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Role of Measurement Incompatibility and Uncertainty in Determining Nonlocality

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Abstract: It has been recently shown that measurement incompatibility and fine grained uncertainty—a particular form of preparation uncertainty relation—are deeply related to the nonlocal feature of quantum mechanics. In particular, the degree of measurement incompatibility in a no-signaling theory determines the bound on the violation of Bell-CHSH inequality, and a similar role is also played by (fine-grained) uncertainty along with steering, a subtle non-local phenomenon. We review these connections, along with comments on the difference in the roles played by measurement incompatibility and uncertainty. We also discuss why the toy model of Spekkens (Phys. Rev. A 75, 032110 (2007)) shows no nonlocal feature even though steering is present in this theory.

Keywords: joint measurability; incompatibility; uncertainty relations

1. Introduction

Quantum theory (QT), admittedly the most accurate mathematical description of physical world, is considered as one of the greatest achievements of 20th century science. It was developed to explain a number of newly-discovered microscopic phenomena which the classical theory could not explain. From a fundamental point of view, a number of concepts of this theory depart in various ways from that of the classical physics. At the time of the birth of quantum mechanics, two central non-classical concepts that appeared were the uncertainty principle [1] and the complementarity principle [2–4]. Then came the most surprising concept, namely, quantum nonlocality [5,6]. These nonclassical features of QT were treated as distinct concepts until very recently. Interestingly, in 2010, Oppenheim and Wehner related nonlocality with the uncertainty principle [7]. In another important piece of work, Banik et al. connected nonlocality with the complementarity principle [8]. More explicitly, the results in [7,8] successfully explain a very peculiar feature of quantum nonlocality, namely its limited strength. The nonlocal strength of quantum correlations is restricted in comparison with other possible correlations compatible with the relativistic causality principle. This restricted behavior is manifested by the amount of optimal violation of the well known Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality [9]. Whereas Cirel'son showed that maximum violation of the Bell-CHSH inequality in quantum theory is limited to $2\sqrt{2}$ [10], Popescu and Rohrlich provided an example of a hypothetical correlation [11] (called the PR-correlation) which satisfies the relativistic causality principle but exhibits stronger nonlocal (with respect to violation of the Bell-CHSH inequality) behavior than quantum correlations. In [7], the authors have introduced a fine-grained version of the uncertainty principle and showed that uncertainty, along with steering, determines the aforesaid restricted nonlocality in quantum theory. On the other hand, in [8], it has been shown that the same restricted behavior can be explained by complementarity principle expressed in terms of measurement incompatibility. In the present article, we critically review these connections, which are not limited only to QT, but extend to a much broader framework of convex operational theories. We then study these connections in a toy theory (introduced by Sppekkens [12]) whose state space does not have a convex structure in strict sense.

The paper is organized as follows: In Section 2, we briefly describe the mathematical framework of convex operational theories, the concept of joint measurability, and the concept of fine-grained uncertainty relation. Section 3 reviews the connection between complementarity principle and nonlocality. Here we also present an alternative proof of a result by Wolf et al., which shows that measurement incompatibility in QT always results in the violation of the Bell-CHSH inequality [13]. The connection between the uncertainty principle and nonlocality has been reviewed in Section 4. Section 5 studies similar connections in the framework of Spekkens' toy theory and draws some interesting conclusions, and Section 6 concludes the article.

2. Mathematical Prerequisites

Before going to the core discussion, in this section, we briefly describe the mathematical framework for convex operational theories followed by the description of joint measurability in quantum theory as well as in the more general convex operational theories. The fine-grained uncertainty relations are discussed towards the end of this section.

2.1. Convex Operational Theories

The framework was initially introduced in the 1960s by researchers in quantum foundations who used it to investigate axiomatic derivations of the Hilbert space formalism of quantum mechanics from operational postulates [14–17]. Due to the emphasis on the convex structure of the set of states and the use of operations to model state transformations, the approach is called the convex state approach. The basic motivation behind this framework is to explain the experimental phenomena in an operational approach. So, the theories considered in this framework are also specified under a common name, called operational theories. Recently, the framework has gained renewed interest from researchers in quantum information science, and the theories encapsulated in this framework are also known as generalized probabilistic theories (GPT's) [18–20].

State space: The set of states Ω, in which a system S can be prepared, is commonly assumed to be a convex subset of a real vector space V. The convexity corresponds to the ability to define a preparation procedure as a probabilistic mixture of different preparation procedures corresponding to other states; i.e., for every two states $\omega_1,\ \omega_2 \in \Omega$, their convex combination $C_{\omega_1,\omega_2} := \{p\omega_1 + (1-p)\omega_2 \mid 0 \leq p \leq 1\}$ is contained in Ω. The extremal points of Ω are referred to as pure states, and the states which can be written as convex combinations of other states are called mixed states. A d-dimensional system (i.e., a system having d degrees of freedom) is classical if and only if its state space Ω is the convex hull of $d-1$ linearly independent pure states (a simplex), in which case Ω can be thought of as the set of probability distributions over $d-1$ distinct possibilities. For a quantum system S, the (convex) set of density operators (states) $\mathcal{D}(\mathcal{H}_S)$ is contained in the real vector space Herm $(\mathcal{H}_S)$ of Hermitian operators on Hilbert space $\mathcal{H}_S$ associated with the system S.

