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Local $C^{0,1}$-Regularity for the Parabolic p-Laplacian Equation on the Group SU(3)

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Abstract: In this article, when $2 \leq p \leq 4$, we establish the $C^{0,1}_{\text{loc}}$-regularity of weak solutions to the degenerate parabolic p-Laplacian equation $\partial_t u = -\sum_{i=1}^{6} X_i^*(|\nabla_{\mathcal{H}} u|^{p-2} X_i u)$ on the group SU(3) granted with horizontal vector fields $X_1, \ldots, X_6$. Compared to the Heisenberg group, $\mathbb{H}^n$, we obtained the optimal range of p; that is, $2 \leq p \leq 4$.

Keywords: p-Laplacian type; $C^{0,1}_{\text{loc}}$-regularity; parabolic p-Laplacian; the group SU(3); range of p; Caccioppoli-type inequality

MSC: 35H20; 35B65

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1. Introduction

The study of the regularity for partial differential equations involving the p-Laplacian operator has always been a hot topic. In the Euclidean space, the $C^{0,1}$, $C^{1,\alpha}$, $W^{2,2}$-regularities and other second-order Sobolev regularities for the p-Laplacian equation have been proved in [1–7]. In recent years, there has been significant progress in the study of the regularity for the p-Laplacian equation in sub-Riemannian manifolds. Many scholars have made outstanding contributions. In the Heisenberg group, $\mathbb{H}^n$, Domokos-Manfredi [8,9], Manfredi-Mingione [10], Migione et al. [11], Ricciotti [12], and Zhong-Mukherjee [13,14] established the $C^{0,1}$ and $C^{1,\alpha}$-regularities for the p-Laplacian equation in the full range $1 < p < \infty$; Domokos [15] and Lie et al. [16] proved the $W^{2,2}$-regularity for the p-Laplacian equation in the range of $1 < p < 3 + \frac{1}{n-1}$ with $n \geq 2$. In the group SU(3), the $C^{0,1}$, $C^{1,\alpha}$, and $W^{2,2}$-regularities of the p-Laplacian equation were established by [17,18]. The method in [13,14] is extended by Citti-Mukherjee [19] to include Hörmander vector fields of step two, and the $C^{0,1}$ and $C^{1,\alpha}$-regularities for the p-Laplacian equation have been successfully established. The $C^{1,\alpha}$-regularity for inhomogeneous quasi-linear equations on the Heisenberg group $\mathbb{H}^n$ were established by [20,21] when $2 - \frac{1}{2n+2} < p < \infty$. New ideas and perspectives behind the development of research on regularity include certain hybrid-type Caccioppoli-type inequalities, as first proposed and introduced by Zhong [13]. In comparison, for the degenerate parabolic p-Laplacian equation, such inequalities are not applicable due to the differences in homogeneity between the time and spatial derivatives. Therefore, we need to find and create new methods and techniques to establish more suitable Caccioppoli-type inequalities.

In this study paper, we propose a new method to construct a crucial Caccioppoli-type inequality. Based on the inequality, when $2 \leq p \leq 4$, we establish the $C^{0,1}$-regularity for the parabolic p-Laplacian equation on the group SU(3). To be specific, we focus on a special type of unitary group composed of 3×3 complex matrices. We denote by SU(3) this unitary group and endow it with horizontal vector fields $X_1, X_2, \ldots, X_6$. More exhaustive geometries and properties of SU(3) are shown in Section 2. We select an open domain Ω in the group SU(3). For $T > 0$, we define a cylinder $\mathcal{Q} = \Omega \times (0, T)$, as first proposed in [22]. In $\mathcal{Q}$, we consider the following equation:

$$\partial_t u = -\sum_{i=1}^{6} X_i^* \mathcal{A}_i(\nabla_{\mathcal{H}} u) \quad \text{in } \mathcal{Q} = \Omega \times (0, T). \tag{1}$$

Here, X_i^* is the formal adjoint of X_i; $\nabla_{\mathcal{H}} = (X_1, X_2, \ldots, X_6)$ is the horizontal gradient; the vector function $\mathcal{A} := (\mathcal{A}_1, \ldots, \mathcal{A}_6) \in C^2(\mathbb{R}^6, \mathbb{R}^6)$ meets the following condition:

$$\begin{cases} v'|\zeta|^{p-2}|\varrho|^2 \leq \sum_{i,j=1}^{6} \mathcal{A}_{i,\zeta_j}(\zeta)\varrho_i\varrho_j \leq \Upsilon'|\zeta|^{p-2}|\varrho|^2, \\ |\mathcal{A}_i(\zeta)| \leq \Upsilon'|\zeta|^{p-1}. \end{cases} \tag{2}$$

for every $\zeta, \varrho \in \mathbb{R}^6$, where $\mathcal{A}_{i,\zeta_j}(\zeta) := \partial_{\zeta_j}\mathcal{A}_i(\zeta)$, $p \in [2, \infty)$ and $0 < v' \leq \Upsilon' < \infty$. If, for every function $\psi \in C_0^\infty(\mathcal{Q})$, the equation

$$\int_0^T \int_\Omega \partial_t u \psi \mathrm{d}x\mathrm{d}t = -\int_0^T \int_\Omega \sum_{i=1}^{6} \mathcal{A}_i(\nabla_{\mathcal{H}} u) X_i \psi \mathrm{d}x\mathrm{d}t \tag{3}$$

holds true, then we name the function $u \in L^p((0,T), W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega))$ as a weak solution to Equation (1). Here, $W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ is the first-order p-th integrable horizontal local Sobolev space, which is composed of total functions $f \in L^p_{\mathrm{loc}}(\Omega)$, whose distributional horizontal gradients are $\nabla_{\mathcal{H}} f \in L^p_{\mathrm{loc}}(\Omega)$. In the classic case, $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$, Equation (1) becomes the parabolic p-Laplacian equation:

$$\partial_t u = -\sum_{i=1}^{6} X_i^*(|\nabla_{\mathcal{H}} u|^{p-2} X_i u).$$

The study of the parabolic p-Laplacian equation originated from DiBenedetto-Friedman [22]. They established the $C^{1,\alpha}$-regularity of the weak solution in the Euclidean space; Wiegner [23] also proved the same result. For more exhaustive results on the parabolic p-Laplacian equation and more general cases in the Euclidean space, we refer to the book by DiBenedetto [24]. For the study of the parabolic p-Laplacian equation in the sub-Riemannian manifold, Capogna et al. [25] established, when $2 \leq p < \infty$, the C^∞-regularity of the weak solution to the non-degenerate parabolic p-Laplacian equation in the Heisenberg group $\mathbb{H}^n$, as follows:

$$\partial_t u = \sum_{i=1}^{2n} X_i((1 + |Xu|^2)^{\frac{p-2}{2}} X_i u).$$

Recently, for the degenerate parabolic p-Laplacian equation in the Heisenberg group, $\mathbb{H}^n$, when $2 \leq p \leq 4$, Capogna et al. [26] established the $C^{0,1}$-regularity of the weak solution.

