mathematics

MDPI

Article

The Double Roman Domination Numbers of Generalized Petersen Graphs $P(n,2)$

Huiqin Jiang [1], Pu Wu [2], Zehui Shao [2], Yongsheng Rao [2] and Jia-Bao Liu [3,*]

1 Key Laboratory of Pattern Recognition and Intelligent Information Processing, Institutions of Higher Education of Sichuan Province, Chengdu University, Chengdu 610106, China; hq.jiang@hotmail.com
2 Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China; puwu1997@126.com (P.W.); zshao@gzhu.edu.cn (Z.S.); rysheng@gzhu.edu.cn (Y.R.)
3 School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China
* Correspondence: liujiabao@ahjzu.edu.cn or liujiabaoad@163.com

Received: 11 September 2018; Accepted: 10 October 2018; Published: 16 October 2018

check for updates

Abstract: A double Roman dominating function (DRDF) f on a given graph G is a mapping from $V(G)$ to $\{0,1,2,3\}$ in such a way that a vertex u for which $f(u)=0$ has at least a neighbor labeled 3 or two neighbors both labeled 2 and a vertex u for which $f(u)=1$ has at least a neighbor labeled 2 or 3. The weight of a DRDF f is the value $w(f)=\sum_{u\in V(G)} f(u)$. The minimum weight of a DRDF on a graph G is called the double Roman domination number $\gamma_{dR}(G)$ of G. In this paper, we determine the exact value of the double Roman domination number of the generalized Petersen graphs $P(n,2)$ by using a discharging approach.

Keywords: double Roman domination; discharging approach; generalized Petersen graphs

1. Introduction

In this paper, only graphs without multiple edges or loops are considered. For two vertices u and v of a graph G, we say $u \sim v$ in G if $uv \in E(G)$. For positive integer k and $u,v \in V(G)$, let $d(u,v)$ be the distance between u and v and $N_k(v)=\{u|d(u,v)=k\}$. The neighborhood of v in G is defined to be $N_1(v)$ (or simply $N(v)$). The closed neighborhood $N[v]$ of v in G is defined to be $N[v]=\{v\}\cup N(v)$. For a vertex subset $S\subseteq V(G)$, we denote by $G[S]$ the subgraph induced by S. For a positive integer n, we denote $[n]=\{1,2,\cdots,n\}$. For a set $S=\{x_1,x_2,\cdots,x_n\}$, if $x_i=x_j$ for some i and j, then S is considered as a multiset. Otherwise, S is an ordinary set.

For positive integer numbers n and k with n at least $2k+1$, the generalized Petersen graph $P(n,k)$ is a graph with its vertex set $\{u_i|i=1,2,\cdots,n\}\cup\{v_i|i=1,2,\cdots,n\}$ and its edge set the union of $\{u_iu_{i+1},u_iv_i,v_iv_{i+k}\}$ for $1\le i\le n$, where subscripts are reduced modulo n (see [1]).

A subset D of the vertex set of a graph G is a dominating set if every vertex in $V(G)\setminus D$ has at least one neighbor in D. The domination number, denoted by $\gamma(G)$, is the minimum number of vertices over all dominating sets of G.

There have been more than 200 papers studying various domination on graphs in the literature [2–6]. Among them, Roman domination and double Roman domination appear to be a new variety of interest [3,7–15].

A double Roman dominating function (DRDF) f on a given graph G is a mapping from $V(G)$ to $\{0,1,2,3\}$ in such a way that a vertex u for which $f(u)=0$ has at least a neighbor labeled 3 or two neighbors both labeled 2 and a vertex u for which $f(u)=1$ has at least a neighbor labeled 2 or 3. The weight of a DRDF f is the value $w(f)=\sum_{u\in V(G)} f(u)$. The minimum weight of a DRDF on a graph G is called the double Roman domination number $\gamma_{dR}(G)$ of G. A DRDF f of G with $w(f)=\gamma_{dR}(G)$

is called a $\gamma_{dR}(G)$-function. Given a DRDF f of G, we denote $E^f_{\{x_1,x_2\}} = \{uv \in E(G) | \{f(u), f(v)\} = \{x_1, x_2\}\}$. A graph G is a double Roman Graph if $\gamma_{dR}(G) = 3\gamma(G)$.

In [7], Beeler et al. obtained the following results:

Proposition 1 ([7])**.** *In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the value one.*

By Proposition 1, we now consider the DRDF of a graph G in which there exists no vertex assigned with one in the following.

Given a DRDF f of a graph G, suppose (V_0^f, V_2^f, V_3^f) is the ordered partition of the vertex set of G induced by f in such a way that $V_i^f = \{v : f(v) = i\}$ for $i = 0, 2, 3$. It can be seen that there is a 1-1 mapping between f and (V_0^f, V_2^f, V_3^f), and we write $f = (V_0^f, V_2^f, V_3^f)$, or simply (V_0, V_2, V_3). Given a DRDF f of $P(n,2)$ and letting $w_i \in \{0,2,3\}$ for $i = 1,2,3$ with $w_1 \geq w_2 \geq w_3$, we write $V_j^{w_1w_2w_3} = \{x \in V(P(n,2)) | f(x) = j, \{w_1, w_2, w_3\} = \{f(x_1), f(x_2), f(x_3)\}\}$, where $N(x) = \{x_1, x_2, x_3\}$.

Now, we will use $f(\cdot) = q^+$ to represent the value scope $f(\cdot) \geq q$ for an integer q. We say a path $t_1t_2 \cdots t_k$ is a path of type $c_1 - c_2 - \cdots - c_k$ if $f(t_i) = c_i$ for $i \in [k]$. Let H be a subgraph induced by five vertices s_1, s_2, s_3, s_4, s_5 with $s_1 \sim s_2, s_2 \sim s_3, s_3 \sim s_4, s_3 \sim s_5$ satisfying $f(s_3) = 0$ and $f(s_1) = a$, $f(s_2) = b$, $f(s_4) = c$, $f(s_5) = d$ for some $a, b, c, d \in \{0,2,3\}$, then we say H is a subgraph of type $a - b - 0^{-c}_{-d}$.