Observables: The set of affine (i.e., non-negative) functionals on Ω forms an ordered linear space $\mathcal{A}(\Omega)$, with the ordering given point-wise: For $f, g \in \mathcal{A}(\Omega)$, we have $f \geq g$ if and only if $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$. $\mathcal{A}(\Omega)$ is an ordered unit space, with unit being defined as u, such that $u(\omega) = 1$ for all $\omega \in \Omega$. The set of effects on Ω is taken to be the unit interval $[0, u] \subset \mathcal{A}(\Omega)$ which is denoted as:

$$\mathcal{E}(\Omega) := \{e \in A(\Omega) \mid 0 \leq e(\omega) \leq 1, \text{ for all } \omega \in \Omega\} \tag{1}$$

$\mathcal{E}(\Omega)$ is the convex hull of the unit effect, the zero effect, and the set of extremal effects; it is a subset of the vector space V^*, which is dual to the vector space V. In this convex framework, one can, however, define unnormalized states as well as unnormalized effects. The collection of unnormalized states forms a convex positive cone lying in V^+. Similarly, the collection of unnormalized effects from the corresponding dual positive cone lying in $V^{*,+}$.

Let us denote the set of measurements or observables, performed on a system by $\mathcal{M}$. Consider, for simplicity, that a measurement $M \in \mathcal{M}$ has some finite set $\mathcal{K}_M = \{1, 2, \ldots, n\}$ of possible outcomes. In the abstract state space formalism, M is represented by a set of functionals $M \equiv \{e_1^M, e_2^M, ..., e_n^M\}$, where $e_j^M \in \mathcal{E}(\Omega)$ is associated with the measurement outcome $j \in \mathcal{K}_M$ for the measurement of M. The value $e_j^M(\omega)$ (lying between 0 and 1) denotes the probability of getting outcome j for a measurement of the observable M on the system prepared in the state ω.

Joint system: The joint system AB will have its own state space, Ω_{AB}, which is convex by definition. Under the assumptions of *no-signalling* and *local tomography*, it can be shown that Ω_{AB} must lie between two extremes, the *maximal* tensor product $\Omega_A \otimes_{max} \Omega_B$ and the *minimal* tensor product $\Omega_A \otimes_{min} \Omega_B$ [21].

A state is called a product state if it can be written in the form $\omega_A \otimes \omega_B$ for some states $\omega_A \in \Omega_A$ and $\omega_B \in \Omega_B$. States that can be written as probabilistic mixtures of product states are separable; i.e., $\omega_{AB}^{sep} = \sum_i p_i \omega_A^i \otimes \omega_B^i$ with $p_i \geq 0\ \forall i$ & $\sum_i p_i = 1$ and $\omega_A^i \in \Omega_A$, $\omega_B^i \in \Omega_B$. States that are not separable are entangled. A state ω_{AB} is entangled *iff* $\omega_{AB} \in \Omega_A \otimes_{max} \Omega_B$ but $\omega_{AB} \notin \Omega_A \otimes_{min} \Omega_B$. If either A or B is classical, then $\Omega_A \otimes_{min} \Omega_B = \Omega_A \otimes_{max} \Omega_B$, and there is no entanglement. In particular, if both are classical, then both $\Omega_A \otimes_{min} \Omega_B$ and $\Omega_A \otimes_{max} \Omega_B$ are the simplices whose vertices are ordered pairs of an extremal point of Ω_A and one of Ω_B. In general $\Omega_A \otimes_{min} \Omega_B \subset \Omega_{AB} \subset \Omega_A \otimes_{max} \Omega_B$; i.e., the inclusions are strict.

2.2. Incompatible Measurement, Unsharpness and Joint Measurability

Bohr's Complementarity principle is one of the central concepts in quantum mechanics [2]. One of the original versions of the complementarity principle states that there are observables in quantum mechanics that do not admit unambiguous joint measurement, and they are called incompatible. With the introduction of the generalized measurement—i.e., with the positive operator-valued measure (POVM)—it was shown that observables that do not admit perfect joint measurement may allow joint measurement if the measurements are made sufficiently fuzzy [22]. Moreover, one can mathematically generalize this notion in the abstract state space formalism introduced above.

Consider a system whose state space is denoted by Ω, and consider a two-outcome measurement M with outcomes denoted by "yes" $(+1)$ and "no" (-1). Let the effects associated with the outcomes be denoted by e_{yes}^M and e_{no}^M, respectively. The *unsharp* version $M^{(\lambda)}$ of a two-outcome measurement M is defined as $M^{(\lambda)} := \{e_{yes}^{M^{(\lambda)}}, e_{no}^{M^{(\lambda)}} \mid e_{yes}^{M^{(\lambda)}}(\omega) + e_{no}^{M^{(\lambda)}}(\omega) = 1 \text{ for all } \omega \in \Omega\}$. Here, $e_{yes}^{M^{(\lambda)}}(\omega)$ denotes the probability of obtaining outcome "yes" whenever the unsharp measurement $M^{(\lambda)}$ is performed on the system prepared in the state ω; similarly, $e_{no}^{M^{(\lambda)}}(\omega)$ denotes the probability of obtaining the other outcome, and these are related to the outcome probabilities of the sharp version M of the unsharp measurement $M^{(\lambda)}$ as follows:

$$e_{yes(no)}^{M^{(\lambda)}}(\omega) = \frac{1+\lambda}{2} e_{yes(no)}^{M}(\omega) + \frac{1-\lambda}{2} e_{no(yes)}^{M}(\omega) \tag{2}$$

with $\lambda \in [0, 1]$, called the "unsharpness parameter". Expectation value of the unsharp measurement $M^{(\lambda)}$ on the state ω is given by,

$$\langle M^{(\lambda)} \rangle_\omega := e_{yes}^{M^{(\lambda)}}(\omega) - e_{no}^{M^{(\lambda)}}(\omega) = \lambda \langle M \rangle_\omega \tag{3}$$

where $\langle M \rangle_\omega$ is the expectation value of the corresponding sharp measurement on the state ω.

