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Controlled Metric Type Spaces and the Related Contraction Principle

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Abstract: In this article, we introduce a new extension of b-metric spaces, called controlled metric type spaces, by employing a control function $\alpha(x,y)$ of the right-hand side of the b-triangle inequality. Namely, the triangle inequality in the new defined extension will have the form, $d(x,y) \leq \alpha(x,z)d(x,z) + \alpha(z,y)d(z,y)$, for all $x,y,z \in X$. Examples of controlled metric type spaces that are not extended b-metric spaces in the sense of Kamran et al. are given to show that our extension is different. A Banach contraction principle on controlled metric type spaces and an example are given to illustrate the usefulness of the structure of the new extension.

Keywords: fixed point; controlled metric type space; extended b-metric space

MSC: 47H10; 54H25

1. Introduction

Fixed point theory has widespread applications in different fields science and Engineering [1–5]. The core of the proof of existence and uniqueness theorems for the solutions of ordinary and fractional initial and boundary value problems depends on applying different fixed point theorems. The most applied fixed point theorem is the Banach contraction principle, which has been generalized by either modifying the contractive type condition or by working on a more generalized metric type space (see [6–11]). The b-metric space [12,13] and its partial versions, which extends the metric space by modifying the triangle equality metric axiom by inserting a constant multiple $s \geq 1$ to the right-hand side, is one of the most applied generalizations for metric spaces (see [14–20]).

Very recently, the authors in [21] introduced a type of extended b-metric spaces by replacing the constant s by a function $\theta(x,y)$ depending on the parameters of the left-hand side of the triangle inequality. In this article, we shall define a different type extension for b-metric spaces by replacing s by a function $\alpha(x,y)$ to act separately on each term in the right-hand side of the triangle inequality as mentioned in the abstract. Then, we give examples to show that this extension is different from extended b-metric spaces in the sense of Kamran et al. [21]. We also prove the corresponding Banach fixed point theorem on controlled metric type spaces and we provide an illustrating example.

2. Preliminary Assertions

In 2017, Kamran et al. [21] initiated the concept of extended b-metric spaces.

Definition 1. *[21] Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$. An extended b-metric is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:*

1. $d(x, y) = 0 \Longleftrightarrow x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq \theta(x, y)[d(x, z) + d(z, y)]$.

We generalize the concept of b-metric spaces in a different way as follows:

Definition 2. *Given a nonempty set X and $\alpha : X \times X \to [1, \infty)$. The function $d : X \times X \to [0, \infty)$ is called a controlled metric type if*
(d1) $d(x, y) = 0$ if and only if $x = y$,
(d2) $d(x, y) = d(y, x)$,
(d3) $d(x, y) \leq \alpha(x, z)d(x, z) + \alpha(z, y)d(z, y)$,
for all $x, y, z \in X$. The pair (X, d) is called a controlled metric type space.

Remark 1. *If, for all x, y in X, $\alpha(x, y) = s \geq 1$, then (X, d) is a b-metric space, which leads us to conclude that every b-metric space is a controlled metric type space. In addition, a controlled metric type space is not in general an extended b-metric space when taking the same function, that is, in the case $\theta = \alpha$. The following examples explain this fact.*

Example 1. *Choose $X = \{1, 2, \ldots\}$. Take $d : X \times X \to [0, \infty)$ as*

$$d(x, y) = \begin{cases} 0, & \Longleftrightarrow \; x = y, \\ \frac{1}{x}, & \text{if } x \text{ is even and } y \text{ is odd}, \\ \frac{1}{y}, & \text{if } x \text{ is odd and } y \text{ is even}, \\ 1, & \text{otherwise}. \end{cases}$$

Consider $\alpha : X \times X \to [1, \infty)$ as

$$\alpha(x, y) = \begin{cases} x, & \text{if } x \text{ is even and } y \text{ is odd}, \\ y, & \text{if } x \text{ is odd and } y \text{ is even}, \\ 1, & \text{otherwise}. \end{cases}$$