In this study paper, we focus on the $C^{0,1}$-regularity of the weak solution u to (3) on SU(3). As a consequence, when $2 \leq p \leq 4$, we establish the $C^{0,1}_{\mathrm{loc}}$-regularity of u; that is, $\nabla_{\mathcal{H}} u \in L^\infty_{\mathrm{loc}}$. See Theorem 1 below for details.

Theorem 1. *Suppose $u \in L^p((0,T), W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega))$ is a weak solution to (1), satisfying condition (2), in $\mathcal{Q} = \Omega \times (0,T)$. Then, $\nabla_{\mathcal{H}} u \in L^\infty_{\mathrm{loc}}(\mathcal{Q})$ for $2 \leq p \leq 4$. Moreover, when $2 \leq p \leq 4$, for every $\mathcal{Q}_{\mu,2r} \subset \mathcal{Q}_{\mu,2r_0} \subset \mathcal{Q}$, we have the following:*

$$\sup_{\mathcal{Q}_{\mu,r}} |\nabla_{\mathcal{H}} u| \leq C\mu^{\frac{1}{2}} \max\left(\left(\frac{1}{\mu r^{N+2}} \int\int_{\mathcal{Q}_{\mu,2r}} (1 + |\nabla_{\mathcal{H}} u|^2)^{\frac{p}{2}} \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}, \mu^{\frac{p}{2(2-p)}}\right), \tag{4}$$

where $C = C(p, v, \Upsilon, r_0) > 0$, $\mathcal{Q}_{\mu,r} := B(x_0, r) \times (t_0 - \mu r^2, t_0)$ *and* $N = 10$ *is the homogeneous dimension of* SU(3).

Consequently, when $2 \le p \le 4$, *the weak solution to the parabolic p-Laplacian equation on* SU(3) *has the* $C^{0,1}$*-regularity and satisfies* (4).

To prove Theorem 1, it requires us to contemplate the following regularized equation:

$$\partial_t u_\sigma = \sum_{i=1}^{6} X_i \mathcal{A}^\sigma(\nabla_{\mathcal{H}} u_\sigma) \quad \text{in } \mathcal{Q}; \quad u_\sigma = u \quad \text{on } \partial_p \mathcal{Q}, \tag{5}$$

where u is a weak solution to (1), and $\partial_p \mathcal{Q} = \Omega \times \{t = 0\} \cup \partial\Omega \times (0, T)$ is the parabolic boundary of the cylinder $\mathcal{Q}$, with the following condition:

$$\begin{cases} v(\sigma + |\zeta|^2)^{\frac{p-2}{2}} |\varrho|^2 \le \sum_{i,j=1}^{6} \mathcal{A}^\sigma_{i,\zeta_j}(\zeta) \varrho_i \varrho_j \le \Upsilon(\sigma + |\zeta|^2)^{\frac{p-2}{2}} |\varrho|^2, \\ |\mathcal{A}^\sigma_i(\zeta)| \le \Upsilon(\sigma + |\zeta|^2)^{\frac{p-1}{2}}. \end{cases} \tag{6}$$

for every $\zeta, \varrho \in \mathbb{R}^6$, where $\sigma \in (0, 1]$, $\mathcal{A}^\sigma_{i,\zeta_j}(\zeta) := \partial_{\zeta_j} \mathcal{A}^\sigma_i(\zeta)$, $p \in [2, \infty)$ and $0 < v \le \Upsilon < \infty$. Here, from [17], since $\{X_i\}_{1 \le i \le 6}$ are the left-invariant vector fields, we have $X_i^* = -X_i$. Simultaneously, we also need to consider the Riemannian approximation equation (see Section 2 for details):

$$\partial_t u_\epsilon = \sum_{i=1}^{8} X_i^\epsilon \mathcal{A}^{\sigma,\epsilon}_i(\nabla^\epsilon u_\epsilon) \quad \text{in } \mathcal{Q}; \quad u_\epsilon = u_\sigma \quad \text{on } \partial_p \mathcal{Q}, \tag{7}$$

where u_σ is a weak solution to (5), with the following condition:

$$\begin{cases} v(\sigma + |\zeta|^2)^{\frac{p-2}{2}} |\varrho|^2 \le \sum_{i,j=1}^{8} \mathcal{A}^{\sigma,\epsilon}_{i,\zeta_j}(\zeta) \varrho_i \varrho_j \le \Upsilon(\sigma + |\zeta|^2)^{\frac{p-2}{2}} |\varrho|^2, \\ |\mathcal{A}^{\sigma,\epsilon}_i(\zeta)| \le \Upsilon(\sigma + |\zeta|^2)^{\frac{p-1}{2}}. \end{cases} \tag{8}$$

for every $\zeta, \varrho \in \mathbb{R}^8$, where $\mathcal{A}^{\sigma,\epsilon}_{i,\zeta_j}(\zeta) := \partial_{\zeta_j} \mathcal{A}^{\sigma,\epsilon}_i(\zeta)$, $p \in [2, \infty)$ and $0 < v \le \Upsilon < \infty$. Above, v, Υ depend only on v', Υ'. Let u_ϵ be a weak solution to (7). When $2 \le p < \infty$, we write $\mathcal{A}^\sigma(\zeta) = \mathcal{A}(\zeta) + v\sigma^{\frac{p-2}{2}} \zeta$ and $\mathcal{A}^{\sigma,\epsilon}_i(\zeta) = \tilde{\mathcal{A}}_i(\zeta_{\mathbb{H}}) + v(\sigma + |\zeta|^2_\epsilon)^{\frac{p-2}{2}} \zeta_i$; see ([26], Section 2) for details. The Riemannian approximation technique has become a mature technique widely used in studying equations; see [17,19,25,26] for the definition and more details of the technique. It is proven in [25,26] that $\mathcal{A}^\sigma \to \mathcal{A}$ and $u_\sigma \to u$ as $\sigma \to 0$, and that $\mathcal{A}^{\sigma,\epsilon} \to \mathcal{A}^\sigma$ and $u_\epsilon \to u_\sigma$ as $\epsilon \to 0$; also see [13,14,17,19] for an example.

Hence, to obtain Theorem 1, we only need to prove that $\{u_\epsilon\}_{\sigma,\epsilon \in (0,1]}$ have the following $C^{0,1}_{\text{loc}}$-regularity uniformly in $\sigma, \epsilon \in (0, 1]$. Finally, letting $\epsilon \to 0, \sigma \to 0$, from the following theorem, we can apply the standard method as [25,26] to derive Theorem 1.