Let W be a subgraph induced by four vertices s_1, s_2, s_3, s_4 with $s_1 \sim s_2, s_2 \sim s_3, s_2 \sim s_4$, satisfying $f(s_1) = a$, $f(s_2) = 0$, $f(s_3) = b$ and $f(s_4) = c$ for some $a, b, c \in \{0,2,3\}$, then we say W is a subgraph of type $a - 0^{-b}_{-c}$.

In the graph $P(n,2)$, we will denote the set of vertices of $\{u_i, v_i\}$ with $L^{(i)}$. For a given DRDF f of $P(n,2)$, let $w_f(L^{(i)})$ denote the weight of $L^{(i)}$, that is $w_f(L^{(i)}) = \sum_{u \in V(L^{(i)})} f(u)$. Let $\mathcal{B}_i$={$L^{(i-2)}$, $L^{(i-1)}$, $L^{(i)}$, $L^{(i+1)}$, $L^{(i+2)}$}, where the subscripts are taken modulo n. We define $w_f(\mathcal{B}_i) = \sum\limits_{j=-2}^{2} w_f(L^{(i+j)})$, and:

$$f(\mathcal{B}_i) = f\begin{pmatrix} u_{i-2} & u_{i-1} & u_i & u_{i+1} & u_{i+2} \\ v_{i-2} & v_{i-1} & v_i & v_{i+1} & v_{i+2} \end{pmatrix}.$$

Motivation: Beeler et al. [7] put forward an open problem about characterizing the double Roman graphs. As an interesting family of graphs, the domination and its variations of generalized Petersen graphs have attracted considerable attention [1,16]. Therefore, it is interesting to characterize the double Roman graphs in generalized Petersen graphs. In this paper, we focus on finding the double Roman graphs in $P(n,2)$.

2. Double Roman Domination Number of $P(n,2)$

2.1. Upper Bound for the Double Roman Domination Number of $P(n,2)$

Lemma 1. *If $n \geq 5$, then:*

$$\gamma_{dR}(P(n,2)) \leq \begin{cases} \lceil \frac{8n}{5} \rceil, & n \equiv 0 \ (\mathrm{mod}\ 5), \\ \lceil \frac{8n}{5} \rceil + 1, & n \equiv 1,2,3,4 \ (\mathrm{mod}\ 5). \end{cases}$$

Proof. We consider the following five cases.

Case 1: $n \equiv 0$ (mod 5).

Let:

$$P_5 = \begin{bmatrix} 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}.$$

Then, by repeating the pattern of P_5, we obtain a DRDF of weight $8k$ of $P(5k, 2)$, and the upper bound is obtained.

Case 2: $n \equiv 1 \pmod 5$.

If $n = 6$, let:

$$P_6 = \begin{bmatrix} 0 & 2 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & 3 \end{bmatrix}.$$

Then, the pattern P_6 induces a DRDF of weight 11 of $P(6, 2)$, and the desired upper bound is obtained.

If $n \geq 11$, let:

$$P_{11} = \begin{bmatrix} 2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 3 & 2 \end{bmatrix}.$$

Then, by repeating the leftmost five columns of the pattern of P_{11}, we obtain a DRDF of weight $8k + 3$ of $P(5k + 1, 2)$, and the desired upper bound is obtained.

Case 3: $n \equiv 2 \pmod 5$.

If $n = 7$, let:

$$P_7 = \begin{bmatrix} 2 & 0 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 \end{bmatrix}.$$

Then, the pattern P_7 induces a DRDF of weight 13 of $P(7, 2)$, and the desired upper bound is obtained.

If $n \geq 12$, let:

$$P_{12} = \begin{bmatrix} 2 & 0 & 2 & 0 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 2 \end{bmatrix}.$$

Then, by repeating the leftmost five columns of the pattern of P_{12}, we obtain a DRDF of weight $8k + 6$ of $P(5k + 2, 2)$, and the desired upper bound is obtained.

Case 4: $n \equiv 3 \pmod 5$.

If $n \geq 8$, let:

$$P_8 = \begin{bmatrix} 2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 2 & 2 \end{bmatrix}.$$

Then, by repeating the leftmost five columns of the pattern of P_8, we obtain a DRDF of weight $8k + 6$ of $P(5k + 3, 2)$, and the desired upper bound is obtained.

Case 5: $n \equiv 4 \pmod 5$.

If $n \geq 9$, let:

$$P_9 = \begin{bmatrix} 2 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 0 & 3 & 2 \end{bmatrix}.$$

Then, by repeating the leftmost five columns of the pattern of P_9, we obtain a DRDF of weight $8k + 8$ of $P(5k + 4, 2)$, and the desired upper bound is obtained. □

2.2. *Lower Bound for Double Roman Domination Number of $P(n, 2)$*

Lemma 2. *Let f be a γ_{dR}-function of $P(n, 2)$ with $n \geq 5$. Then, $w_f(\mathcal{B}_i) \geq 4$.*

Proof. Since u_i, v_i, u_{i+1} and u_{i-1} need to be double Roman dominated by vertices in $\mathcal{B}_i$, we have $w_f(\mathcal{B}_i) \geq 3$. Now, we will show that $w_f(\mathcal{B}_i) \neq 3$. Otherwise, it is clear that $f(u_i) = 3$, and $f(x) = 0$ for any $x \in \mathcal{B}_i \setminus \{u_i\}$. Since $v_{i\pm1}, u_{i\pm2}$ and $v_{i\pm2}$ need to be double Roman dominated, we have

$f(u_{i\pm3}) = f(v_{i\pm3}) = f(v_{i\pm4}) = 3$. Now, we can obtain a DRDF f' from f by letting $f'(u_{i-2}) = 2$, $f'(u_{i-3}) = 0$ and $f'(v) = f(v)$ for $v \in V(P(n,2)) \setminus \{u_{i-2}, u_{i-3}\}$. Then, we have $w(f') < w(f)$, a contradiction (see Figure 1). Therefore, $w_f(\mathcal{B}_i) \geq 4$. □