(d1) and (d2) are clearly true. We shall prove that (d3) holds.
Case 1: If $z = x$ or $z = y$, (d3) is satisfied.
Case 2: If $z \neq x$ and $z \neq y$, (d3) holds when $x = y$. From now on, suppose that $x \neq y$. Then, we have $x \neq y \neq z$. It is also obvious that (d3) holds in all following possible subcases:
Subcase 1: x, z are even and y is odd;
Subcase 2: x is even and y, z are odd;
Subcase 3: x, z are odd and y is even;
Subcase 4: x, z are even and y is odd;
Subcase 5: x, y, z are even;
Subcase 6: x, y are even and z is odd;
Subcase 7: x, y are odd and z is even;
Subcase 8: x, y, z are odd.
Consequently, d is a controlled metric type.

On the other hand, for $n = 2, 3, \ldots$, we have

$$d(2n + 1, 4n + 1) = 1 > \frac{1}{n} = \alpha(2n + 1, 4n + 1)[d(2n + 1, 2n) + d(2n, 4n + 1)],$$

that is, d is not an extended b-metric for the same function $\alpha = \theta$.

Example 2. *Take $X = \{0, 1, 2\}$. Consider the function d given as*

$$d(0,0) = d(1,1) = d(2,2) = 0,$$

and

$$d(0,1) = d(1,0) = 1, \quad d(0,2) = d(2,0) = \frac{1}{2}, \quad d(1,2) = d(2,1) = \frac{2}{5}.$$

Take $\alpha : X \times X \to [1, \infty)$ to be symmetric and be defined by

$$\alpha(0,0) = \alpha(1,1) = \alpha(2,2) = \alpha(0,2) = 1, \quad \alpha(1,2) = \frac{5}{4}, \quad \alpha(0,1) = \frac{11}{10}.$$

It is easy to see that d is a controlled metric type.
Note that

$$d(0,1) = 1 > \frac{99}{100} = \alpha(0,1)[d(0,2) + d(2,1)].$$

Thus, d is not an extended b-metric for the same function $\alpha = \theta$.

We define Cauchy and convergent sequences in controlled metric type spaces as follows:

Definition 3. *Let (X, d) be a controlled metric type space and $\{x_n\}_{n \geq 0}$ be a sequence in X.*
(1) We say that the sequence $\{x_n\}$ converges to some x in X, if, for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(2) We say that the sequence $\{x_n\}$ is Cauchy, if, for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$.
(3) The controlled metric type space (X, d) is called complete if every Cauchy sequence is convergent.

Definition 4. *Let (X, d) be a controlled metric type space. Let $x \in X$ and $\varepsilon > 0$.*
(i) The open ball $B(x, \varepsilon)$ is

$$B(x, \varepsilon) = \{y \in X, d(x, y) < \varepsilon\}.$$

(ii) The mapping $T : X \to X$ is said continuous at $x \in X$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $T(B(x, \delta)) \subseteq B(Tx, \varepsilon)$.

Clearly, if T is continuous at x in the controlled metric type space (X, d), then $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to \infty$.

3. Main Results

Now, we are ready to prove our main result corresponding to the Banach contraction principle on controlled metric type spaces.

Theorem 1. *Let (X, d) be a complete controlled metric type space. Let $T \colon X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq kd(x, y), \tag{1}$$

for all $x, y \in X$, where $k \in (0, 1)$. For $x_0 \in X$, take $x_n = T^n x_0$. Suppose that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})} \alpha(x_{i+1}, x_m) < \frac{1}{k}. \tag{2}$$

In addition, assume that, for every $x \in X$, we have

$$\lim_{n\to\infty} \alpha(x_n, x) \text{ and } \lim_{n\to\infty} \alpha(x, x_n) \text{ exist and are finite.} \tag{3}$$

Then, T has a unique fixed point.