Two dichotomic (i.e., two-outcome) observables M_1 and M_2 are jointly measurable if there exists a four outcome measurement $M_{12} \equiv \{e_{i,j}^{M_{12}};\ i,j \in \{yes, no\} \mid \sum_{i,j \in \{yes,no\}} e_{i,j}^{M_{12}}(\omega) = 1$ for all $\omega \in \Omega\}$ whose measurement statistics reproduces the measurement statistics of the observables M_1 and M_2 as marginals for every possible state of the system; i.e.,

$$\begin{aligned} \sum_{i \in \{yes,no\}} e_{i,j}^{M_{12}}(\omega) &= e_j^{M_2}(\omega)\ \forall\ j \\ \sum_{j \in \{yes,no\}} e_{i,j}^{M_{12}}(\omega) &= e_i^{M_1}(\omega)\ \forall\ i \end{aligned} \tag{4}$$

whatever be the choice of $\omega \in \Omega$.

Whenever the effects $e_{i,j}^{M_{12}}$ clicks in the joint measurement, it implies (according to the construction of M_{12}) that the effect $e_i^{M_1}$ clicks for the measurement M_1 and effect $e_j^{M_2}$ clicks for the second one.

Degree of incompatibility: It may be possible that observables which are not jointly measurable in a theory may admit joint measurement for their unsharp counterparts within that theory. For two given observables, the largest value of the unsharpness parameter up to which the joint measurement of their unsharp counterparts is still possible depends on the observables and is considered the degree of incompatibility of the two observables. If two observables are jointly measurable for the value λ of the unsharpness parameter, then they are also jointly measurable for any value of the unsharpness parameter within the interval $[0, \lambda]$. The value λ_{opt} that guarantees the existence of joint measurement for all possible pairs of dichotomic observables in a theory can be considered as the "degree of incompatibility" of the theory.

2.3. Fine-Grained Uncertainty Relation

The uncertainty principle postulates the existence of certain observables (such as position and momentum, spin of a particle in two different directions), all of which cannot be simultaneously but arbitrarily well defined in a quantum mechanical state. Conventionally, uncertainty relations in quantum theory have been expressed in terms of commutators and standard deviations [1]. Later, entropic measures were used to express the uncertainty relations [23]. Recently, in [7], the authors have introduced more fine-grained uncertainty relations formulated in terms of the probabilities of particular sets of possible outcomes for given sets of measurements on a quantum system.

Let $p(j^{(M)}|M)_\omega$ be the probability of getting outcome j for the measurement of the observable M on the system prepared in state ω. Consider a set of possible outcomes which we write as a string $\vec{x} := (x^{(M_1)}, x^{(M_2)}, \ldots, x^{(M_n)})$ for a set of measurement $\mathcal{M} := (M_1, M_2, \ldots, M_n)$ chosen with some probability distribution $\mathcal{D} = \{p(M_k)\}_k$. The uncertainty relation introduced in [23] is of the following form

$$P(\omega; \vec{x}) := \sum_{k=1}^{n} p(M_k) p(x^{(M_k)}|M_k)_\omega \leq \xi_{\vec{x}}(\mathcal{M}, \mathcal{D}) \tag{5}$$

where $P(\omega; \vec{x})$ is the probability of the string $\vec{x}$ corresponding to the set of measurements $\mathcal{M}$ (chosen with the probability distribution $\mathcal{D}$) on the system prepared in the state ω; $\xi_{\vec{x}}(\mathcal{M}, \mathcal{D})$ is obtained by a maximization over all possible states of the system under consideration:

$$\xi_{\vec{x}}(\mathcal{M}, \mathcal{D}) = \max_{\omega} \sum_{k=1}^{n} p(M_k) p(x^{(M_k)}|M_k)_\omega \tag{6}$$

It is noteworthy here that (5), in fact, represents a series of inequalities—one for each combination $\vec{x}$ of possible outcomes. These inequalities say that one cannot obtain a measurement outcome with certainty for all measurements simultaneously whenever $\xi_{\vec{x}}(\mathcal{M}, \mathcal{D}) < 1$. As an example, for a spin-1/2 quantum system if we consider binary spin observables σ_x and σ_z (chosen with equal probability), then the above relations reads:

$$\frac{1}{2}p(m|\sigma_x)_\rho + \frac{1}{2}p(n|\sigma_z)_\rho \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}, \; \forall \, \vec{x} = (m,n) \tag{7}$$

where $m,n \in \{+1,-1\}$ denote the outcomes of spin measurements for any state ρ of the system. The maximally certain states (i.e., the states achieving the right-hand side bound of the above equation) are given by the eigenstates of $(\sigma_x \pm \sigma_z)/\sqrt{2}$.

3. Measurement Incompatibility and Nonlocality

Undoubtedly, one of the most fundamental contradictions of quantum mechanics (QM) with classical physics gets manifested in its nonlocal behavior. This bizarre feature of QM was first established in the seminal work of J.S. Bell [5], where he showed that QM is incompatible with the local-realistic world view of classical physics. Bell showed this by means of an empirically testable inequality derived under the joint assumption of local-realism. There exists quantum states and observable-settings for which this inequality gets violated, and hence the corresponding correlations (measurement statistics) cannot be explained in any local-realistic theory. On one hand, quantum theory shows surprising nonlocal behavior; on the other hand, its nonlocal strength is peculiarly restricted in comparison to other nonlocal correlations compatible with the relativistic causality principle. This restricted feature is exhibited in the limited violation of the Bell-CHSH inequality. Is there a principle that determines this limited violation? In an interesting piece of work, Banik et al. [8] showed that the complementarity principle (incompatibility of two observables beyond a certain degree of unsharpness) can be one of the reasons for this limited behavior. They linked the "degree of incompatibility" of two dichotomic observables with the maximum violation of the Bell-CHSH inequality in a theory. The pivotal for this work of Banik et al. is a work by Andersson et al. [24] where the Bell-CHSH inequality was derived under an assumption different from that of local-realism. In this section, we present a comprehensive review on all these developments and supplement it with alternative derivations of some known results in this area of research.

