s_3])$	$([s_3, s_5], [s_1, s_3])$	$([s_3, s_5], [s_2, s_3])$
$A_4$	$([s_3, s_4], [s_2, s_3])$	$([s_1, s_2], [s_3, s_4])$	$([s_3, s_5], [s_1, s_2])$	$([s_5, s_6], [s_1, s_2])$

Step 2: Each attribute is of the profit type, so there is no need for normalization.

Step 3: Calculate the  $T_{kt}^{(q)}, q = 1, 2, 3$ , and obtain:

$$T_{kt}^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad T_{kt}^{(2)} = \begin{bmatrix} 0.5938 & 0.6875 & 0.6250 & 0.6250 \\ 0.5938 & 0.5938 & 0.4688 & 0.4688 \\ 0.7188 & 0.7813 & 0.5625 & 0.6563 \\ 0.6250 & 0.4063 & 0.6250 & 0.8438 \end{bmatrix}$$

$$T_{kt}^{(3)} = \begin{bmatrix} 0.3340 & 0.4727 & 0.4102 & 0.4297 \\ 0.3711 & 0.2598 & 0.2051 & 0.2930 \\ 0.4043 & 0.5859 & 0.3340 & 0.3691 \\ 0.4297 & 0.1396 & 0.3320 & 0.4746 \end{bmatrix}$$

Step 4: W.l.o.g., we utilize the LIVIFPOWA operator, i.e., Equation (17), to fuse the information of each expert. Their values are represented in Table 4.

**Table 4.** Aggregated linguistic interval-valued intuitionistic fuzzy decision matrix  $R$  by using the LIVIFPOWA operator.

	$G_1$	$G_2$	$G_3$	$G_4$
$A_1$	$([s_{2.5415}, s_{4.5545}], [s_{1.4327}, s_{2.6478}])$	$([s_{3.6289}, s_{4.6452}], [s_{1.1638}, s_{2.3275}])$	$([s_{3.8160}, s_{5.0000}], [s_{1.4227}, s_{3.0000}])$	$([s_{4.1006}, s_{5.6880}], [s_{1.0000}, s_{2.0000}])$
$A_2$	$([s_{2.6719}, s_{4.8325}], [s_{1.4055}, s_{3.0000}])$	$([s_{3.0736}, s_{4.3748}], [s_{1.0000}, s_{2.2040}])$	$([s_{2.0000}, s_{3.8891}], [s_{1.5562}, s_{3.3684}])$	$([s_{2.7123}, s_{4.4742}], [s_{1.3495}, s_{2.2387}])$
$A_3$	$([s_{3.3398}, s_{4.9889}], [s_{1.2645}, s_{2.4784}])$	$([s_{4.5956}, s_{5.7889}], [s_{1.1872}, s_{1.6499}])$	$([s_{3.0000}, s_{4.8441}], [s_{1.3877}, s_{3.0000}])$	$([s_{3.5216}, s_{4.8385}], [s_{1.4204}, s_{3.0000}])$
$A_4$	$([s_{3.8089}, s_{4.8140}], [s_{1.4273}, s_{2.4627}])$	$([s_{1.0000}, s_{2.6675}], [s_{3.0000}, s_{4.0814}])$	$([s_{3.0000}, s_{4.7284}], [s_{1.1248}, s_{2.4386}])$	$([s_{5.0974}, s_{6.2108}], [s_{1.0000}, s_{1.4831}])$

Step 5: Calculate  $T_{kt}$  ( $k, t = 1, 2, 3, 4$ ) as:

$$T_{kt} = \begin{bmatrix} 1.0000 & 0.5942 & 0.3859 & 0.2460 \\ 1.0000 & 0.5968 & 0.3776 & 0.2002 \\ 1.0000 & 0.6433 & 0.4734 & 0.2878 \\ 1.0000 & 0.6479 & 0.2548 & 0.1606 \end{bmatrix}$$