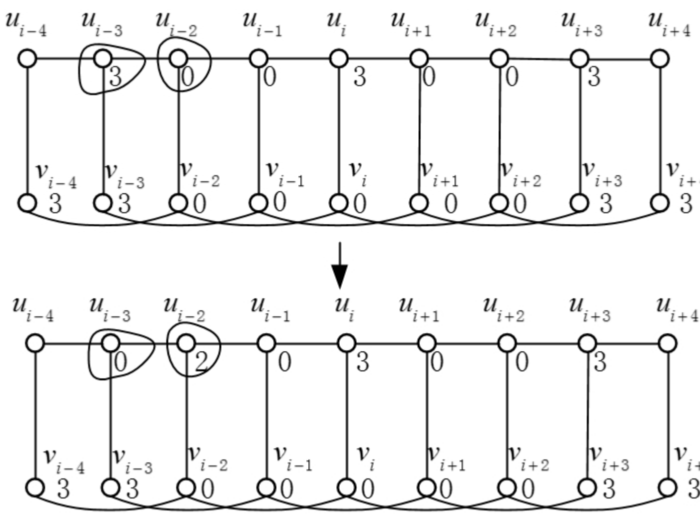

Figure 1. Construct a function f' from f used in Lemma 2.

Lemma 3. *Let f be a γ_{dR}-function of $P(n,2)$ with $n \geq 5$. Then, for any $i \in [n]$, it is impossible that $f(v_{i-1}) = f(v_i) = f(v_{i+1}) = 3$ and $f(x) = 0$ for any $x \in \mathcal{B}_i \setminus \{v_{i-1}, v_i, v_{i+1}\}$.*

Proof. Suppose to the contrary that $f(v_{i-1}) = f(v_i) = f(v_{i+1}) = 3$ and $f(x) = 0$ for $x \in \mathcal{B}_i \setminus \{v_{i-1}, v_i, v_{i+1}\}$. Then, we have $f(u_{i\pm3}) = 3$. Now, we can obtain a DRDF f' from f by letting $f'(u_{i-1}) = 2$, $f'(v_{i-1}) = 0$ and $f'(v) = f(v)$ for $v \in V(P(n,2)) \setminus \{v_{i-1}, u_{i-1}\}$. Then, we have $w(f') < w(f)$, a contradiction (see Figure 2). □

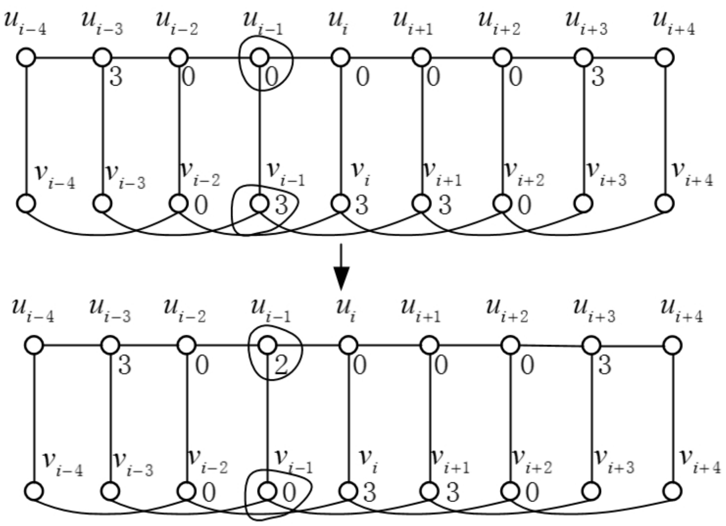

Figure 2. Construct a function f' from f in Lemma 3.

Lemma 4. *Let f be a γ_{dR}-function of $P(n,2)$ with $n \geq 5$. Then, for each $x \in V_3^{000}$, there exists a neighbor y of x such that $y \in V_0^{320} \cup V_0^{330} \cup V_0^{322} \cup V_0^{332} \cup V_0^{333}$, or equivalently, it is impossible that for any $x \in V_3^{000}$, $f(z) = 0$ for any $z \in N_2(x)$.*

Proof. Suppose to the contrary that there is a vertex $x \in V_3^{000}$ such that $y \in V_0^{300}$ for every neighbor y of x. Now, it is sufficient to consider the following two cases.

Case 1: $x = u_i$ for some i.

In this case, we have $f(u_i) = 3$ and $f(x) = 0$ for $x \in \mathcal{B}_i \setminus \{u_i\}$. Then, we have $w_f(\mathcal{B}_i) = 3 < 4$, contradicting Lemma 2.

Case 2: $x = v_i$ for some i.

In this case, since $u_{i\pm1}$ and $u_{i\pm2}$ need to be double Roman dominated, we have $f(v_{i\pm1}) = 3$ and $f(u_{i\pm3}) = 3$. By Lemma 3, such a case is impossible. □

Discharging procedure: Let f be a DRDF of $P(n, 2)$. We set the initial charge of every vertex x be $s(x) = f(x)$. We use the discharging procedure, leading to a final charge s', defined by applying the following rules:

R1: Each $s(x)$ for which $s(x) = 3$ transmits 0.8 charge to each neighbor y with $y \in V_0^{300}$ transmits 0.6 charge to each neighbor y with $y \in V_0^{320} \cup V_0^{330} \cup V_0^{322} \cup V_0^{332} \cup V_0^{333}$.
R2: Each $s(x)$ for which $s(x) = 2$ transmits 0.4 charge to each neighbor y with $y \in V_0$.

Proposition 2. *If $n \geq 5$, then $\gamma_{dR}(P(n, 2)) \geq \lceil \frac{8n}{5} \rceil$.*

Proof. Assume f is a γ_{dR}-function of $P(n, 2)$. We use the above discharging procedure. Now, it is sufficient to consider the following three cases.

Case 1: By Lemma 4, there exists a vertex z with $f(z) \geq 2$ for some $z \in N_2(x)$, for any $x \in V_3^{000}$. Therefore, by rule R1, for each $v \in V_3^{000}$, the final charge $s'(v)$ is at least $3 - 0.6 - 0.8 - 0.8 = 0.8$. For each $v \in V_3 \setminus V_3^{000}$, then the final charge $s'(v)$ is at least $3 - 0.8 - 0.8 = 1.4$.