Theorem 2. *Assume that* $u_\epsilon \in L^p((0, T), W^{1,p}_{\mathcal{H},\text{loc}}(\Omega))$ *is a weak solution to* (7) *with condition* (8)*, in* $\mathcal{Q} = \Omega \times (0, T)$*. If* $2 \le p \le 4$*, then* $\nabla^\epsilon u \in L^\infty_{\text{loc}}(\mathcal{Q})$*. Moreover, when* $2 \le p \le 4$*, for any* $\mathcal{Q}_{\mu,2r} \subset \mathcal{Q}_{\mu,2r_0} \subset \mathcal{Q}$*, we have the following:*

$$\sup_{\mathcal{Q}_{\mu,r}} |\nabla^\epsilon u_\epsilon| \le C\mu^{\frac{1}{2}} \max\left(\left(\frac{1}{\mu r^{N+2}} \int\int_{\mathcal{Q}_{\mu,2r}} (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p}{2}} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}}, \mu^{\frac{p}{2(2-p)}} \right), \tag{9}$$

where $C = C(p, v, \Upsilon, r_0) > 0$ *and* $\mathcal{Q}_{\mu,r} := B_\epsilon(x_0, r) \times (t_0 - \mu r^2, t_0)$.

The proof of Theorem 2 relies on Moser's iteration; see Section 4 for details. The key point, by the approach in [25,26], is to establish a crucial Caccioppoli-type estimate for

$\nabla^\epsilon u_\epsilon$ involving $\nabla^\epsilon \nabla^\epsilon u_\epsilon$ (see Lemma 6). To obtain the crucial Caccioppoli-type estimate, when $2 \leq p \leq 4$, we establish two Caccioppoli-type inequalities for $\nabla^\epsilon \nabla^\epsilon u_\epsilon$ and $\nabla_{\mathcal{R}} u_\epsilon$ in Lemmas 4 and 5, proven in Section 3. Applying Lemma 5 to re-estimate the integral terms on the right hand of (26) in Lemma 4, we prove the crucial Caccioppoli-type estimate in Section 3.

Consequently, we construct a crucial Caccioppoli-type inequality (38). Based on the inequality we establish, when $2 \leq p \leq 4$, the $C^{0,1}$-regularity for the parabolic p-Laplacian equation on the group SU(3). Compared to the Heisenberg group $\mathbb{H}^n$, our new result achieves the same range of p as [26]. Unfortunately, the $C^{0,1}$-regularity for the range $p \in (1,2) \cup (4,\infty)$ cannot be achieved with our current technology because our argument rests in a crucial way on Lemma 5 with the condition $p \in [2,4]$. The difficulties in the proof arise from handling and estimating integral terms involving $\nabla_{\mathcal{R}} \nabla^\epsilon u_\epsilon$. In the Heisenberg group $\mathbb{H}^n$, there exists the property that $[X_i, R] = 0$; however, it does not hold true on SU(3). For example, $[X_1, R_7] = 4X_2$ (see Table 1). This means that we need to handle more integral terms when estimating integral terms involving $\nabla_{\mathcal{R}} \nabla^\epsilon u_\epsilon$. Our approach can also be applied to more general sub-Riemannian manifolds. For instance, it can be used with a special class of semi-simple Lie groups as proposed in [17], and Hörmander vector fields of step two as discussed in [19], to establish the regularity for the parabolic p-Laplacian equation. Technically speaking, our method can also be extended to other types of partial differential equations, for example, the non-homogeneous equation $\partial_t u = -\sum_{i=1}^{6} X_i^* \mathcal{A}_i(\nabla_{\mathcal{H}} u) + \mathcal{B}(x,t,u,\nabla_{\mathcal{H}} u)$. The establishment of the regularity for the range of $p \in (1,2) \cup (4,\infty)$ will be the focus and difficulty of our next work.

Table 1. Lie bracket on SU(3).

	X_1	X_2	X_3	X_4	X_5	X_6	R_7	R_8
X_1	0	$-R_7$	X_5	$-X_6$	$-X_3$	X_4	$4X_2$	$2X_2$
X_2	R_7	0	X_6	X_5	$-X_4$	$-X_3$	$-4X_1$	$-2X_1$
X_3	$-X_5$	$-X_6$	0	$-R_8$	X_1	X_2	$2X_4$	$4X_4$
X_4	X_6	$-X_5$	R_8	0	X_2	$-X_1$	$-2X_3$	$-4X_3$
X_5	X_3	X_4	$-X_1$	$-X_2$	0	$R_8 - R_7$	$2X_6$	$-2X_6$
X_6	$-X_4$	X_3	$-X_2$	X_1	$R_7 - R_8$	0	$-2X_5$	$2X_5$
R_7	$-4X_2$	$4X_1$	$-2X_4$	$2X_3$	$-2X_6$	$2X_5$	0	0
R_8	$-2X_2$	$2X_1$	$-4X_4$	$4X_3$	$2X_6$	$-2X_5$	0	0

2. Preliminaries

The group SU(3) is a special type of unitary group composed of 3×3 complex matrices; that is,

$$\mathrm{SU}(3) := \{A \in \mathrm{GL}(3,\mathbb{C}) : A \cdot A^* = E, \det A = 1\},$$

where E is the identity matrix. The Lie algebra of SU(3) is defined by the following:

$$\mathrm{su}(3) := \{B \in \mathrm{gl}(3,\mathbb{C}) : B + B^* = 0, \mathrm{tr}B = 0\}$$

granted with the inner product $\langle B, C\rangle := -\frac{1}{2}\mathrm{tr}(BC)$.

The two-dimensional maximal torus on the group SU(3) is provided by the following:

$$\mathbb{S} := \left\{ \begin{pmatrix} e^{is_1} & 0 & 0 \\ 0 & e^{is_2} & 0 \\ 0 & 0 & e^{is_3} \end{pmatrix} : s_1, s_2, s_3 \in \mathbb{R}, s_1 + s_2 + s_3 = 0 \right\},$$

whose Lie algebra is as follows:

$$\mathcal{S} := \left\{ \begin{pmatrix} is_1 & 0 & 0 \\ 0 & is_2 & 0 \\ 0 & 0 & is_3 \end{pmatrix} : s_1, s_2, s_3 \in \mathbb{R}, s_1 + s_2 + s_3 = 0 \right\}$$

is selected as the Cartan subalgebra. The following Gell–Mann matrices form a set of the orthogonal basis of su(3), namely the following:

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -\frac{i}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{i}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2i}{\sqrt{3}} \end{pmatrix}.$$

The following two vector fields are generated from $[X_1, X_2]$ and$[X_3, X_4]$, respectively; that is,

$$R_7 = -[X_1, X_2] = \begin{pmatrix} -2i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R_8 = -[X_3, X_4] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}.$$

Since $S_1 = \frac{1}{2} R_7$ and $S_2 = \frac{1}{2\sqrt{3}} R_7 - \frac{1}{\sqrt{3}} R_8$, the vertical vector fields R_7, R_8 form a set of orthogonal basis of $\mathcal{S}$. Hence, the vertical gradient is defined by $\nabla_{\mathcal{R}} := (R_7, R_8)$.