Quantum nonlocality is based on two central features of quantum theory, namely entanglement and incompatible measurements—i.e., observation of quantum nonlocality implies the presence of both entanglement and incompatible measurements. Conversely, entanglement does not always imply nonlocality. There exist entangled states for which no form of quantum nonlocality can be demonstrated using non-sequential measurements [25–27]. However, in an important development, Wolf et al. showed that every pair of incompatible dichotomic quantum observables always leads to the violation of the Bell-CHSH inequality in QT [13]. That is, there exists a bipartite quantum state and a set of dichotomic observables at an added site together with which the given observables violate the Bell-CHSH inequality. In the following, we give an alternative proof of this result following the approach made in Reference [28]. This proof makes the bound on nonlocality in QT immediate.

Consider two binary measurements $\{E,(\mathbf{I}-E)\}$ and $\{F,(\mathbf{I}-F)\}$ with $0 \leq E,F \leq \mathbf{I}$, and consider their unsharp counterparts,

$$\begin{aligned} &\{E_\lambda \equiv \frac{1+\lambda}{2}E + \frac{1-\lambda}{2}(\mathbf{1}-E), \;\; (\mathbf{I}-E_\lambda)\} \\ &\{F_\lambda \equiv \frac{1+\lambda}{2}F + \frac{1-\lambda}{2}(\mathbf{1}-F), \;\; (\mathbf{I}-F_\lambda)\} \end{aligned} \tag{8}$$

Let λ_{EF} be the maximum value of λ such that the unsharp versions are jointly measurable. λ_{EF} can be found from the following cone-linear optimization problem,

$$\begin{aligned} &\text{Maximize} \;\; : \;\; \lambda \\ &\text{Subject to}: \;\; 0 \leq G \leq E_\lambda, F_\lambda \qquad \& \; E_\lambda + F_\lambda - \mathbf{I} \leq G \end{aligned} \tag{9}$$

The unsharp observable can also be expressed as $E_\lambda = \lambda E + \frac{1-\lambda}{2}\mathbf{I}$, which implies $E_\lambda/\lambda = E + \epsilon\mathbf{I}$, where $\epsilon = \frac{1-\lambda}{2\lambda}$. Let ϵ_{EF} be the minimum value of ϵ such that the unsharp observables are jointly measurable, and this value can be obtained from the following optimization,

$$\begin{aligned} &\text{Minimize} \quad : \quad \epsilon \\ &\text{Subject to}: \; 0 \leq G \leq E + \epsilon\mathbf{I}, F + \epsilon\mathbf{I} \qquad \& \; E + F - \mathbf{I} \leq G \end{aligned} \tag{10}$$

We have $\epsilon_{EF} = \frac{1-\lambda_{EF}}{2\lambda_{EF}}$. The optimization problem (10) can be cast into the following dual optimization problem,

$$\begin{aligned} \text{Maximize} \quad &: \quad \mathrm{Tr}[\sigma_3(E + F - \mathbf{I})] - \mathrm{Tr}[\sigma_2 E] - \mathrm{Tr}[\sigma_1 F] \\ \text{Subject to}: \quad & \quad \sigma_3 \leq \sigma_1 + \sigma_2 = \rho \text{ (a density operator)} \\ & \quad \sigma_1, \sigma_2, \sigma_3 \geq 0 \end{aligned} \tag{11}$$

$|\psi_{AB}\rangle$ is a bipartite state such that $\mathrm{Tr}_A[|\psi_{AB}\rangle\langle\psi_{AB}|] = \rho$. Let $\{P, (\mathbf{I} - P)\}$ be the projective measurement corresponding to the observable $A_1 = (+1)P + (-1)(\mathbf{I} - P) = \mathbf{I} - 2P\}$; similarly, let $\{Q, (\mathbf{I} - Q)\}$ be the projective measurement corresponding to the observable $A_2 = (+1)Q + (-1)(\mathbf{I} - Q) = \mathbf{I} - 2Q\}$, both being performed on the subsystem A such that $\mathrm{Tr}_A[(P \otimes \mathbf{I})|\psi_{AB}\rangle\langle\psi_{AB}|] = \sigma_1$ and $\mathrm{Tr}_A[(Q \otimes \mathbf{I})|\psi_{AB}\rangle\langle\psi_{AB}|] = \sigma_3$. Consider two measurements B_1 and B_2 on B subsystem such that $B_1 = (+1)(\mathbf{I} - F) + (-1)F = 1 - 2F$ and $B_2 = (+1)E + (-1)(\mathbf{I} - E) = 2E - 1$. The Bell-CHSH expression for the set of measurements takes the form,

$$\langle Bell - CHSH \rangle = \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \tag{12}$$

$\langle\ \rangle$ denotes expectation, and here expectations are taken on the state $|\psi_{AB}\rangle$. We have thus,

$$\begin{aligned} \langle A_1 B_1 \rangle &= \langle\psi_{AB}|(\mathbf{I} - 2P) \otimes (\mathbf{I} - 2F)|\psi_{AB}\rangle \\ &= \langle\psi_{AB}|\mathbf{I} \otimes \mathbf{I} - 2P \otimes \mathbf{I} - \mathbf{I} \otimes 2F + 2P \otimes 2F)|\psi_{AB}\rangle \\ &= 1 - 2\mathrm{Tr}[\sigma_1] - 2\mathrm{Tr}[\rho F] + 4\mathrm{Tr}[\sigma_1 F] \end{aligned}$$

Similar expressions can be calculated for other $\langle A_i B_j \rangle$, that finally give,

$$\langle Bell - CHSH \rangle = 2 + 4\left(\mathrm{Tr}[\sigma_3(E + F - \mathbf{I})] - \mathrm{Tr}[\sigma_2 E] - \mathrm{Tr}[\sigma_1 F]\right) \tag{13}$$

which, by using the dual optimization problem in Equation (11), takes the form: $\langle Bell - CHSH \rangle = 2(1 + 2\epsilon_{EF}) = 2/\lambda_{EF}$.