Case 2: By rule R2, for each $v \in V_2$, the final charge $s'(v)$ is at least $2 - 0.4 - 0.4 - 0.4 = 0.8$.

Case 3: For each $v \in V_0^{300}$, the final charge $s'(v)$ is 0.8 by rule R1. For each $v \in V_0 \setminus V_0^{300}$, the final charge $s'(v)$ is at least 0.8 by rules R1 and R2.

From the above, we have:

$$s'(v) \geq 0.8 \text{ for any } v \in P(n, 2). \tag{1}$$

Hence, $w(f) = \sum_{v \in V(P(n,2))} s(v) = \sum_{v \in V(P(n,2))} s'(v) \geq 0.8 \times 2n = \frac{8n}{5}$. Since $w(f)$ is an integer, we have $w(f) \geq \lceil \frac{8n}{5} \rceil$. □

By using the above discharging rules, we have the following lemma immediately, and the proof is omitted.

Lemma 5. *Let f be a γ_{dR}-function of $P(n, 2)$ with $n \geq 5$. If we use the above discharging procedure for f on $P(n, 2)$, then:*

(a) *if there exists a path P of type $2 - 2 - 2$, or type $2^+ - 3$, or type $2 - 2 - 0 - 3$, or type $3 - 0 - 2^+ - 0 - 3 - 0 - 2^+ - 0 - 3$, or type $3 - 0 - 2^+ - 0 - 3 - 0 - 3$, or type $3 - 0 - 3 - 0 - 3$, or type $2^+ - 0 - 3 - 0 - 3 - 0 - 2^+$ or a subgraph P of type $3 - 0_{-3}^{-3}$, then $\sum_{v \in V(P)} (s'(v) - 0.8) \geq 1$.*
(b) *if there exist a path P_1 of type $2 - 2$ and a path P_2 of type $2^+ - 0 - 3$, then $\sum_{v \in V(P_1) \cup V(P_2)} (s'(v) - 0.8) \geq 1$.*
(c) *if there exists a subgraph H of type $2 - 2 - 0_{-2}^{-2}$, then $\sum_{v \in V(H)} (s'(v) - 0.8) \geq 1.2$.*
(d) *if there exist a path P of type $3 - 0 - 3$, together with a subgraph H of type $2^+ - 0 - 3 - 0 - 2^+$ or type $3 - 0_{-2^+}^{-2^+}$, then $\sum_{v \in V(P) \cup V(H)} (s'(v) - 0.8) \geq 1$.*
(e) *if there exist three paths P_1, P_2, P_3 of type $3 - 0 - 3$, then $\sum_{v \in V(P_1) \cup V(P_2) \cup V(P_3)} (s'(v) - 0.8) \geq 1.2$.*

Lemma 6. *Let f be a γ_{dR}-function of $P(n,2)$ with weight $\lceil \frac{8n}{5} \rceil$, then there exists no edge $uv \in E(P(n,2))$ for which $uv \in E^f_{\{2,2\}} \cup E^f_{\{2,3\}} \cup E^f_{\{3,3\}}$.*

Proof. First, we have:

$$\gamma_{dR}(P(n,2)) = w(f) = \lceil \frac{8n}{5} \rceil \leq \frac{8n+4}{5} = \frac{8n}{5} + 0.8,$$

and so:

$$w(f) - \frac{8n}{5} \leq 0.8.$$

We use the above discharging procedure for f on $P(n,2)$, and similar to the proof of Proposition 2, we have:

$$w(f) = \sum_{v \in V(P(n,2))} s'(v),$$

and so:

$$\sum_{v \in V(P(n,2))} (s'(v) - \frac{4}{5}) \leq 0.8 \tag{2}$$

By Lemma 5a and Equation (2), we have that there exists no edge $uv \in E^f_{\{2,3\}} \cup E^f_{\{3,3\}}$.

Now, suppose to the contrary that there exists an edge $uv \in E^f_{\{2,2\}}$, and it is sufficient to consider the following three cases.

Case 1: $f(u_i) = f(u_{i+1}) = 2$.

We have $f(u_{i-1}) = f(u_{i+2}) = f(v_{i+1}) = f(v_i) = 0$. Otherwise, there exists a path P of type $2-2-2$ or type $2^+ - 3$. By Lemma 5a, we have $\sum_{v \in V(P)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

Since u_{i+2} needs to be double Roman dominated, we have $\{f(u_{i+3}), f(v_{i+2})\} = \{0,2\}$. Otherwise, $f(x) = 3$ for some $x \in \{u_{i+3}, v_{i+2}\}$ or $f(u_{i+3}) = f(v_{i+2}) = 2$.

If $f(x) = 3$ for some $x \in \{u_{i+3}, v_{i+2}\}$, there exists a path P of type $2-2-0-3$. By Lemma 5a, we have $\sum_{v \in V(P)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

If $f(u_{i+3}) = f(v_{i+2}) = 2$, there exists a subgraph H of type $2-2-0^{-2}_{-2}$. By Lemma 5c, we have $\sum_{v \in V(H)} (s'(v) - 0.8) \geq 1.2$, contradicting Equation (2).

Now, it is sufficient to consider the following two cases.

Case 1.1: $f(v_{i+2}) = 2$, $f(u_{i+3}) = 0$.

To double Roman dominate v_{i+1}, we have $f(v_{i+3}) \geq 2$ or $f(v_{i-1}) \geq 2$. First, we have $f(v_{i+3}) \neq 3$ and $f(v_{i-1}) \neq 3$. Otherwise, $u_i u_{i+1} v_{i+1} v_{i+3}$ or $u_i u_{i+1} v_{i+1} v_{i-1}$ is a path P of type $2-2-0-3$. By Lemma 5a, we have $\sum_{v \in V(P)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

Now, we have that it is impossible $f(v_{i+3}) = f(v_{i-1}) = 2$. Otherwise, the set $\{u_i, u_{i+1}, v_{i+1}, v_{i+3}, v_{i-1}\}$ induces a subgraph H of type $2-2-0^{-2}_{-2}$. By Lemma 5c, we have $\sum_{v \in V(H)} (s'(v) - 0.8) \geq 1.2$, contradicting Equation (2).