We recall the Riemannian approximation technique. Given $\epsilon \in (0, 1]$, we define the Riemannian approximation to the vector fields $X_1, X_2, \ldots, X_6$, as

$$X_1^\epsilon = X_1, X_2^\epsilon = X_2, \ldots, X_6^\epsilon = X_6, X_7^\epsilon = \epsilon R_7, X_8^\epsilon = \epsilon R_8.$$

From which, we denote $\nabla^\epsilon = (X_1, \ldots, X_6, \epsilon R_7, \epsilon R_8)$ as the gradient,

The following table ([17], Table 2.1), shows the total Lie bracket for any two vector fields belonging to $\{X_1, \ldots, X_6, R_7, R_8\}$.

Table 1 shows that

$$[X_i, X_j] = v_{i,j}^{(k)} X_k + \theta_{i,j}^{(l)} R_l, \quad [X_i, R_j] = \vartheta_{i,j}^{(k)} X_k, \quad [R_i, R_j] = 0, \tag{10}$$

and that

$$[X_i^\epsilon, X_j^\epsilon] = v_{i,j}^{(k)} X_k + \theta_{i,j}^{(l)} R_l, \tag{11}$$

where $v_{i,j}^{(k)}, \theta_{i,j}^{(l)}, \vartheta_{i,j}^{(k)} \in \mathbb{R}$ are constants determined entirely by Table 1. From Table 1, it is not difficult for us to discover that the horizontal subspace $\mathcal{H}$ in SU(3) is generated by the set of orthogonal bases $\{X_1, X_2, \ldots, X_6\}$ satisfying the Hörmander condition. Hence, the horizontal gradient is defined by $\nabla_{\mathcal{H}} = (X_1, X_2, \ldots, X_6)$. Here, the basis $\{X_1, X_2, \ldots, X_6\}$ is left-invariant due to the left-invariance of the Gell–Mann matrices. To summarize, the basis $\{X_1, X_2, \ldots, X_6\}$ generates the horizontal distribution of a sub-Riemannian manifold.

3. Several Caccioppoli-Type Inequalities and a Crucial Caccioppoli-Type Estimate

In this section, we establish the crucial Caccioppoli-type estimate for $\nabla^\epsilon u_\epsilon$ involving $\nabla^\epsilon\nabla^\epsilon u_\epsilon$ and some Caccioppoli-type inequalities, which are uniform in $\sigma, \epsilon \in (0,1]$. The following two lemmas are prerequisites for the proofs of subsequent lemmas.

Lemma 1. *Suppose u_ϵ is a weak solution to (7). Then, $v_l^\epsilon = X_l^\epsilon u_\epsilon$, with $l = 1, \ldots, 8$, solves*

$$\partial_t v_l^\epsilon = \sum_{i,j=1}^{8} X_i^\epsilon(\mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) X_l^\epsilon X_j^\epsilon u_\epsilon) + \sum_{i=1}^{8} [X_l^\epsilon, X_i^\epsilon] \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon). \tag{12}$$

Proof. From (7), by the Lie bracket, we have the following:

$$\begin{aligned}
\partial_t v_l^\epsilon &= X_l^\epsilon \partial_t u_\epsilon = \sum_{i=1}^{8} X_l^\epsilon(X_i^\epsilon \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon)) \\
&= \sum_{i=1}^{8} X_i^\epsilon(X_l^\epsilon \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon)) + \sum_{i=1}^{8} [X_l^\epsilon, X_i^\epsilon] \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) \\
&= \sum_{i=1}^{8} X_i^\epsilon(\mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) X_l^\epsilon X_j^\epsilon u_\epsilon) + \sum_{i,j=1}^{8} [X_l^\epsilon, X_i^\epsilon] \mathcal{A}_i^{\delta,\epsilon}(\nabla^\epsilon u_\epsilon)
\end{aligned}$$

□

Lemma 2. *Suppose u_ϵ is a weak solution to (7). Then, $R_l u_\epsilon$, with $l = 7, 8$ solves the following:*

$$\begin{aligned}
\partial_t R_l u_\epsilon = &\sum_{i,j=1}^{8} X_i^\epsilon(\mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) X_j^\epsilon R_l u_\epsilon) + \sum_{i,j=1}^{8} X_i^\epsilon(\mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon)[R_l, X_j^\epsilon] u_\epsilon) \\
&+ \sum_{i=1}^{8} [R_l, X_i^\epsilon] \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon).
\end{aligned} \tag{13}$$

Proof. Letting $v_l^\epsilon = \epsilon R_l u_\epsilon$ in Lemma 1, we have the following:

$$\partial_t R_l u_\epsilon = \sum_{i,j=1}^{8} X_i^\epsilon(\mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) R_l X_j^\epsilon u_\epsilon) + \sum_{i=1}^{8} [R_l, X_i^\epsilon] \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon).$$

From this, by $R_l X_j^\epsilon = X_j^\epsilon R_l + [R_l, X_j^\epsilon]$, we obtain (13). □

3.1. Several Caccioppoli-Type Inequalities

The following lemma provides a Caccioppoli-type inequality for $\nabla_{\mathcal{R}} u_\epsilon$ involving $\nabla^\epsilon \nabla_{\mathcal{R}} u_\epsilon$.

Lemma 3. *Suppose u_ϵ is a weak solution to (7). Then, when $p \in (1, \infty)$, for every $\gamma \geq 0$ and every $\varrho \in C^1([0,T], C_0^\infty(\Omega))$, we have the following:*

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla^\epsilon \nabla_{\mathcal{R}} u_\epsilon|^2 \varrho^{4+\gamma} |\nabla_{\mathcal{R}} u_\epsilon|^\gamma \mathrm{d}x\mathrm{d}t \\
&\leq C \int_{t_1}^{t_2} \int_\Omega |\nabla^\epsilon \nabla_{\mathcal{R}} u_\epsilon|^{\gamma+2} \varrho^{3+\gamma} |\partial_t \varrho| \mathrm{d}x\mathrm{d}t \\
&\quad + C(\gamma+1)^2 \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla^\epsilon \varrho|^2 \varrho^{2+\gamma} |\nabla_{\mathcal{R}} u_\epsilon|^{\gamma+2} \mathrm{d}x\mathrm{d}t \\
&\quad + C(\gamma+1)^2 \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p}{2}} \varrho^{4+\gamma} |\nabla_{\mathcal{R}} u_\epsilon|^\gamma \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{14}$$

where $C = C(v, \Upsilon) > 0$.