Proof. Consider the sequence $\{x_n = T^n x_0\}$. By using (1), we get

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) \quad \text{for all } n \geq 0.$$

For all natural numbers $n < m$, we have

$$\begin{aligned}
d(x_n, x_m) &\leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m) d(x_{n+1}, x_m) \\
&\leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m)\alpha(x_{n+1}, x_{n+2}) d(x_{n+1}, x_{n+2}) \\
&+ \alpha(x_{n+1}, x_m)\alpha(x_{n+2}, x_m) d(x_{n+2}, x_m) \\
&\leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m)\alpha(x_{n+1}, x_{n+2}) d(x_{n+1}, x_{n+2}) \\
&+ \alpha(x_{n+1}, x_m)\alpha(x_{n+2}, x_m)\alpha(x_{n+2}, x_{n+3}) d(x_{n+2}, x_{n+3}) \\
&+ \alpha(x_{n+1}, x_m)\alpha(x_{n+2}, x_m)\alpha(x_{n+3}, x_m) d(x_{n+3}, x_m) \\
&\leq \dots \\
&\leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) d(x_i, x_{i+1}) \\
&+ \prod_{k=n+1}^{m-1} \alpha(x_k, x_m) d(x_{m-1}, x_m) \\
&\leq \alpha(x_n, x_{n+1}) k^n d(x_0, x_1) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) k^i d(x_0, x_1) \\
&+ \prod_{i=n+1}^{m-1} \alpha(x_i, x_m) k^{m-1} d(x_0, x_1) \\
&\leq \alpha(x_n, x_{n+1}) k^n d(x_0, x_1) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) k^i d(x_0, x_1) \\
&+ \left(\prod_{i=n+1}^{m-1} \alpha(x_i, x_m) \right) k^{m-1} \alpha(x_{m-1}, x_m) d(x_0, x_1) \\
&= \alpha(x_n, x_{n+1}) k^n d(x_0, x_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^{i} \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) k^i d(x_0, x_1) \\
&\leq \alpha(x_n, x_{n+1}) k^n d(x_0, x_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^{i} \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) k^i d(x_0, x_1).
\end{aligned}$$

Above, we make use of that $\alpha(x, y) \geq 1$. Let

$$S_p = \sum_{i=0}^{p} \left(\prod_{j=0}^{i} \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) k^i.$$

Hence, we have

$$d(x_n, x_m) \leq d(x_0, x_1) \left[k^n \alpha(x_n, x_{n+1}) + (S_{m-1} - S_n) \right]. \tag{4}$$

Condition (2), by using the ration test, guarantees that $\lim_{n\to\infty} S_n$ exists and hence the real sequence $\{S_n\}$ is Cauchy. Finally, if we take the limit in the inequality (4) as $n, m \to \infty$, we deduce that

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0, \tag{5}$$

that is, $\{x_n\}$ is a Cauchy sequence in the complete controlled metric type space (X, d), so $\{x_n\}$ converges to some $u \in X$. We shall show that u is a fixed point of T. The triangle inequality yields that

$$d(u, x_{n+1}) \leq \alpha(u, x_n)d(u, x_n) + \alpha(x_n, x_{n+1})d(x_n, x_{n+1}).$$

Using (2), (3) and (5), we deduce that

$$\lim_{n\to\infty} d(u, x_{n+1}) = 0. \tag{6}$$

Using again the triangle inequality and (1),

$$\begin{aligned} d(u, Tu) &\leq \alpha(u, x_{n+1})d(u, x_{n+1}) + \alpha(x_{n+1}, Tu)d(x_{n+1}, Tu) \\ &\leq \alpha(u, x_{n+1})d(u, x_{n+1}) + k\alpha(x_{n+1}, Tu)d(x_n, u). \end{aligned}$$

Taking the limit as $n \to \infty$ and taking (3) and (6) into view, we deduce that $d(u, Tu) = 0$, that is, $Tu = u$. Finally, assume that T has two fixed points, say u and v. Thus,

$$d(u, v) = d(Tu, Tv) \leq kd(u, v),$$

which holds unless $d(u, v) = 0$, so $u = v$. Hence, T has a unique fixed point. □

We illustrate Theorem 1 by the following example.