Therefore, we have $\{f(v_{i+3}), f(v_{i-1})\} = \{0,2\}$. Now, it is sufficient to consider the following two cases.

Case 1.1.1: $f(v_{i+3}) = 2$, $f(v_{i-1}) = 0$.

Since v_{i-1} and u_{i-1} need to be double Roman dominated, we have $f(v_{i-3}) = 3$, $f(u_{i-2}) = 2^+$. Then, there exists a path P_1 of type $2-2$ and a path P_2 of type $2^+ - 0 - 3$. By Lemma 5b, we have $\sum_{v \in V(P_1) \cup V(P_2)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

Case 1.1.2: $f(v_{i+3}) = 0$, $f(v_{i-1}) = 2$.

Since u_{i+3} and v_{i+3} need to be double Roman dominated, we have $f(u_{i+4}) = f(v_{i+5}) = 3$. Then, there exist a path P_1 of type $2-2$ and a path P_2 of type $3-0-3$. By Lemma 5b, $\sum_{v\in V(P_1)\cup V(P_2)}(s'(v)-0.8) \geq 1$, contradicting Equation (2).

Case 1.2: $f(v_{i+2}) = 0$, $f(u_{i+3}) = 2$.

Since v_{i+2} needs to be double Roman dominated, we have $f(v_{i+4}) = 3$. Then, there exist a path P_1 of type $2-2$ and a path P_2 of type $2-0-3$. By Lemma 5b, $\sum_{v\in V(P_1)\cup V(P_2)}(s'(v)-0.8) \geq 1$, contradicting Equation (2).

Case 2: $f(v_i) = f(u_i) = 2$.

We have $f(u_{i\pm1}) = f(v_{i\pm2}) = 0$. Otherwise, there exists a path P of type $2-2-2$ or type 2^+-3. By Lemma 5a, we have $\sum_{v\in V(P)}(s'(v)-0.8) \geq 1$, contradicting Equation (2).

Since u_{i+1} needs to be double Roman dominated, we have $\{f(u_{i+2}), f(v_{i+1})\} = \{0,2\}$. Otherwise, by Lemma 5a or Lemma 5c, we obtain a contradiction with Equation (2).

Now, we consider the following two subcases.

Case 2.1: $f(v_{i+1}) = 2$, $f(u_{i+2}) = 0$.

Since u_{i+2} needs to be double Roman dominated, we have $f(u_{i+3}) = 3$. Then, there exist a path P_1 of type $2-2$ and a path P_2 of type $2-0-3$. By Lemma 5b, $\sum_{v\in V(P_1)\cup V(P_2)}(s'(v)-0.8) \geq 1$, contradicting Equation (2).

Case 2.2: $f(v_{i+1}) = 0$, $f(u_{i+2}) = 2$.

Since v_{i+1} needs to be double Roman dominated, we have $f(x) = 3$ for some $x \in \{v_{i+3}, v_{i-1}\}$ or $f(v_{i+3}) = f(v_{i-1}) = 2$. If $f(x) = 3$ for some $x \in \{v_{i+3}, v_{i-1}\}$, there exist a path P_1 of type $2-2$ and a path P_2 of type $2-0-3$. By Lemma 5b, $\sum_{v\in V(P_1)\cup V(P_2)}(s'(v)-0.8) \geq 1$, contradicting Equation (2).

If $f(v_{i+3}) = f(v_{i-1}) = 2$, then by Lemma 5b,c, we have $u_{i-2} = 0$. Since u_{i-2} needs to be double Roman dominated, we have $f(u_{i-3}) = 3$. Then, there exist a path P_1 of type $2-2$ and a path P_2 of type $2-0-3$. By Lemma 5b, $\sum_{v\in V(P_1)\cup V(P_2)}(s'(v)-0.8) \geq 1$, contradicting Equation (2).

Case 3: $f(v_{i+1}) = f(v_{i-1}) = 2$.

We have $f(u_{i\pm1}) = f(v_{i\pm3}) = 0$. Otherwise, there exists a path P of type $2-2-2$ or type 2^+-3. By Lemma 5a, we have $\sum_{v\in V(P)}(s'(v)-0.8) \geq 1$, contradicting Equation (2).

Since u_i needs to be double Roman dominated, we have $f(u_i) = 2$ or $f(v_i) = 3$.

Case 3.1: $f(u_i) = 2$, $f(v_i) = 0$.

By Lemma 5b,c and Equation (2), we have $f(u_{i\pm2}) = 0$. Since v_i needs to be double Roman dominated, we have $\{f(v_{i-2}), f(v_{i+2})\} = \{0,2\}$. Considering isomorphism, we without loss of generality assume $f(v_{i+2}) = 2$ and $f(v_{i-2}) = 0$. Since u_{i-2} needs to be double Roman dominated, $f(u_{i-3}) = 3$. Then, there exist a path P_1 of type $2-2$ and a path P_2 of type $2-0-3$. By Lemma 5b, $\sum_{v\in V(P_1)\cup V(P_2)}(s'(v)-0.8) \geq 1$, contradicting Equation (2).

Case 3.2: $f(u_i) = 0$, $f(v_i) = 3$.

By Lemma 5a and Equation (2), we have $f(v_{i\pm2}) = 0$. Since u_{i+1} needs to be double Roman dominated, we have $f(u_{i+2}) = 2$. Then, there exist a path P_1 of type $2-2$ and a path P_2 of type $2-0-3$. By Lemma 5b, $\sum_{v\in V(P_1)\cup V(P_2)}(s'(v)-0.8) \geq 1$, contradicting Equation (2).