If the observables B_1 and B_2 are not jointly measurable, $\lambda_{EF} < 1$, which implies the violation of the Bell-CHSH inequality. Later, in this section, we show how the bound on quantum nonlocality (i.e., the Cirelson's bound) easily follows from this alternative proof.

3.1. Connection between Incompatibility and Nonlocality in a Generalized No-Signaling Theory

As a consequence of this sufficiency of incompatibility of the observables for quantum nonlocality, Wolf et al. concluded that these observables must remain incompatible within any generalized no-signalling theory [13]. In this context, one can ask whether there is any general connection between incompatibility of observables and nonlocality in a generalized no-signaling theory. It is also important whether the nonlocality of such a theory (in terms of optimal Bell violation) can be quantified by the degree of incompatibility (defined for a pair of observables) of the theory. We discuss some of the works which answer this below.

Bell's inequalities are usually derived under the assumptions of local-realism. In [24], Andersson et al. have derived the Bell-CHSH inequality under a different set of assumptions, namely under the assumption of the existence of joint measurement of observables for one of the parties along with the assumption of no signaling. It is important to note that if joint measurement exists on both sides, then the four (joint) probability distribution exists, and the BI follows immediately from Fine's result [29].

Consider a composite system composed of two subsystems shared between the two observers Alice and Bob with the composite and the individual state spaces denoted by Ω_{AB} and Ω_A, Ω_B respectively.

- *Andersson et al.*: Let M_1 and M_2 be the two dichotomic ($\{+1,-1\}$-valued) observables on Alice's side which are jointly measurable, and Bob on his side can measure any of the two dichotomic observables N_1 and N_2 ($\{+1,-1\}$-valued), and if the probabilities for the results that Alice obtains do not depend on what Bob measures (no-signalling), then

$$|\langle M_1N_1\rangle_{\eta_{AB}} + \langle M_1N_2\rangle_{\eta_{AB}} + \langle M_2N_1\rangle_{\eta_{AB}} - \langle M_2N_2\rangle_{\eta_{AB}}| \leq 2 \tag{14}$$

for any composite system state $\eta_{AB} \in \Omega_{AB}$, where $\langle MN\rangle_{\eta_{AB}} := e^{M,N}_{yes,yes}(\eta_{AB}) - e^{M,N}_{yes,no}(\eta_{AB}) - e^{M,N}_{no,yes}(\eta_{AB}) + e^{M,N}_{no,no}(\eta_{AB})$.

From this, it follows that (justified below):

- For any pair of dichotomic observables M_1 and M_2 on Alice's side (independent of whether they are jointly measurable) and for any measurement pair N_1 and N_2 (both dichotomic) on Bob's side under the constraint of no-signalling,

$$|\langle M_1N_1\rangle_{\eta_{AB}} + \langle M_1N_2\rangle_{\eta_{AB}} + \langle M_2N_1\rangle_{\eta_{AB}} - \langle M_2N_2\rangle_{\eta_{AB}}| \leq 2/\lambda^{M_1,M_2} \tag{15}$$

where λ^{M_1,M_2} is the degree of incompatibility between the two observables M_1 and M_2 (maximum allowed unsharpness parameter for the pair M_1 and M_2 up to which the joint measurement of their unsharp counterparts is still possible, cf. Section 2.2).

- For any pair of dichotomic measurements M_1 and M_2 on Alice's side and for any measurement pair N_1 and N_2 on Bob's side, in a no-signalling theory

$$|\langle M_1N_1\rangle_{\eta_{AB}} + \langle M_1N_2\rangle_{\eta_{AB}} + \langle M_2N_1\rangle_{\eta_{AB}} - \langle M_2N_2\rangle_{\eta_{AB}}| \leq 2/\lambda_{opt} \tag{16}$$

where λ_{opt} is the degree of incompatibility of the underlying theory (cf. Section 2.2).

Equations (15) and (16) can be obtained from Equation (14) by using the fact that $\langle M^{(\lambda)}N\rangle_{\eta_{AB}} = \lambda\langle MN\rangle_{\eta_{AB}}$, where $M^{(\lambda)}$ is the unsharp version of the measurement M (we refer to [8] for the detailed proof).

From inequality (16), it is clear that the amount of Bell violation is upper bounded by the incompatibility of a theory quantified by the quantity λ_{opt}. For example, in classical theory, joint measurement of any two dichotomic observables is possible, which means $\lambda_{opt} = 1$. Contrary to this, in [8], it was shown that, in quantum mechanics, the value of λ_{opt} is $1/\sqrt{2}$, and hence the amount of nonlocality in quantum theory is restricted to the Cirel'son bound—i.e., to $2\sqrt{2}$. Earlier, in this section, we showed that the Bell-CHSH function for a bipartite quantum system is related to λ_{EF} (the maximum value of unsharp parameter for a pair of observables B_1 and B_2 such that their unsharp versions are jointly measurable) as $\langle Bell - CHSH\rangle = 2/\lambda_{EF}$. If we set $\lambda_{EF} = \lambda_{opt} = \frac{1}{\sqrt{2}}$, the Cirel'son bound will immediately follow. On the other hand, in [28], the authors have shown that for PR-box theory, the value of λ_{opt} is $1/2$, and correspondingly, the Bell-CHSH value is 4, the algebraic optimal of the expression.

At this point it should be noted that the degree of incompatibility of a theory puts a limit on the maximum strength of CHSH inequality violations of the theory via the relation expressed in Equation (16). However, this result does not tell whether a theory saturates this bound or not. In [28], the authors have found a sufficient condition when the optimal value is saturated. They have shown that under an additional assumption on the physical theory—namely, that it supports a sufficient degree of steering (more specifically, if the theory supports *uniform universal steering*)—the bound can be saturated.