Step 6: Use  $T_{kt}$  and the LIVIFPWA operator; we get the aggregated value  $\gamma_k$  for each alternative as:

$$\gamma_1 = ([s_{3.2667}, s_{4.8041}], [s_{1.3010}, s_{2.5344}]), \quad \gamma_2 = ([s_{2.6804}, s_{4.5267}], [s_{1.2981}, s_{2.8325}]),$$

$$\gamma_3 = ([s_{3.6762}, s_{5.1855}], [s_{1.2840}, s_{2.3613}]), \quad \gamma_4 = ([s_{3.1098}, s_{4.4074}], [s_{1.7022}, s_{2.7710}])$$

Step 7: Using Equation (29), we get  $S(\gamma_1) = 5.0589, S(\gamma_2) = 4.7691, S(\gamma_3) = 5.3041, S(\gamma_4) = 4.7610$ .

Step 8: Since  $S(\gamma_3) > S(\gamma_1) > S(\gamma_2) > S(\gamma_4)$ , therefore  $A_3 \succ A_1 \succ A_2 \succ A_4$ . Hence,  $A_3$  is the best candidate.

If we utilize weighted geometric AOs instead of averaging operators to aggregate the numbers in Steps 3 and 5, then the following steps are executed

Step 1: The preference values of each expert are given in Tables 1–3, respectively.

Step 2: Each attribute is of the profit type, so there is no need for normalization.

Step 3: Calculate the  $T_{kt}^{(q)}$ ,  $q = 1, 2, 3$ , and get:

$$T_{kt}^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad T_{kt}^{(2)} = \begin{bmatrix} 0.5938 & 0.6875 & 0.6250 & 0.6250 \\ 0.5938 & 0.5938 & 0.4688 & 0.4688 \\ 0.7188 & 0.7813 & 0.5625 & 0.6563 \\ 0.6250 & 0.4063 & 0.6250 & 0.8438 \end{bmatrix}$$

$$T_{kt}^{(3)} = \begin{bmatrix} 0.3340 & 0.4727 & 0.4102 & 0.4297 \\ 0.3711 & 0.2598 & 0.2051 & 0.2930 \\ 0.4043 & 0.5859 & 0.3340 & 0.3691 \\ 0.4297 & 0.1396 & 0.3320 & 0.4746 \end{bmatrix}$$

Step 4: Utilize the LIVIFPOWG operator as given in Equation (19) to aggregate the preference of each decision-maker. The results corresponding to it are summarized in Table 5.

**Table 5.** Aggregated linguistic interval-valued intuitionistic fuzzy decision matrix  $R$  by using the LIVIFPOWG operator.

	$G_1$	$G_2$	$G_3$	$G_4$
$A_1$	$([s_{2.4682}, s_{4.4909}], [s_{1.5380}, s_{2.7112}])$	$([s_{3.4371}, s_{4.4712}], [s_{1.2322}, s_{2.5094}])$	$([s_{3.7747}, s_{5.0000}], [s_{1.5279}, s_{3.0000}])$	$([s_{3.8467}, s_{5.5122}], [s_{1.0000}, s_{2.0000}])$
$A_2$	$([s_{2.4379}, s_{4.7937}], [s_{1.5103}, s_{3.0000}])$	$([s_{2.6379}, s_{4.0941}], [s_{1.0000}, s_{2.3314}])$	$([s_{2.0000}, s_{3.8615}], [s_{1.8868}, s_{3.4296}])$	$([s_{2.4991}, s_{4.3280}], [s_{1.4513}, s_{2.2882}])$
$A_3$	$([s_{3.1801}, s_{4.5836}], [s_{1.3559}, s_{2.5516}])$	$([s_{4.4061}, s_{5.7352}], [s_{1.2621}, s_{1.8788}])$	$([s_{3.0000}, s_{4.8073}], [s_{1.4919}, s_{3.0000}])$	$([s_{3.4579}, s_{4.8007}], [s_{1.5255}, s_{3.0000}])$
$A_4$	$([s_{3.7664}, s_{4.7720}], [s_{1.5325}, s_{2.5360}])$	$([s_{1.0000}, s_{2.5998}], [s_{3.0000}, s_{4.1026}])$	$([s_{3.0000}, s_{4.5849}], [s_{1.1807}, s_{2.5118}])$	$([s_{4.4839}, s_{5.5642}], [s_{1.0000}, s_{1.5875}])$