Therefore, the proof is complete. □

Lemma 7. *Let f be a γ_{dR}-function of $P(n,2)$ with weight $\lceil \frac{8n}{5} \rceil$, $v \in V_3^{000}$ and $S = \{x | x \in N_2(v), f(x) \geq 2\}$, then $1 \leq |S| \leq 2$.*

Proof. We use the above discharging procedure for f on $P(n,2)$. By Lemma 4, we have $|S| \geq 1$. Now, suppose to the contrary that $|S| \geq 3$. By rules R1 and R2 and Equation (1), we have:

$$\sum_{v \in V(P(n,2))} (s'(v) - \frac{4}{5}) \quad \geq \quad \sum_{x \in N[v] \cup N_2(v)} (s'(x) - \frac{4}{5}) \geq 1,$$

contradicting Equation (2). □

Lemma 8. *If $n \geq 5$ and f is a γ_{dR}-function of $P(n,2)$ with $f(u_i) = 3$ for some $i \in [n]$, then $w(f) \geq \lceil \frac{8n}{5} \rceil + 1$.*

Proof. Suppose to the contrary that there exists a γ_{dR}-function f with $w(f) = \lceil \frac{8n}{5} \rceil$ such that $f(u_i) = 3$ for some $i \in [n]$. By Lemma 6, we have $f(v_i) = f(u_{i\pm1}) = 0$. Let $S = \{x | x \in N_2(v), f(x) \geq 2\}$. By Lemma 7, we have $|S| \in \{1, 2\}$. Therefore, we just need to consider the following two cases.

Case 1: $|S| = 1$.

We may w.l.o.g assume that $\{f(u_{i-2}), f(v_{i-1}), f(v_{i-2})\}$ =$\{0,0,2\}$ or $\{0,0,3\}$ and $f(v_{i+1}) = f(v_{i+2}) = f(u_{i+2}) = 0$. Since u_{i+2}, v_{i+2} need to be double Roman dominated, we have $f(u_{i+3}) = f(v_{i+4}) = 3$, and thus, $f(v_{i+3}) = 0$. Since v_{i+1} needs to be double Roman dominated, we have $f(v_{i-1}) = 3$. Thus, $f(u_{i-2}) = f(v_{i-2}) = 0$. Since u_{i-2}, v_{i-2} need to be double Roman dominated, we have $f(u_{i-3}) = f(v_{i-4}) = 3$. Then, there exist three paths P_1, P_2, P_3 of type $3-0-3$. By Lemma 5e, we have $\sum_{v \in V(P_1) \cup V(P_2) \cup V(P_3)} (s'(v) - 0.8) \geq 1.2$, contradicting Equation (2).

Case 2: $|S| = 2$.

It is sufficient to consider the following cases.

Case 2.1: $S \subseteq \{v_{i-1}, v_{i-2}, u_{i-2}\}$ and $f(v_{i+1}) = f(v_{i+2}) = f(u_{i+2}) = 0$.

Since u_{i+2}, v_{i+2} need to be double Roman dominated, we have $f(u_{i+3}) = f(v_{i+4}) = 3$. Then, there exist a path P of type $3-0-3$, and a subgraph H of type $2^+ - 0 - 3 - 0 - 2^+$ or type $3 - 0^{-2^+}_{-2^+}$. By Lemma 5d, we have $\sum_{v \in V(P) \cup V(H)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

Case 2.2: $S = \{s_1, s_2\}$, $s_1 \in \{v_{i-1}, v_{i-2}, u_{i-2}\}$ and $s_2 \in \{v_{i+1}, v_{i+2}, u_{i+2}\}$.

First, we have $f(v_{i\pm1}) = 0$. Otherwise, we may without loss of generality assume that $f(v_{i+1}) \geq 2$. Since u_{i+2}, v_{i+2} need to be double Roman dominated, we have $f(u_{i+3}) = f(v_{i+4}) = 3$. Then, there exist a path P of type $3-0-3$, and a path H of type $2^+ - 0 - 3 - 0 - 2^+$. By Lemma 5d, we have $\sum_{v \in V(P) \cup V(H)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

Then, since v_{i+1}, v_{i-1} need to be double Roman dominated, we have $f(v_{i+3}) = f(v_{i-3}) = 3$. By Lemma 6, we have $f(u_{i+3}) = f(u_{i-3}) = 0$. Since $u_{i\pm2}$ need to be double Roman dominated, we have $(f(u_{i-2}), f(v_{i-2})) \in \{(0,3), (2,0), (3,0)\}$ and $(f(u_{i+2}), f(v_{i+2})) \in \{(0,3), (2,0), (3,0)\}$.

It is impossible that $f(v_{i+2}) + f(u_{i+2}) = 3$ and $f(v_{i-2}) + f(u_{i-2}) = 3$. Otherwise, there exists a path P of type $3-0-3-0-3$ or a subgraph P of type $3 - 0^{-3}_{-3}$. By Lemma 5a, we have $\sum_{v \in V(P)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

It is impossible $f(u_{i\pm2}) \geq 2$. Otherwise, there exists a path P of type $3 - 0 - 2^+ - 0 - 3 - 0 - 2^+ - 0 - 3$. By Lemma 5a, we have $\sum_{v \in V(P)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

Then, we may without loss of generality assume that $f(u_{i+2}) = 2$ and $f(v_{i-2}) = 3$. Then, there exists a path P of type $3-0-2-0-3-0-3$. By Lemma 5a, we have $\sum_{v \in V(P)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2). □

Lemma 9. *If $n \geq 5$ and f is a γ_{dR}-function of $P(n,2)$ with $f(v_i) = 3$ for some $i \in [n]$, then $w(f) \geq \lceil \frac{8n}{5} \rceil + 1$.*

Proof. Suppose to the contrary that there exists a γ_{dR}-function f with $w(f) = \lceil \frac{8n}{5} \rceil$ such that $f(v_i) = 3$ for some $i \in [n]$. By Lemma 6, we have $f(u_i) = f(v_{i\pm2}) = 0$. Let $S = \{x | x \in N_2(v), f(x) \geq 2\}$. By Lemma 7, we have $1 \leq |S| \leq 2$, and we just need to consider the following two cases.

Case 1: $|S| = 1$.