The concepts of measurement incompatibility can be extended for more than two observables with an arbitrary number of outcomes for each observable. However, it has been shown very recently that in this generalized scenario, measurement incompatibility does not always imply Bell nonlocality, but implies a weaker form of nonlocality—namely, steering [30–33].

4. Fine-Grained Uncertainty and Nonlocality

In order to establish the link between the strength of nonlocality and uncertainty in a particular theory, Oppenheim and Wehner considered the famous Bell-CHSH inequality and cast it in the form of a game [7]. In a typical Bell game, two parties, Alice and Bob, receive questions $s \in \mathcal{S}$ and $t \in \mathcal{T}$, respectively, from a third party (verifier). These questions are chosen according to some input distribution $p(s,t)$, which, for the sake of simplicity, we take as $p(s,t) = p(s)p(t)$. Alice and Bob then return answers $a \in \mathcal{A}$ and $b \in \mathcal{B}$ to the verifier, who then, according to some fixed set of rules, decides whether Alice and Bob win by giving answers a and b to questions s and t. To win the game, Alice and Bob may agree on any strategy beforehand, but can no longer communicate once the game starts. The CHSH game is an example of the simplest Bell game where the questions received by Alice and Bob are binary and so are their answers; i.e., $\mathcal{S} = \{0,1\}$ and also $\mathcal{T} = \{0,1\}$. The verifier declares them as the winners if their answers satisfy $a \oplus b = s.t$.

Consider now those runs where Alice gets the question $s = 0$. In these runs, Bob needs to give the same answer as that of Alice in order to win the game. Similarly, for $s = 1$, Bob needs to give the same answer as Alice if he receives $t = 0$, but the opposite answer if $t = 1$. We represent Bob's answer by $x_{s,a}^{(t)}$. Then, the winning answers would be

$$
\begin{aligned}
x_{0,0}^{(0)} = x_{0,0}^{(1)} = x_{1,0}^{(0)} = x_{1,1}^{(1)} = 0 \\
x_{0,1}^{(0)} = x_{0,1}^{(1)} = x_{1,0}^{(1)} = x_{1,1}^{(0)} = 1
\end{aligned}
$$

As mentioned above, Alice and Bob may agree on any strategy beforehand, but can no longer communicate once the game starts. In any theory, such a strategy consists of a choice of shared state σ_{AB} as well as measurements. For any particular strategy, the probability of Alice and Bob winning the game, thus given as

$$
P^{\text{game}}(\mathcal{S},\mathcal{T},\sigma_{A,B}) = \sum_{s,t} p(s,t) \sum_{a} p(a, b = x_{s,a}^{(t)} | s,t)_{\sigma_{AB}} \tag{17}
$$

where in the right hand side $p(a, b = x_{s,a}^{(t)} | s,t)_{\sigma_{AB}}$ denotes the probability of Alice and Bob giving the answers a and $b = x_{s,a}^{(t)}$ (where $x_{s,a}^{(t)}$ are according to Equation (17)), respectively, when they receive the questions s and t, respectively, from the verifier. The maximum winning probability (maximized over all possible strategies for Alice and Bob) is thus given by

$$
P_{\max}^{\text{game}} = \max_{\mathcal{S},\mathcal{T},\sigma_{AB}} P^{\text{game}}(\mathcal{S},\mathcal{T},\sigma_{AB}) \tag{18}
$$

The maximum winning probability, $P_{\max}^{\text{game}}$ quantifies the strength of nonlocality for any theory. For the Bell-CHSH inequality, $P_{\max}^{\text{game}} = \frac{3}{4}$ in classical theories, $P_{\max}^{\text{game}} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ quantum mechanically and $P_{\max}^{\text{game}} = 1$ for PR-box theory.

In order to connect the nonlocality with the uncertainty, Oppenheim and Wehner [7] rewrote the probability of Alice and Bob winning the game (i.e., Equation (17)) as

$$\begin{aligned} P^{\text{game}}(\mathcal{S},\mathcal{T},\sigma_{A,B}) &= \sum_{\substack{s\\a}} p(s)p(a|s)\sum_t p(t)p(b=x_{s,a}^{(t)}|s,t,a)_{\sigma_{AB}} \\ &= \sum_{\substack{s\\a}} p(s)p(a|s)\{\sum_t p(t)p(b=x_{s,a}^{(t)}|t)\}_{\sigma_B^{s,a}} \end{aligned} \tag{19}$$

where $\sigma_B^{s,a}$ denotes the reduced state of Bob's system for the setting s and outcome a of Alice (Here $p(s,t)=p(s)p(t)$ has also been used). It is noteworthy here that the term in the parentheses in Equation (19) is upper bounded by the fine grained uncertainty relation (5) as follows

$$\sum_t p(t)p(b=x_{s,a}^{(t)}|t)\}_{\sigma_B^{s,a}} \leq \xi_{\vec{x}_{s,a}}(\mathcal{T},\mathcal{D}) \tag{20}$$

This, in fact, is a fine-grained uncertainty relation for Bob's system for a given (s,a); $\xi_{\vec{x}_{s,a}}(\mathcal{T},\mathcal{D})$ denotes the maximum of the left hand side of the above equation over all possible states of Bob's system. The fine-grained uncertainty relation (Equation (5)) thus limits the winning probability as

$$P^{\text{game}}(\mathcal{S},\mathcal{T},\sigma_{A,B}) \leq \sum_{\substack{s\\a}} p(s)p(a|s)\xi_{\vec{x}_{s,a}}(\mathcal{T}_{\text{opt}},\mathcal{D}) \tag{21}$$

Let $\{\sigma_B^{s,a}\}$ be the set of states that achieve the maximum value of the uncertainty expressions (these states are called maximally certain states) for each (s,a) when Bob's optimal measurements are given as $\mathcal{T}_{\text{opt}}$. Thus, to achieve the upper bound for winning the game, Alice, by her act of measurement on the system she possess, should be able to prepare Bob's system to these maximally certain states. Thus, the degree of nonlocality of any theory is determined by two factors— the strength of uncertainty relations (for Bob's optimal measurements) and strength of *steering* (which determines which states Alice can prepare at Bob's location by performing measurements on her system).