Step 5: The values of  $T_{kt}$ ,  $k, t = 1, 2, 3, 4$ , are computed as:

$$T_{kt} = \begin{bmatrix} 1.0000 & 0.5847 & 0.3685 & 0.2331 \\ 1.0000 & 0.5850 & 0.3547 & 0.1834 \\ 1.0000 & 0.6205 & 0.4460 & 0.2692 \\ 1.0000 & 0.6397 & 0.2498 & 0.1553 \end{bmatrix}$$

Step 6: Utilize the LIVIFPWG operator, as given in Equation (26), to obtain the overall value of each alternative, and get:

$$\gamma_1 = ([s_{3.0373}, s_{4.6685}], [s_{1.3998}, s_{2.6366}]), \quad \gamma_2 = ([s_{2.4155}, s_{4.3880}], [s_{1.4342}, s_{2.9272}]),$$

$$\gamma_3 = ([s_{3.4627}, s_{4.9356}], [s_{1.3772}, s_{2.5263}]), \quad \gamma_4 = ([s_{2.4515}, s_{3.9731}], [s_{1.9579}, s_{3.0213}])$$

Step 7: By using Equation (29), the score values of each alternative are  $S(\gamma_1) = 4.9173$ ,  $S(\gamma_2) = 4.6105$ ,  $S(\gamma_3) = 5.1237$  and  $S(\gamma_4) = 4.3614$ .

Step 8: Since  $S(\gamma_3) > S(\gamma_1) > S(\gamma_2) > S(\gamma_4)$ , therefore  $A_3 \succ A_1 \succ A_2 \succ A_4$ . Hence,  $A_3$  is the best candidate.

However, apart from the above analysis, in order to analyze the effect of the different aggregation operators on to the DM steps, we perform an experiment where we set up the different operators in Step 4 and Step 6 according to the decision-maker preferences: the optimism or pessimism point of way. For instance, initially, we utilize the LIVIFPOWA operator during the implementation of Step 4 of the proposed algorithm, to aggregate the different preferences of each expert and by varying all the other algorithms in Step 6. The results corresponding to each alternative are summarized in Table 6. Under this analysis, it is assumed that the expert/decision-maker wants an optimism behavior towards the expert preferences and a different behavior towards the criteria. Similarly, if the decision-maker can chooses the pessimistic behavior towards the expert preferences, then we utilize the LIVIFPOWG operator during the implementation of Step 4 of the proposed algorithm. Further, by selecting the nature of the decision-maker in Step 6 of the algorithm, we vary all the proposed operators and compute the final score values for each operator. The results corresponding to it are

summarized in Table 6 along with the final ranking order. The first four rows of Table 6 depict an optimistic approach towards the aggregation, while the last four rows depict the pessimistic approach by taking the geometric aggregation operator. Furthermore, it is clearly seen from the table that the final score values of the alternatives are less for the pessimistic approach than the optimistic approach. The final ranking order suggests the effect of the proposed operators on the best alternative as per the decision-maker preferences.

**Table 6.** The effect of the operators on the alternatives and ranking order. LIVIFPOWA, linguistic interval-valued intuitionistic fuzzy prioritized ordered weighted averaging; LIVIFPWA, LIVIF prioritized weighted averaging; LIVIFPWG, LIVIF prioritized weighted geometric; LIVIFPOWG, LIVIF prioritized ordered weighted geometric.