We may without loss of generality assume that $\{f(u_{i-1}), f(u_{i-2}), f(v_{i-4})\}$=$\{0,0,2\}$ or $\{0,0,3\}$ and $f(u_{i+1}) = f(u_{i+2}) = f(v_{i+4}) = 0$. Since u_{i+1} and u_{i+2} need to be double Roman dominated, we have $f(v_{i+1}) = f(u_{i+3}) = 3$, contradicting Lemma 8.

Case 2: $|S| = 2$.

Now, it is sufficient to consider the following two cases.

Case 2.1: $S \subseteq \{u_{i-1}, u_{i-2}, v_{i-4}\}$ and $f(u_{i+1}) = f(u_{i+2}) = f(v_{i+4}) = 0$.

Since u_{i+1}, u_{i+2} need to be double roman dominated, we have $f(v_{i+1}) = f(u_{i+3}) = 3$, contradicting Lemma 8.

Case 2.2: $S = \{s_1, s_2\}$, where $s_1 \in \{u_{i-1}, u_{i-2}, v_{i-4}\}$ and $s_2 \in \{u_{i+1}, u_{i+2}, v_{i+4}\}$.

By Lemma 8, $f(u_k) \neq 3$ for each $k \in \{1, 2, \cdots, n\}$, and thus, $\{f(u_{i+1}), f(u_{i+2}), f(u_{i-2}), f(u_{i-1})\} = \{0, 2\}$.

Then, we have $f(v_{i+4}) = f(v_{i-4}) = 0$. Otherwise, $f(v_{i+4}) \neq 0$ or $f(v_{i-4}) \neq 0$. By symmetry, we may assume without loss of generality that $f(v_{i+4}) \neq 0$. Thus, we have $f(u_{i+1}) = f(u_{i+2}) = 0$. Since u_{i+1}, u_{i+2} need to be double Roman dominated, we have $f(v_{i+1}) = f(u_{i+3}) = 3$, contradicting Lemma 8.

Now, it is sufficient to consider the following three cases.

Case 2.2.1: $f(u_{i+1}) = f(u_{i-1}) = 2$.

By Lemma 6, we have $f(u_{i\pm2}) = f(v_{i\pm1}) = 0$.

Since u_{i+2} needs to be double Roman dominated and by Lemma 8, we have $f(u_{i+3}) = 2$. Since v_{i+1} needs to be double Roman dominated, we have $f(v_{i+3}) \geq 2$. Thus, there exists an edge $e \in E^f_{\{2,2^+\}}$, a contradiction with Lemma 6.

Case 2.2.2: $f(u_{i+2}) = f(u_{i-2}) = 2$.

By Lemma 6, we have $f(u_{i\pm3}) = f(u_{i\pm1}) = 0$.

Since u_{i+1}, u_{i-1} need to be double Roman dominated, we have $f(v_{i\pm1}) = 2$. Thus, there exists an edge $e \in E^f_{\{2,2\}}$, a contradiction with Lemma 6.

Case 2.2.3: $f(u_{i+1}) = f(u_{i-2}) = 2$.

By Lemma 6, we have $f(u_{i-3}) = f(v_{i+1}) = f(u_{i+2}) = 0$.

Since u_{i+2} needs to be double Roman dominated, we have $f(u_{i+3}) = 2$. By Lemma 6, we have $f(v_{i+3}) = f(u_{i+4}) = 0$. Since u_{i+4} needs to be double Roman dominated and by Lemma 8, we have $f(u_{i+5}) = 2$. Since v_{i+3} needs to be double Roman dominated, we have $f(v_{i+5}) \geq 2$. Thus, there exists an edge $e \in E^f_{\{2,2^+\}}$, a contradiction with Lemma 6. □

Lemma 10. *Let $n \geq 5$ and $n \not\equiv 0 \ (\mathrm{mod}\ 5)$. If f is a γ_{dR}-function of $P(n,2)$, then $w(f) \geq \lceil \frac{8n}{5} \rceil + 1$.*

Proof. Suppose to the contrary that $w(f) = \lceil \frac{8n}{5} \rceil$. By Lemmas 8 and 9, we have $|V_3| = 0$. Now, we have:

Claim 1. $|V_2 \cap N(v)| = 2$ *for any $v \in V(P(n,2))$ with $f(v) = 0$.*

Proof. Suppose to the contrary that there exists a vertex $v \in V(P(n,2))$ with $f(v) = 0$ and $|V_2 \cap N(v)| = 3$. We consider the following two cases.

Case 1: $v = u_i$ for some $i \in [n]$.

Since $|V_2 \cap N(v)| = 3$, we have $f(u_{i-1}) = f(u_{i+1}) = f(v_i) = 2$. By Lemma 6, we have $f(u_{i\pm 2}) = 0$, $f(v_{i\pm 1}) = 0$ and $f(v_{i\pm 2}) = 0$. Since v_{i+1} needs to be double Roman dominated, we have $f(v_{i+3}) = 2$. Since u_{i+2} needs to be double Roman dominated, we have $f(u_{i+3}) = 2$. Since $v_{i+3}u_{i+3} \in E^f_{\{2,2\}}$, contradicting Lemma 6.

Case 2: $v = v_i$ for some $i \in [n]$.

Since $|V_2 \cap N(v)| = 3$, we have $f(v_{i-2}) = f(v_{i+2}) = f(u_i) = 2$. By Lemma 6, we have $f(u_{i\pm 1}) = f(u_{i\pm 2}) = f(v_{i\pm 4}) = 0$. Since u_{i+1} needs to be double Roman dominated, we have $f(v_{i+1}) = 2$. Since u_{i-1} needs to be double Roman dominated, we have $f(v_{i-1}) = 2$. Since $v_{i+1}v_{i-1} \in E^f_{\{2,2\}}$, contradicting Lemma 6. □