Proof. Applying $\psi = \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma R_l u_\epsilon$ to test (13), we obtain the following:

$$
\begin{aligned}
L^l &= \int_{t_1}^{t_2} \int_\Omega \partial_t R_l u_\epsilon \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma R_l u_\epsilon \mathrm{d}x\mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{8} X_i^\epsilon (\mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) X_j^\epsilon R_l u_\epsilon) \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma R_l u_\epsilon \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{8} X_i^\epsilon (\mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) [R_l, X_j^\epsilon] u_\epsilon) \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma R_l u_\epsilon \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{8} [R_l, X_i^\epsilon] \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma R_l u_\epsilon \mathrm{d}x\mathrm{d}t = S_1^l + S_2^l + S_3^l.
\end{aligned}
\tag{15}
$$

For L^l, integrating by parts, we have the following:

$$
\sum_{l=7}^{8} L^l = \frac{1}{\gamma+2} \int_{t_1}^{t_2} \int_\Omega \partial_t (|\nabla_{\mathcal{R}} u_\epsilon|^{\gamma+2}) \varrho^2 \mathrm{d}x\mathrm{d}t = -\frac{2}{\gamma+2} \int_{t_1}^{t_2} \int_\Omega |\nabla_{\mathcal{R}} u_\epsilon|^{\gamma+2} \varrho \partial_t \varrho \mathrm{d}x\mathrm{d}t,
$$

which yields

$$
|\sum_{l=7}^{8} L^l| \le \frac{2}{\gamma+2} \int_{t_1}^{t_2} \int_\Omega |\nabla_{\mathcal{R}} u_\epsilon|^{\gamma+2} \varrho |\partial_t \varrho| \mathrm{d}x\mathrm{d}t. \tag{16}
$$

For S_1^l, integrating by parts, we have the following:

$$
\begin{aligned}
\sum_{l=7}^{8} S_1^l =& - \sum_{l=7}^{8} \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{8} \mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) X_j^\epsilon R_l u_\epsilon 2\varrho X_i^\epsilon \varrho |\nabla_{\mathcal{R}} u_\epsilon|^\gamma R_l u_\epsilon \mathrm{d}x\mathrm{d}t \\
& - \sum_{l=7}^{8} \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{8} \mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) X_j^\epsilon R_l u_\epsilon \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma X_i^\epsilon R_l u_\epsilon \mathrm{d}x\mathrm{d}t \\
& - \gamma \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{8} \mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) X_j^\epsilon (|\nabla_{\mathcal{R}} u_\epsilon|^2) \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^{\gamma-2} X_i^\epsilon (|\nabla_{\mathcal{R}} u_\epsilon|^2) \mathrm{d}x\mathrm{d}t \\
=& - S_{11} - S_{12} - S_{13}.
\end{aligned}
\tag{17}
$$

For S_2^l, integrating by parts, we have the following:

$$
\begin{aligned}
S_2^l =& - \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{8} \mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) [R_l, X_j^\epsilon] u_\epsilon 2\varrho X_i^\epsilon \varrho |\nabla_{\mathcal{R}} u_\epsilon|^\gamma R_l u_\epsilon \mathrm{d}x\mathrm{d}t \\
& - \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{8} \mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) [R_l, X_j^\epsilon] u_\epsilon \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma X_i^\epsilon R_l u_\epsilon \mathrm{d}x\mathrm{d}t \\
& - \gamma \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{8} \mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) [R_l, X_j^\epsilon] u_\epsilon \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^{\gamma-2} \sum_{k=7}^{8} R_k u_\epsilon X_i^\epsilon R_k u_\epsilon \mathrm{d}x\mathrm{d}t \\
=& - S_{21}^l - S_{22}^l - S_{23}^l.
\end{aligned}
\tag{18}
$$

For S_2^l, integrating by parts, we have the following:

$$\begin{aligned}S_3^l = & -\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^{8} \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) 2\varrho [R_l, X_i^\epsilon]\varrho |\nabla_{\mathcal{R}} u_\epsilon|^\gamma R_l u_\epsilon \mathrm{d}x\mathrm{d}t \\ & -\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^{8} \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon)\varrho^2 [R_l, X_i^\epsilon] R_l u_\epsilon \mathrm{d}x\mathrm{d}t \\ & -\gamma\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^{8} \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon)\varrho^2 R_l u_\epsilon |\nabla_{\mathcal{R}} u_\epsilon|^{\gamma-2} \sum_{k=7}^{8} R_k u_\epsilon [R_l, X_i^\epsilon] R_k u_\epsilon \mathrm{d}x\mathrm{d}t \\ = & -S_{31}^l - S_{32}^l - S_{33}^l. \end{aligned} \tag{19}$$

Combining (15) and (17)–(19), we obtain the following:

$$S_{12} + S_{13} = -\sum_{l=7}^{8} L^l - S_{11} - \sum_{l=7}^{8}\sum_{k=1}^{3}(S_{2k}^l + S_{3k}^l). \tag{20}$$

Now, we use the condition inequality to estimate each term in (20) separately. Applying condition (8) to estimate S_{12}, we obtain the following:

$$S_{12} \geq v \int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla^\epsilon \nabla_{\mathcal{R}} u_\epsilon|^2 \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma \mathrm{d}x\mathrm{d}t. \tag{21}$$

Applying condition (8) to estimate S_{13}, we obtain the following:

$$S_{13} \geq \frac{\gamma v}{4}\int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla^\epsilon (|\nabla_{\mathcal{R}} u_\epsilon|^2)|^2 \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^{\gamma-2} \mathrm{d}x\mathrm{d}t \geq 0. \tag{22}$$

Applying condition (8) to estimate S_{13}, we obtain the following:

$$|S_{11}| \leq 2\Upsilon \int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla^\epsilon \nabla_{\mathcal{R}} u_\epsilon| \varrho |\nabla^\epsilon \varrho| |\nabla_{\mathcal{R}} u_\epsilon|^{\gamma+1} \mathrm{d}x\mathrm{d}t. \tag{23}$$

Applying condition (8) to estimate S_{21}^l and S_{31}^l, we obtain the following:

$$|\sum_{l=7}^{8} S_{21}^l| + |\sum_{l=7}^{8} S_{31}^l| \leq 4\Upsilon \int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-1}{2}} \varrho |\nabla^\epsilon \varrho| |\nabla_{\mathcal{R}} u_\epsilon|^{\gamma+1} \mathrm{d}x\mathrm{d}t. \tag{24}$$

Applying condition (8) to estimate S_{22}^l, S_{32}^l, S_{23}^l, and S_{33}^l, we obtain the following:

$$\begin{aligned}& |\sum_{l=7}^{8} S_{22}^l| + |\sum_{l=7}^{8} S_{32}^l| + |\sum_{l=7}^{8} S_{23}^l| + |\sum_{l=7}^{8} S_{33}^l| \\ & \leq 4\Upsilon(\gamma+1)\int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-1}{2}} \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma |\nabla^\epsilon \nabla_{\mathcal{R}} u_\epsilon| \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{25}$$

Combining (16) and (20)–(25), by Young's inequality, we obtain the following:

$$\begin{aligned}& \int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla^\epsilon \nabla_{\mathcal{R}} u_\epsilon|^2 \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma \mathrm{d}x\mathrm{d}t \\ & \leq \frac{C}{\gamma+2}\int_{t_1}^{t_2}\int_\Omega |\nabla^\epsilon \nabla_{\mathcal{R}} u_\epsilon|^{\gamma+2} \varrho |\partial_t \varrho| \mathrm{d}x\mathrm{d}t \\ & \quad + C\int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla^\epsilon \varrho|^2 |\nabla_{\mathcal{R}} u_\epsilon|^{\gamma+2} \mathrm{d}x\mathrm{d}t \\ & \quad + C(\gamma+1)^2\int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p}{2}} \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon|^\gamma \mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $C = C(v, Y) > 0$. Setting $\varrho \to \varrho^{2+\gamma/2}$ in the above inequality, we obtain (14). □

The following lemma provides a Caccioppoli-type inequality for $\nabla^\epsilon\nabla^\epsilon u_\epsilon$.