Operators Used in Step 4	Operators Used in Step 6	Rating Values				Ranking
		A <sub>1</sub>	A <sub>2</sub>	A <sub>3</sub>	A <sub>4</sub>	
LIVIFPOWA	LIVIFPWA	5.0589	4.7691	5.3041	4.7610	A <sub>3</sub> > A <sub>1</sub> > A <sub>2</sub> > A <sub>4</sub>
	LIVIFPOWA	5.3820	4.8557	5.4304	5.6292	A <sub>4</sub> > A <sub>3</sub> > A <sub>1</sub> > A <sub>2</sub>
	LIVIFPWG	5.0145	4.7343	5.2486	4.4213	A <sub>3</sub> > A <sub>1</sub> > A <sub>2</sub> > A <sub>4</sub>
	LIVIFPOWG	5.3369	4.8223	5.3636	5.3839	A <sub>4</sub> > A <sub>3</sub> > A <sub>1</sub> > A <sub>2</sub>
LIVIFPOWG	LIVIFPWA	4.9582	4.6438	5.1737	4.6656	A <sub>3</sub> > A <sub>1</sub> > A <sub>4</sub> > A <sub>2</sub>
	LIVIFPOWA	5.2861	4.7079	5.3151	5.3997	A <sub>4</sub> > A <sub>3</sub> > A <sub>1</sub> > A <sub>2</sub>
	LIVIFPWG	4.9173	4.6105	5.1237	4.3614	A <sub>3</sub> > A <sub>1</sub> > A <sub>2</sub> > A <sub>4</sub>
	LIVIFPOWG	5.2416	4.6760	5.2556	5.2233	A <sub>3</sub> > A <sub>1</sub> > A <sub>4</sub> > A <sub>2</sub>

To examine whether the proposed approach is stable under the different criteria, we have investigated the following tests as proposed by Wang and Triantaphyllou [39].

5.2. Validity Test

The following three test criteria were established by Wang and Triantaphyllou [39] to validate the MAGDM methods.

Test 1: “An effective MAGDM method does not change the index of the best alternative by replacing a non optimal alternative with a worse alternative without shifting the corresponding importance of every decision attribute”.

Test 2: “To an effective MAGDM method must be satisfy transitive property”.

Test 3: “If we decomposed a MAGDM problem into the sub DM problems and same MAGDM method is utilized on sub problems to rank alternatives, collective ranking of alternatives must be identical to ranking of un-decomposed DM problem”.

5.2.1. Validity by Test 1

Under this test, if we replace the values of non-optimal alternative A<sub>4</sub> with its worse one A'<sub>4</sub> for each decision-maker (given in Table 7), then by performing the steps of the proposed approach on them, we get the final score values of each alternative as 5.0589, 4.7691, 5.3041 and 3.2519. Thus, we conclude that the ranking order is A<sub>3</sub> > A<sub>1</sub> > A<sub>2</sub> > A'<sub>4</sub>, which coincides with the original ordering. Therefore, Test 1 is valid for the approach.

**Table 7.** Worse alternative A'<sub>4</sub> for every decision-maker.

	G <sub>1</sub>	G <sub>2</sub>	G <sub>3</sub>	G <sub>4</sub>
D <sup>(1)</sup>	([s <sub>2</sub> , s <sub>3</sub> ], [s <sub>3</sub> , s <sub>4</sub> ])	([s <sub>0</sub> , s <sub>3</sub> ], [s <sub>4</sub> , s <sub>5</sub> ])	([s <sub>2</sub> , s <sub>4</sub> ], [s <sub>2</sub> , s <sub>4</sub> ])	([s <sub>3</sub> , s <sub>4</sub> ], [s <sub>2</sub> , s <sub>3</sub> ])
D <sup>(2)</sup>	([s <sub>2</sub> , s <sub>3</sub> ], [s <sub>3</sub> , s <sub>4</sub> ])	([s <sub>0</sub> , s <sub>1</sub> ], [s <sub>4</sub> , s <sub>6</sub> ])	([s <sub>1</sub> , s <sub>2</sub> ], [s <sub>3</sub> , s <sub>4</sub> ])	([s <sub>1</sub> , s <sub>2</sub> ], [s <sub>3</sub> , s <sub>4</sub> ])
D <sup>(3)</sup>	([s <sub>1</sub> , s <sub>2</sub> ], [s <sub>3</sub> , s <sub>5</sub> ])	([s <sub>1</sub> , s <sub>1</sub> ], [s <sub>4</sub> , s <sub>5</sub> ])	([s <sub>2</sub> , s <sub>4</sub> ], [s <sub>3</sub> , s <sub>4</sub> ])	([s <sub>2</sub> , s <sub>3</sub> ], [s <sub>2</sub> , s <sub>3</sub> ])