We assume without loss of generality that $f(u_i) = 2$. By Lemma 6, we have $f(u_{i-1}) = 0$, $f(v_i) = 0$ and $f(u_{i+1}) = 0$. Since v_i needs to be double Roman dominated, we assume without loss of generality that $f(v_{i-2}) = 2$. By Claim 1, we have $f(v_{i+2}) = 0$. Since $f(v_{i-2}) = 2$, together with Lemma 6, we have $f(u_{i-2}) = 0$. Since u_{i-1} needs to be double Roman dominated, we have $f(v_{i-1}) = 2$. Then, by Lemma 6, we have $f(v_{i+1}) = 0$. Since v_{i+2} needs to be double Roman dominated, we have $f(u_{i+2}) = 2$. That is to say, we have:

$$f(\mathcal{B}_i) = f\begin{pmatrix} u_{i-2} & u_{i-1} & u_i & u_{i+1} & u_{i+2} \\ v_{i-2} & v_{i-1} & v_i & v_{i+1} & v_{i+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

By repeatedly applying Claim 1 and Lemma 6, $f(x)$ can be determined for each $x \in \mathcal{B}_{i+5}$, and we have $f(\mathcal{B}_i) = f(\mathcal{B}_{i+5})$. It is straightforward to see that $w(f) = \lceil \frac{8n}{5} \rceil$ only if $n \equiv 0 \pmod 5$, a contradiction. □

3. Conclusions

By Lemma 1, Proposition 2 and Lemma 10, we have

Theorem 1. *If $n \geq 5$, then:*

$$\gamma_{dR}(P(n,2)) = \begin{cases} \lceil \frac{8n}{5} \rceil, & n \equiv 0 \pmod 5, \\ \lceil \frac{8n}{5} \rceil + 1, & n \equiv 1,2,3,4 \pmod 5. \end{cases}$$

Remark 1. *Beeler et al. [7] proposed the concept of the double Roman domination. They showed that $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$. Moreover, they suggested to find double Roman graphs.*

In [17], it was proven that:

Theorem 2. *If $n \geq 5$, then $\gamma(P(n,2)) = \lceil \frac{3n}{5} \rceil$.*

Therefore, we have that $P(n,2)$ is not double Roman for all $n \geq 5$.

In fact, there exist many double Roman graphs among Petersen graph $P(n,k)$. For example, in [12], it was shown that $P(n,1)$ is a double Roman graph for any $n \not\equiv 2 \pmod 4$. Therefore, it is interesting to find other Petersen graphs that are double Roman.

Author Contributions: Z.S. contributes for supervision, methodology, validation, project administration and formal analysing. H.J., P.W., Y.R., J.-B.L. contribute for resources, some computations and wrote the initial draft of the paper which were investigated and approved by Z.S. and J.-B.L. wrote the final draft.

Funding: This research work is supported by Key Supported Disciplines of Guizhou Province Computer Application Technology (No. QianXueWeiHeZi ZDXX[2016]20); Anhui Province Key Laboratory of Intelligent Building & Building Energy Saving.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Xu, G. 2-rainbow domination in generalized Petersen graphs $P(n,3)$. *Discret. Appl. Math.* **2009**, *157*, 2570–2573. [CrossRef]
2. Garey, M.R.; Johnson, D.S. *Computers and Intractability: A Guide to the Theory of NP-Completeness*; W. H. Freeman and Co.: San Francisco, CA, USA, 1979.
3. Henning, M.A. A Characterization of Roman trees. *Discuss. Math. Graph Theory* **2002**, *22*, 325–334. [CrossRef]
4. Ahangar, H.A.; Amjadi, J.; Sheikholeslami, S.M.; Volkmann, L.; Zhao, Y. Signed Roman edge domination numbers in graphs. *J. Comb. Optim.* **2016**, *31*, 333–346. [CrossRef]
5. Ore, O. *Theory of Graphs*; American Mathematical Society: Providence, RI, USA, 1967.
6. Shao, Z.; Xu, J.; Sheikholeslami, S.M.; Wang, S. The domination complexity and related extremal values of large 3D torus. *Complexity* **2018**, 3041426. [CrossRef]
7. Beeler, R.A.; Haynes, T.W.; Hedetniemi, S.T. Double Roman domination. *Discret. Appl. Math.* **2016**, *211*, 23–29. [CrossRef]
8. Liu, C.H.; Chang, G.J. Roman domination on 2-connected graphs. *SIAM J. Discret. Math.* **2012**, *26*, 193–205. [CrossRef]
9. Liu, C.H.; Chang, G.J. Upper bounds on Roman domination numbers of graphs. *Discret. Math.* **2012**, *312*, 1386–1391. [CrossRef]
10. Liu, C.H.; Chen, S.H.; Chang, G.J. Edge Roman domination on graphs. *Graphs Comb.* **2016**, *32*, 1731–1747.
11. Pushpam, P.R.; Mai, T.N. Roman domination in unicyclic graphs. *J. Discret. Math. Sci. Cryptogr.* **2012**, *15*, 237–257. [CrossRef]
12. Shao, Z.; Jiang, H.; Li, Z.; Wu, P.; Zerovnik, J.; Zhang, X. Discharging approach for double Roman domination in graphs. *IEEE Access* **2018**, accepted for publication.
13. Shao, Z.; Klavžar, S.; Li, Z.; Wu, P.; Xu, J. On the signed Roman k-domination: Complexity and thin torus graphs. *Discret. Appl. Math.* **2017**, *233*, 175–186. [CrossRef]
14. Zhang, X.; Li, Z.; Jiang, H.; Shao, Z. Double Roman domination in trees. *Inf. Process. Lett.* **2018**, *134*, 31–34. [CrossRef]
15. Li, Z.; Shao, Z.; Lang, F.; Zhang, X.; Liu, J.B. Computational complexity of outer-Independent total and total Roman domination numbers in trees. *IEEE Access* **2018**, *6*, 35544–35550. [CrossRef]
16. Tong, C.; Lin, X.; Yang, Y.; Luo, M. 2-rainbow domination of generalized Petersen graphs $P(n,2)$. *Discret. Appl. Math.* **2009**, *157*, 1932–1937. [CrossRef]
17. Ebrahimi, B.J.; Jahanbakht, N.; Mahmoodian, E.S. Vertex domination of generalized Petersen graphs. *Discret. Math.* **2009**, *309*, 4355–4361. [CrossRef]