As an example, we consider the quantum theory, where for a spin-1/2 system, Bob's optimal measurements are σ_x and σ_z for which, as described in Section 2.3, $\xi_{\vec{x}_{s,a}}=\frac{1}{2}+\frac{1}{2\sqrt{2}}$. The maximally certain states are given by the eigenstates of $(\sigma_x\pm\sigma_z)/\sqrt{2}$. Thus, if Alice could steer Bob's state to these maximally certain states, they would be able to achieve the maximum winning probability—i.e., the degree of nonlocality would be determined solely by the strength of the uncertainty relation. This, indeed, is the case in quantum theory. If, as a part of their strategy, Alice and Bob share the singlet state then by measuring her system in the basis given by the eigenstates of $(\sigma_x+\sigma_z)/\sqrt{2}$ or $(\sigma_x-\sigma_z)/\sqrt{2}$, Alice can steer Bob's state to the said maximally certain states. On the other hand, classical theories are fully certain, but there is no steering present and hence these theories are local. PR box theories are fully steerable and fully certain, and hence the maximum value of the Bell-CHSH expression in these theories is 4 (and correspondingly the maximum winning probability of the CHSH game in these theories is 1).

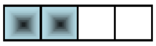
It is noteworthy that the connections discussed in this section and in the previous section hold good not only in Hilbert space quantum mechanics, but also in a more general convex structure discussed earlier. However, in the following, we will discuss these connections in a toy model introduced by Spekkens, which, truly speaking, does not belong to the said convex framework.

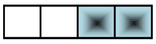
5. Spekkens' Toy Theory: Steerable But Local

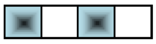
Spekkens has introduced a toy theory in order to argue for an epistemic view of quantum states; i.e., to argue that quantum states are states of knowledge rather than the states of reality [12]. The latter is called the ontic view for quantum states. This theory is based on a principle—namely, the *knowledge balance principle*—according to which the number of questions about the physical state of a system that are answered must always be equal to the number that are unanswered in a state of maximal knowledge.

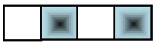
The most fine-grained description of a system in this toy theory is its ontic state, but we might not know exactly which of the ontic states the system is in, and hence our knowledge about the system is described by a probability distribution over the ontic states. This probability distribution is our epistemic state. The "knowledge balance principle" puts restrictions on the set of epistemic states that may be assigned to the system.

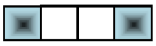
Elementary system: For the elementary system, the number of questions in the canonical set is two, and consequently the number of *ontic* states is four. Denote the four ontic states as "1", "2", "3", and "4". One can fully specify the ontic state of the system by asking the following set of "yes-no" questions: "Is it in the set $\{1,2\}$—yes or no?" and "Is it in the set $\{1,3\}$—yes or no?". An *epistemic* state is nothing but a probability distribution $\{(p_1, p_2, p_3, p_4)|\ p_i \geq 0\ \forall i\ \&\ \sum_{i=1}^{4} p_i = 1\}$ over the ontic states. Denoting disjunction by the symbol "$\vee$" (read as *or*), the six possible states of maximal knowledge (termed as the *pure epistemic* states) allowed by the knowledge balance principle, read as:

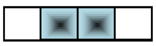
$$1 \vee 2 \leftrightarrow \{\frac{1}{2}, \frac{1}{2}, 0, 0\} =$$


$$3 \vee 4 \leftrightarrow \{0, 0, \frac{1}{2}, \frac{1}{2}\} =$$


$$1 \vee 3 \leftrightarrow \{\frac{1}{2}, 0, \frac{1}{2}, 0\} =$$


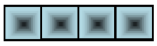
$$2 \vee 4 \leftrightarrow \{0, \frac{1}{2}, 0, \frac{1}{2}\} =$$


$$1 \vee 4 \leftrightarrow \{\frac{1}{2}, 0, 0, \frac{1}{2}\} =$$


$$2 \vee 3 \leftrightarrow \{0, \frac{1}{2}, \frac{1}{2}, 0\} =$$


For an epistemic state of the form $a \vee b$ (with $a \neq b$), a and b are its ontic supports.

For such a system, one has less than maximal knowledge if both questions in the canonical set are unanswered. This corresponds to the epistemic mixed state:

$$1 \vee 2 \vee 3 \vee 4 \leftrightarrow \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\} =$$


The mixed state has following different convex decompositions:

$$1 \vee 2 \vee 3 \vee 4 = (1 \vee 2) +_{cx} (3 \vee 4) \tag{22}$$
$$= (1 \vee 3) +_{cx} (2 \vee 4) \tag{23}$$
$$= (1 \vee 4) +_{cx} (2 \vee 3) \tag{24}$$

where $+_{cx}$ denotes the convex sum. The above set of decompositions can be thought of as different preparation procedures for the same mixed state—a phenomenon that can be observed in quantum theory. However, the convex combinations in this toy theory are not defined for arbitrary probability distributions, and hence the theory lacks the general mathematical structure discussed in Section 2.1.