Example 3. *Consider $X = \{0, 1, 2\}$. Let d be symmetric and given as*

$$d(x, x) = 0 \quad \text{for each } x \in X,$$

and

$$d(0, 1) = 1, \quad d(0, 2) = \frac{11}{20}, \quad d(1, 2) = \frac{2}{5}.$$

Take $\alpha : X \times X \to [1, \infty)$ to be symmetric and be defined by

$$\alpha(0, 0) = \alpha(1, 1) = \alpha(2, 2) = \alpha(0, 2) = 1, \quad \alpha(1, 2) = \frac{49}{40}, \quad \alpha(0, 1) = \frac{11}{10}.$$

Clearly, d is an α-b-metric. Consider the self map T on X as

$$T0 = 2 \quad \text{and} \quad T1 = T2 = 1.$$

Choose $k = \frac{4}{5}$. Clearly, (1) holds. For any $x_0 \in X$, (2) is satisfied. All hypotheses of Theorem 1 hold, and so the T has a unique fixed point, which is $u = 1$.

Note that we can not apply the standard Banach contraction principle on metric spaces.

Definition 5. *Let $T : X \longrightarrow X$. For some $x_0 \in X$, let $O(x_0) = \{x_0, Tx_0, T^2x_0, ...\}$ be the orbit of x_0. A function $H : X \longrightarrow \mathbb{R}$ is said to be T-orbitally lower semi-continuous at $v \in X$ if for $\{x_n\} \subset O(x_0)$ such that $x_n \longrightarrow v$, we have $H(v) \leq \lim_{n\to\infty} \inf H(x_n)$.*

Similar to [21], we can employ Definition 5, to state the following consequent of Theorem 1, which generalizes Theorem 1 in [22].

Corollary 1. *Let (X, d) be a complete controlled metric type space. Let $x_0 \in X$ and $T\colon X \to X$ be a given mapping. Suppose that there exists $k \in (0,1)$ such that*

$$d(Tz, T^2z) \leq kd(z, Tz), \; for \; each \; z \in O(x_0). \tag{7}$$

Take $x_n = T^n x_0$. Suppose that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})} \alpha(x_{i+1}, x_m) < \frac{1}{k}. \tag{8}$$

Then, $x_n \to u \in X$ (as $n \to \infty$). Moreover, such u verifies $Tu = u$ if and only if the functional $x \mapsto d(x, Tx)$ is $T-$orbitally lower semi-continuous at u.

Remark 2. *Notice that the condition (3) is not needed in Corollary 1. It is replaced by the lower semi-continuity condition. In Theorem 2 of Kamran et al. [21], the continuity of d was used. As we see in Theorem 1, the continuity of d is not required and it is replaced by condition (3). In addition, (2) is the analogue of the condition on θ in Theorem 2 of [21] is rewritten differently, but in the correct form.*

4. Perspectives

It is an open question regarding the treatment of the cases of Kannan contraction, Chatterjee contraction, Hardy-Rogers contraction, Ćirić contraction and Suzuki contraction.

5. Conclusions

We summarize our conclusions as follow.

(1) As an extension of b-metric spaces, we defined a controlled metric type space by employing a control function $\alpha(x, y) \geq 1$ to the right-hand side of the triangle inequality.
(2) We gave an example of a controlled metric type space which is not an extended b-metric space in the sense of Kamran et al. in [21].
(3) We proved a contraction principle in the newly defined controlled meetric type space and concluded a fixed point result under T-orbitally lower semi-continuity assumption.
(4) An example is given to illustrate the proven contraction principle.
(5) Open problems have been stated in the Perspectives Section for possible future works.

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