Lemma 4. *Suppose u_ϵ is a weak solution to (7). Then, when $p \in (1, \infty)$, for every $\gamma \geq 0$ and every $\varrho \in C^1([0, T], C_0^\infty(\Omega))$, we have the following:*

$$\begin{aligned}
&\frac{1}{\gamma+2}\sup_{t_1<t<t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma+2}{2}}\varrho^2\mathrm{d}x+\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2+\gamma}{2}}|\nabla^\epsilon\nabla^\epsilon u_\epsilon|^2\varrho^2\mathrm{d}x\mathrm{d}t\\
&\leq C(\gamma+1)^2\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}}(\varrho^2+|\nabla^\epsilon\varrho|^2+\varrho|\nabla_{\mathcal{R}}\varrho|)\mathrm{d}x\mathrm{d}t\\
&\quad+C(\gamma+1)^4\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2+\gamma}{2}}|\nabla_{\mathcal{R}}u_\epsilon|^2\varrho^2\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{C}{\gamma+2}\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma+2}{2}}|\partial_t\varrho|\varrho\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{26}$$

where $C = C(n, p, v, Y) > 0$.

Proof. Applying $\psi = \varrho^2(\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}} X_l^\epsilon u_\epsilon$ to test (12), then integrating by parts, we obtain the following:

$$\begin{aligned}
L^l =&\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}}\partial_t((X_l^\epsilon u_\epsilon)^2)\varrho^2\mathrm{d}x\mathrm{d}t\\
=&-\int_{t_1}^{t_2}\int_\Omega\sum_{i,j=1}^{8}\mathcal{A}^{\sigma,\epsilon}_{i,\xi_j}(\nabla^\epsilon u_\epsilon)X_l^\epsilon X_j^\epsilon u_\epsilon 2\varrho X_i^\epsilon\varrho(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}}X_l^\epsilon u_\epsilon\mathrm{d}x\mathrm{d}t\\
&-\int_{t_1}^{t_2}\int_\Omega\sum_{i,j=1}^{8}\mathcal{A}^{\sigma,\epsilon}_{i,\xi_j}(\nabla^\epsilon u_\epsilon)X_l^\epsilon X_j^\epsilon u_\epsilon\varrho^2(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}}X_i^\epsilon X_l^\epsilon u_\epsilon\mathrm{d}x\mathrm{d}t\\
&-\frac{\gamma}{2}\int_{t_1}^{t_2}\int_\Omega\sum_{i,j=1}^{8}\mathcal{A}^{\sigma,\epsilon}_{i,\xi_j}(\nabla^\epsilon u_\epsilon)X_l^\epsilon X_j^\epsilon u_\epsilon\varrho^2X_l^\epsilon u_\epsilon(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma-2}{2}}X_i^\epsilon(|\nabla^\epsilon u_\epsilon|^2)\mathrm{d}x\mathrm{d}t\\
&-\int_{t_1}^{t_2}\int_\Omega\sum_{i=1}^{8}\mathcal{A}^{\sigma,\epsilon}_{i}(\nabla^\epsilon u_\epsilon)2\varrho[X_l^\epsilon,X_i^\epsilon]\varrho(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}}X_l^\epsilon u_\epsilon\mathrm{d}x\mathrm{d}t\\
&-\int_{t_1}^{t_2}\int_\Omega\sum_{i=1}^{8}\mathcal{A}^{\sigma,\epsilon}_{i}(\nabla^\epsilon u_\epsilon)\varrho^2(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}}[X_l^\epsilon,X_i^\epsilon]X_l^\epsilon u_\epsilon\mathrm{d}x\mathrm{d}t\\
&-\gamma\int_{t_1}^{t_2}\int_\Omega\sum_{i=1}^{8}\mathcal{A}^{\sigma,\epsilon}_{i}(\nabla^\epsilon u_\epsilon)\varrho^2X_l^\epsilon u_\epsilon(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma-2}{2}}\sum_{k=1}^{8}X_k^\epsilon u_\epsilon[X_l^\epsilon,X_i^\epsilon]X_k^\epsilon u_\epsilon\mathrm{d}x\mathrm{d}t\\
&=-S_1^l-S_2^l-S_3^l-S_4^l-S_5^l-S_6^l.
\end{aligned} \tag{27}$$

For S_2^l, we use $X_l^\epsilon X_j^\epsilon = X_j^\epsilon X_l^\epsilon + [X_l^\epsilon, X_j^\epsilon]$ to obtain the following:

$$\begin{aligned}
S_2^l=&\int_{t_1}^{t_2}\int_\Omega\sum_{i,j=1}^{8}\mathcal{A}^{\sigma,\epsilon}_{i,\xi_j}(\nabla^\epsilon u_\epsilon)X_j^\epsilon X_l^\epsilon u_\epsilon\varrho^2(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}}X_i^\epsilon X_l^\epsilon u_\epsilon\mathrm{d}x\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_\Omega\sum_{i,j=1}^{8}\mathcal{A}^{\sigma,\epsilon}_{i,\xi_j}(\nabla^\epsilon u_\epsilon)[X_l^\epsilon,X_j^\epsilon]u_\epsilon\varrho^2(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}}X_i^\epsilon X_l^\epsilon u_\epsilon\mathrm{d}x\mathrm{d}t\\
=&S_{21}^l+S_{22}^l.
\end{aligned} \tag{28}$$

For S_3^l, we use $X_l^\epsilon X_j^\epsilon = X_j^\epsilon X_l^\epsilon + [X_l^\epsilon, X_j^\epsilon]$ to obtain the following:

$$\begin{aligned}S_3^l =&\frac{\gamma}{4}\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{8} \mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon)X_j^\epsilon((X_l^\epsilon u_\epsilon)^2)\varrho^2(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma-2}{2}}X_i^\epsilon(|\nabla^\epsilon u_\epsilon|^2)\mathrm{d}x\mathrm{d}t\\ &\frac{\gamma}{2}\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{8} \mathcal{A}_{i,\xi_j}^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon)[X_l^\epsilon, X_j^\epsilon]u_\epsilon\varrho^2 X_l^\epsilon u_\epsilon(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma-2}{2}}X_i^\epsilon(|\nabla^\epsilon u_\epsilon|^2)\mathrm{d}x\mathrm{d}t\\ =&S_{31}^l + S_{32}^l.\end{aligned} \tag{29}$$

Combining (27)–(29), we obtain

$$L^l + S_{21}^l + S_{31}^l = -S_1^l - S_{22}^l - S_{32}^l - S_4^l - S_5^l - S_6^l. \tag{30}$$