### 5.2.2. Validity by Tests 2 and 3

Under this test, we decompose the original problem into three smaller sub-problems, which contain alternatives  $\{A_1, A_2, A_4\}$ ,  $\{A_1, A_3, A_4\}$  and  $\{A_2, A_3, A_4\}$ . Now, to each sub-problem, we apply the proposed approach, and the final ranking order of them is computed as  $A_1 \succ A_2 \succ A_4, A_3 \succ A_1 \succ A_4$  and  $A_3 \succ A_2 \succ A_4$ , respectively. Thus, by combining these orderings, we get  $A_3 \succ A_1 \succ A_2 \succ A_4$ , which is similar to the original problem. Thus, the approach is true under this test, as well.

### 5.3. Comparative Study

In order to check the performance of the proposed approach with existing ones, firstly based on the given information, we take the weight matrix for each expert and each criterion from Step 3 and Step 5 of the proposed approach. The computed weight vector for each decision-maker with respect to each attribute is summarized as:

$$w = \begin{matrix} & D^{(1)} & D^{(2)} & D^{(3)} \\ G_1 & (0.4867 & 0.3042 & 0.2091) \\ G_2 & (0.5676 & 0.2661 & 0.1663) \\ G_3 & (0.4937 & 0.3240 & 0.1823) \\ G_4 & (0.4313 & 0.3639 & 0.2047) \end{matrix}$$

while the weight decision matrix for each attribute is summarized as:

$$\omega = \begin{matrix} & G_1 & G_1 & G_1 & G_1 \\ A_1 & (0.4492 & 0.2669 & 0.1734 & 0.1105) \\ A_2 & (0.4599 & 0.2745 & 0.1736 & 0.0921) \\ A_3 & (0.4159 & 0.2675 & 0.1969 & 0.1197) \\ A_4 & (0.4847 & 0.3140 & 0.1235 & 0.0778) \end{matrix}$$

Based on these matrices, we compare the proposed results with some of the existing methods' results under LIVIFS [36] and LIFS environment [23–25,27,29,30,40]. Since under the LIFS environment, each linguistic term has one intuitionistic fuzzy membership, we convert our LIVIFS into LIFS by considering  $s_{(\tau+\eta)/2}$  and  $s_{(\theta+v)/2}$  as the linguistic membership degrees. Based on this information and the given preferences, we perform some of the existing approaches [23–25,27,29,30,40] on this considered data, and then the final scores of the alternatives  $A_k(k = 1, 2, 3, 4)$  are computed and summarized in Table 8. From this table, we can get similar ranking results as the proposed method. However, the computational procedure of the proposed approach is entirely different from the existing ones.

Table 8. Comparison with existing approaches.

Author	Rating Values				Ranking
	A <sub>1</sub>	A <sub>2</sub>	A <sub>3</sub>	A <sub>4</sub>	
Zhang [23]	5.0484	4.7612	5.3255	4.7123	$A_3 \succ A_1 \succ A_2 \succ A_4$
Chen et al. [24]	2.6685	2.3250	3.5830	1.7842	$A_3 \succ A_1 \succ A_2 \succ A_4$
Garg and Kumar [27]	4.4910	4.3547	4.6060	4.2208	$A_3 \succ A_1 \succ A_2 \succ A_4$
Liu and Qin [40]	−6.8114	−7.0609	−6.7229	−7.0373	$A_3 \succ A_1 \succ A_4 \succ A_2$
Liu and Wang [25]	1.8488	0.8771	2.5409	1.1646	$A_3 \succ A_1 \succ A_4 \succ A_2$
Liu and Liu [29]	3.6675	3.0294	3.8132	3.0307	$A_3 \succ A_1 \succ A_4 \succ A_2$
Garg and Kumar [30]	0.2524	0.2140	0.3078	0.2258	$A_3 \succ A_1 \succ A_4 \succ A_2$
Garg and Kumar [36]	1.6218	1.1837	1.8719	0.9893	$A_3 \succ A_1 \succ A_4 \succ A_2$

5.4. Further Discussion

In the following, we give some characteristic comparisons of our proposed method and the aforementioned methods [10,11,14,23–25,27,29,30,36,40], which are listed in Table 9.