Measurement: The knowledge balance principle imposes restrictions on the sort of possible measurements that can be implemented in this model. Compatible with this principle, the smallest number of ontic states that can be associated with a single outcome of a measurement is two. Thus, the only valid reproducible measurements are those which partition the four ontic states into two sets of two ontic states. There are only three such partitionings:

$$\begin{aligned} M_1 &\equiv \{1 \vee 2, 3 \vee 4\} \\ M_2 &\equiv \{1 \vee 3, 2 \vee 4\} \\ M_3 &\equiv \{1 \vee 4, 2 \vee 3\} \end{aligned} \tag{25}$$

If the initial epistemic state has its ontic support inside the ontic support of a particular outcome, then that outcome is certain to occur; otherwise, the outcome is not determined by the initial epistemic state. For instance, suppose the epistemic state is $1 \vee 2$, and the measurement M_1 is performed. Then, the first outcome is certain to occur. On the other hand, if the measurement M_2 is performed, then two outcomes occur with equal frequency. (At this point please note that, to define the measurement procedure completely, one needs to define the update rule in the toy theory which should be compatible with the *knowledge balance* principle. Such an appropriate rule is given in the original paper [12]. However, for our purpose this is not required.) Denoting the outcomes for the measurement M_k as $m_k \in \{0, 1\}$, the expressions of Equation (7) in this toy theory become

$$\frac{1}{2} p(m_k|M_k) + \frac{1}{2} p(m_l|M_l) \leq \frac{3}{4}, \; \forall \, \vec{x} = (m_k, m_l) \tag{26}$$

for any two measurements M_k and M_l, $k \neq l$. The bound is saturated on those epistemic states which have their ontic supports inside the ontic support of any of the outcomes of any measurement M_k or M_l. For example, if we consider the measurement M_1 and M_2, then the bound is saturated on the epistemic states $1 \vee 2$, $3 \vee 4$, $1 \vee 3$, and $2 \vee 4$. On the other hand, for the two epistemic states (i.e., on $1 \vee 4$ and $2 \vee 3$), the right hand side of the above inequality takes the value $1/2$.

Steering in toy theory: Compatible with the knowledge balance principle, there are two types of (pure) epistemic states for a pair of elementary systems:

(1) $(a \vee b).(c \vee d) \equiv (a.c) \vee (a.d) \vee (b.c) \vee (b.d)$; where $a, b, c, d \in \{1, 2, 3, 4\}$ and $a \neq b$, $c \neq d$.

(2) $(a.e) \vee (b.f) \vee (c.g) \vee (d.f)$; where $a, b, c, d, e, f, g, h \in \{1, 2, 3, 4\}$ and a, b, c, d are all different and same is for e, f, g, h,

where the symbol "." represents conjunction (read as "and").

For the second type of states, the state for marginal elementary systems (both) is $1 \vee 2 \vee 3 \vee 4$ (see [12] for detail). Let Alice share with Bob a bipartite elementary system prepared in the state $(1.1) \vee (2.2) \vee (3.3) \vee (4.4)$. If Alice implements the measurement that distinguishes $1 \vee 2$ from $3 \vee 4$ on her part of the system, then she will be able to remotely prepare Bob's system in decomposition of Equation (22). Similarly, implementing measurements that distinguishes $1 \vee 3$ from $2 \vee 4$ and $1 \vee 4$ from $2 \vee 3$, she can prepare the other two decompositions—i.e., the decompositions of Equations (23) and (24), respectively, which establishes steering-like phenomena for the toy-bit theory.

It has been recently established that in quantum mechanics a set of observables does not admit joint measurement if and only if it can be used to demonstrate steering [30–33]. From the spirit of the proof, it is obvious that if there is steering in any no signaling theory, then there must be incompatible measurements. Hence, in Spekkens' toy theory, the measurements that demonstrate steering would not admit joint measurement. With this we can say that nonlocality of this theory (violation of Bell-CHSH inequality) is bounded by a value $\frac{2}{\lambda_{opt}^{Toy}}$ which must be greater than 2 as $\lambda_{opt}^{Toy} < 1$. The necessary condition for achieving this bound in a no signaling theory is not known, though a sufficient condition has been provided for those theories whose state space has convex structure. Spekkens' toy model lacks this structure, and hence the analysis is not applicable. In this context, we show that though there is steering, the high amount of uncertainty (as expressed in Equation (26)) constrains the toy theory to satisfy Bell-CHSH inequality, and hence the toy bit theory is local. To get the optimal Bell-CHSH violation in this toy theory, consider a situation where Alice and Bob share a bipartite steerable state and Alice performs any of the two measurements of Equation (25). Now, to get the maximum certain value on the conditional states of his part steered by Alice, Bob needs to perform the same pair of measurements chosen by Alice, and Equation (26) shows that the Bell-CHSH value in this theory is correspondingly bounded by 3/4—i.e., the local bound of this inequality.

6. Concluding Remarks

The concept of uncertainty and incompatibility of measurements emerge with the birth of quantum mechanics, whereas quantum nonlocality, in its precise sense, was discovered long after the birth of QM. It is surprising that though apparently these three concepts seem to be uncorrelated, there are deep connections among them. Uncertainty (fine grained version) refers to dispersion property of the state, and the amount of maximal violation of Bell's inequality in a no signaling theory can be found by optimizing the Bell quantity over all possible steering and uncertainty of states. On the other hand, measurement incompatibility refers to the structure of observables over which the theory has to be built. In this sense, it is the stronger condition which is manifested by its capacity to offer the bound on maximal violation of Bell's inequality without any reference to steering. Steering here decides whether that bound will be achieved or not. In quantum mechanics, optimization over steering and uncertainty exactly reproduce the bound for Bell violation set by the degree of incompatibility in quantum mechanics, whereas Spekkens' toy theory is a nice example where uncertainty destroys the possibility of nonlocality created by the phenomena of steering and measurement incompatibility.

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Abbreviations

The following abbreviations are used in this manuscript:

Bell-CHSH	Bell-Clauser-Horne-Shimony-Holt
POVM	positive operator valued measure
QT	quantum theory
QM	quantum mechanics
BI	Bell's inequality

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