Now, we use the condition inequality to estimate each term in (30) separately. To bound the first term in the left hand of (26), we note the following:

$$\frac{1}{\gamma+2}\left(\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma+2}{2}}\varrho^2\mathrm{d}x\right)\Bigg|_{t_1}^{t_2} = 2L^l + \frac{2}{\gamma+2}\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma+2}{2}}\varrho\partial_t\varrho\mathrm{d}x\mathrm{d}t.$$

Applying condition (8) to estimate S_{21}^l, we obtain the following:

$$\sum_{l=1}^{8} S_{21}^l \geq v\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2+\gamma}{2}}|\nabla^\epsilon\nabla^\epsilon u_\epsilon|^2\varrho^2\mathrm{d}x\mathrm{d}t.$$

Applying condition (8) to estimate S_{31}^l, we obtain the following:

$$\sum_{l=1}^{8} S_{31}^l \geq \frac{v\gamma}{4}\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p-4+\gamma}{2}}|\nabla^\epsilon(|\nabla^\epsilon u_\epsilon|^2)|^2\varrho^2\mathrm{d}x\mathrm{d}t.$$

Applying condition (8) to estimate S_1^l, we obtain the following:

$$|S_1^l| \leq C\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p-1+\gamma}{2}}|\nabla^\epsilon\nabla^\epsilon u_\epsilon|\varrho|\nabla^\epsilon\varrho|\mathrm{d}x\mathrm{d}t,$$

where $C = C(\Upsilon) > 0$. Applying condition (8) to estimate S_{22}^l and S_{32}^l, we obtain the following:

$$|S_{22}^l| + |S_{32}^l| \leq C(\gamma+1)\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2+\gamma}{2}}(|\nabla^\epsilon u_\epsilon| + |\nabla_{\mathcal{R}}u_\epsilon|)\varrho^2|\nabla^\epsilon\nabla^\epsilon u_\epsilon|\mathrm{d}x\mathrm{d}t,$$

where $C = C(\Upsilon) > 0$. Applying condition (8) to estimate S_4^l, we obtain the following:

$$|S_4^l| \leq C\int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}}\varrho(|\nabla^\epsilon\varrho| + |\nabla_{\mathcal{R}}\varrho|)\mathrm{d}x\mathrm{d}t,$$

where $C = C(\Upsilon) > 0$.

Below, we estimate S_5^l. We use (11) to S_5^l and obtain the following:

$$\begin{aligned}S_5^l =&v_k\int_{t_1}^{t_2}\int_\Omega\sum_{i=1}^{8}\mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon)\varrho^2(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}}X_k X_l^\epsilon u_\epsilon\mathrm{d}x\mathrm{d}t\\ &+\theta_m\int_{t_1}^{t_2}\int_\Omega\sum_{i=1}^{8}\mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon)\varrho^2(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}}R_m X_l^\epsilon u_\epsilon\mathrm{d}x\mathrm{d}t\\ =&v_k S_{51}^l + \theta_m S_{52}^l,\end{aligned}$$

where v_k, θ_m are constants completely determined by Table 1. Applying condition (8) to estimate S^l_{51}, we obtain the following:

$$|S^l_{51}| \le C \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-1+\gamma}{2}} |\nabla^\epsilon \nabla^\epsilon u_\epsilon| \varrho^2 \mathrm{d}x\mathrm{d}t,$$

where $C = C(Y) > 0$. For S^l_{52}, by $R_m X^\epsilon_l = X^\epsilon_l R_m + [R_m, X^\epsilon_l]$, we obtain the following:

$$\begin{aligned} S^l_{52} = & \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{8} \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) \varrho^2 (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}} X^\epsilon_l R_m u_\epsilon \mathrm{d}x\mathrm{d}t \\ & + \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{8} \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) \varrho^2 (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}} [R_m, X^\epsilon_l] u_\epsilon \mathrm{d}x\mathrm{d}t \\ = & S^l_{521} + S^l_{522}. \end{aligned}$$

Using (10) to S^l_{522}, by condition (8), we obtain the following:

$$|S^l_{522}| \le C \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}} \varrho^2 \mathrm{d}x\mathrm{d}t,$$

where $C = C(Y) > 0$. For S^l_{521}, integrating by parts, we have the following:

$$\begin{aligned} S^l_{521} = & - \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{8} \mathcal{A}^{\sigma,\epsilon}_{i,\xi_j}(\nabla^\epsilon u_\epsilon) X^\epsilon_l X^\epsilon_j u_\epsilon \varrho^2 (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}} R_m u_\epsilon \mathrm{d}x\mathrm{d}t \\ & - \frac{\gamma}{2} \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{8} \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) \varrho^2 (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma-2}{2}} \sum_{k=1}^{8} X^\epsilon_k u_\epsilon X^\epsilon_l X^\epsilon_k u_\epsilon R_m u_\epsilon \mathrm{d}x\mathrm{d}t \\ & - 2 \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{8} \mathcal{A}_i^{\sigma,\epsilon}(\nabla^\epsilon u_\epsilon) \varrho X^\epsilon_l \varrho (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma}{2}} R_m u_\epsilon \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Applying condition (8) to estimate S^l_{521}, we obtain the following:

$$\begin{aligned} |S^l_{521}| \le & C(\gamma+1) \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2+\gamma}{2}} |\nabla^\epsilon \nabla^\epsilon u_\epsilon| \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon| \mathrm{d}x\mathrm{d}t \\ & + C \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-1+\gamma}{2}} \varrho |\nabla^\epsilon \varrho| |\nabla_{\mathcal{R}} u_\epsilon| \mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $C = C(Y) > 0$. Combining these estimates, we obtain the estimate of S^l_5, as follows:

$$\begin{aligned} |S^l_5| \le & C \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-1+\gamma}{2}} |\nabla^\epsilon \nabla^\epsilon u_\epsilon| \varrho^2 \mathrm{d}x\mathrm{d}t \\ & + C \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}} \varrho^2 \mathrm{d}x\mathrm{d}t \\ & + C(\gamma+1) \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2+\gamma}{2}} |\nabla^\epsilon \nabla^\epsilon u_\epsilon| \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon| \mathrm{d}x\mathrm{d}t \\ & + C \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-1+\gamma}{2}} \varrho |\nabla^\epsilon \varrho| |\nabla_{\mathcal{R}} u_\epsilon| \mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $C = C(Y) > 0$.

For S_6^l, we use the same method as estimating S_5^l and obtain the following:

$$
\begin{aligned}
|S_6^l| \leq & C\gamma \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-1+\gamma}{2}} |\nabla^\epsilon \nabla^\epsilon u_\epsilon| \varrho^2 \mathrm{d}x\mathrm{d}t \\
& + C\gamma \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}} \varrho^2 \mathrm{d}x\mathrm{d}t \\
& + C\gamma(\gamma+1) \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2+\gamma}{2}} |\nabla^\epsilon \nabla^\epsilon u_\epsilon| \varrho^2 |\nabla_{\mathcal{R}} u_\epsilon| \mathrm{d}x\mathrm{d}t \\
& + C\gamma \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-1+\gamma}{2}} \varrho |\nabla^\epsilon \varrho| |\nabla_{\mathcal{R}} u_\epsilon| \mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where $C = C(\Upsilon) > 0$.