Table 9. The characteristic comparisons of different methods.

Methods Properties	Whether Describes Information Using Linguistic Information	Whether Describes Information by Interval-Valued Numbers	Whether Considers More Than One Decision-Maker	Considers Priority Relations Between Input Arguments
Xu and Yager [14]	×	×	×	×
Xu and Chen [10]	×	✓	×	×
Xu [11]	×	✓	×	×
Zhang [23]	✓	×	✓	×
Chen et al. [24]	✓	×	✓	×
Garg and Kumar [27]	✓	×	✓	×
Liu and Qin [40]	✓	×	✓	×
Liu and Wang [25]	✓	×	✓	×
Liu and Liu [29]	✓	×	✓	×
Garg and Kumar [30]	✓	×	✓	×
Garg and Kumar [36]	✓	✓	✓	×
The proposed method	✓	✓	✓	✓

The method proposed by [14] adopts IFNs to aggregate the uncertain information using geometric operators only in a quantitative manner. On the other hand, the method described by the author in [10,11] represents a wider range of information in terms of the interval-valued membership degrees. However, that approach is also limited to only quantitative aspects and does not apply the linguistic information. Apart from this, the method proposed by Zhang [23], Chen et al. [24] adopted LIFNs to describe the uncertainties in the data as a crisp number. In Garg and Kumar [27], the authors analyzed the problem by using the linguistic connection number of the set pair analysis theory under the LIFS environment. In [25,29,40], the authors studied the LIFS environment by proposing AOs using the Bonferroni mean, Maclaurin symmetric mean and some improved aggregation operators. In [30], the authors aggregated the preferences by using Einstein-based AOs under the LIFS environment only. In [36], the authors presented a possibility degree measure to rank the different LIVIFNs. However, in the present study, we proposed aggregation operators for the collection of LIVIFNs to describe the uncertainties in terms of linguistic interval pairs of the membership degrees, which can easily express the information in a more semantic and concise way, hence being able to reduce the information loss.

In addition, LIVIFNs used in the new method can model the uncertain and fuzzy information with more flexibility by its linguistic interval-valued intuitionistic fuzzy numbers during the evaluation process, which can reflect the inherent thoughts of decision-makers more accurately. Further, it has been analyzed that the set defined by the author in [23] can be considered as a special case of the proposed set by setting the lower and upper bound of membership degrees as equal. Thus, the proposed method is more generalized and captures more information during the analysis.

6. Conclusions

In real-world problems, the decision-makers give their preferences in terms of qualitative values rather than crisp numbers. To handle this decision, IVIFS theory is a more efficient tool to deal with imprecise data, while the linguistic approach expresses the uncertainty in the qualitative aspect. By taking advantage of both, in this paper, LIVIFS is considered, which is the generalization of LIFS in which membership and non-membership degrees are represented by the interval-valued linguistic terms in order to better deal with fuzzy information under the qualitative aspect. Afterward, based on the basic operational laws on LIVIFNs, we developed some prioritized averaging and geometric aggregation operators, namely LIVIFPWA, LIVIFPOWA, LIVIFPWG, and LIVIFPOWG. The main advantage of these operators is that they consider the prioritized relationship between the input arguments. Various desirable properties have also been discussed in detail. To demonstrate the applicability of the proposed operators to the MAGDM process, we present an algorithm for

solving the MAGDM problems based on the proposed operator. A real-life numerical example is given to demonstrate the approach, and comparative studies with some existing approaches demonstrate the feasibility and reliability of the proposed operators. The superiority of the proposed work has been justified with a validity test. Further, the impact of the different AOs on the DM process—optimistic and pessimistic—is analyzed in the study. Based on the computed results and methodologies, a decision-maker can see the influence of his/her various choices towards the aggregation process, and hence based on this, he/she can choose the desirable alternative(s). From the study, we conclude that the proposed approach presents a better and easier way to solve the uncertainties of real-life problems. Future research will focus on introducing the various aggregation operators to address some more complicated problems involving the Pythagorean FS [41,42] and other uncertain environments [43–47].

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