Combining all estimates to (30), then by Young's inequality, we obtain (26). □

Based on Lemma 3, we obtain the following lemma, which provides two Caccioppoli-type inequalities for $\nabla_{\mathcal{R}} u_\epsilon$.

Lemma 5. *Suppose u_ϵ is a weak solution to (7). Then, when $p \in [2,4]$, for every $\gamma \geq 0$ and every $\varrho \in C^1([0,T], C_0^\infty(\Omega))$, we have the following:*

$$
\begin{aligned}
& \int_{t_1}^{t_2} \int_\Omega |\nabla_{\mathcal{R}} u_\epsilon|^{p+\gamma} \varrho^{p+\gamma} \mathrm{d}x\mathrm{d}t \\
& \leq C(p+\gamma) \|\nabla^\epsilon \varrho\|_{L^\infty} \int \int_{\mathrm{spt}(\varrho)} (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}} \mathrm{d}x\mathrm{d}t \\
& \quad + C(p+\gamma) \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{R}} u_\epsilon|^\gamma |\nabla^\epsilon \nabla^\epsilon u_\epsilon|^2 \varrho^{4+\gamma} \mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{31}
$$

where $C = C(p) > 0$;

$$
\begin{aligned}
& \left(\int_{t_1}^{t_2} \int_\Omega |\nabla_{\mathcal{R}} u_\epsilon|^{p+\gamma} \varrho^{p+\gamma} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{p+\gamma}} \\
& \leq C(p+\gamma)^2 (\|\nabla^\epsilon \varrho\|_{L^\infty} + \|\varrho\|_{L^\infty}) \left(\int \int_{\mathrm{spt}(\varrho)} (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{p+\gamma}} \\
& \quad + C(p+\gamma) \|\varrho \partial_t \varrho\|_{L^\infty}^{\frac{1}{2}} |\mathrm{spt}(\varrho)|^{\frac{p-2}{2(p+\gamma)}} \left(\int \int_{\mathrm{spt}(\varrho)} (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}} \mathrm{d}x\mathrm{d}t \right)^{\frac{4-p}{2(p+\gamma)}},
\end{aligned}
\tag{32}
$$

where $C = C(v, \Upsilon) > 0$.

Proof. First, we prove (31). Denote

$$
U := \int_{t_1}^{t_2} \int_\Omega |\nabla_{\mathcal{R}} u_\epsilon|^{p+\gamma} \varrho^{p+\gamma} \mathrm{d}x\mathrm{d}t, \quad V := \int \int_{\mathrm{spt}(\varrho)} (\sigma + |\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}} \mathrm{d}x\mathrm{d}t.
$$

According to Table 1, we write the following:

$$
\begin{aligned}
R_7 u_\epsilon &= -[X_1, X_2] u_\epsilon = X_2 X_1 u_\epsilon - X_1 X_2 u_\epsilon \\
R_8 u_\epsilon &= -[X_3, X_4] u_\epsilon = X_4 X_3 u_\epsilon - X_3 X_4 u_\epsilon.
\end{aligned}
$$

From this and $\nabla_{\mathcal{R}} u_\epsilon = (R_7 u_\epsilon, R_8 u_\epsilon)$, we rewrite $|\nabla_{\mathcal{R}} u_\epsilon|^{p+\gamma}$ as

$$
\begin{aligned}
|\nabla_{\mathcal{R}} u_\epsilon|^{p+\gamma} = & |\nabla_{\mathcal{R}} u_\epsilon|^{p-2+\gamma} ((R_7 u_\epsilon)^2 + (R_8 u_\epsilon)^2) \\
= & |\nabla_{\mathcal{R}} u_\epsilon|^{p-2+\gamma} (R_7 u_\epsilon (X_2 X_1 u_\epsilon - X_1 X_2 u_\epsilon) + R_8 u_\epsilon (X_4 X_3 u_\epsilon - X_3 X_4 u_\epsilon)).
\end{aligned}
$$

Then U can be written as follows:

$$U = \int_{t_1}^{t_2}\int_\Omega |\nabla_\mathcal{R} u_\epsilon|^{p-2+\gamma} R_7 u_\epsilon (X_2X_1u_\epsilon - X_1X_2u_\epsilon)\varrho^{p+\gamma}\mathrm{d}x\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\int_\Omega |\nabla_\mathcal{R} u_\epsilon|^{p-2+\gamma} R_8 u_\epsilon (X_4X_3u_\epsilon - X_3X_4u_\epsilon)\varrho^{p+\gamma}\mathrm{d}x\mathrm{d}t = U_1 + U_2.$$

For U_1, we integrate by parts to obtain the following:

$$\begin{aligned} U_1 = &-(p-2-\gamma)\int_{t_1}^{t_2}\int_\Omega |\nabla_\mathcal{R} u_\epsilon|^{p-3+\gamma} R_7 u_\epsilon \varrho^{p+\gamma}(X_2|\nabla_\mathcal{R} u_\epsilon|X_1u_\epsilon - X_1|\nabla_\mathcal{R} u_\epsilon|X_2u_\epsilon)\mathrm{d}x\mathrm{d}t \\ &- \int_{t_1}^{t_2}\int_\Omega |\nabla_\mathcal{R} u_\epsilon|^{p-2+\gamma}\varrho^{p+\gamma}(X_2R_7u_\epsilon X_1u_\epsilon - X_1R_7u_\epsilon X_2u_\epsilon)\mathrm{d}x\mathrm{d}t \\ &-(p+\gamma)\int_{t_1}^{t_2}\int_\Omega |\nabla_\mathcal{R} u_\epsilon|^{p-2+\gamma} R_7 u_\epsilon \varrho^{p-1+\gamma}(X_2\varrho X_1u_\epsilon - X_1\varrho X_2u_\epsilon)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Thus,

$$\begin{aligned} U_1 \le &2(p+\gamma)\int_{t_1}^{t_2}\int_\Omega |\nabla^\epsilon u_\epsilon||\nabla_\mathcal{R} u_\epsilon|^{p-2+\gamma}|\nabla^\epsilon\nabla_\mathcal{R} u_\epsilon|\varrho^{p+\gamma}\mathrm{d}x\mathrm{d}t \\ &+ 2(p+\gamma)\int_{t_1}^{t_2}\int_\Omega |\nabla^\epsilon u_\epsilon||\nabla_\mathcal{R} u_\epsilon|^{p-1+\gamma}|\nabla^\epsilon\nabla_\mathcal{R} u_\epsilon||\nabla^\epsilon\varrho|\varrho^{p-1+\gamma}\mathrm{d}x\mathrm{d}t. \end{aligned}$$

In the same way, we obtain the estimate of U_2, as follows:

$$\begin{aligned} U_2 \le &2(p+\gamma)\int_{t_1}^{t_2}\int_\Omega |\nabla^\epsilon u_\epsilon||\nabla_\mathcal{R} u_\epsilon|^{p-2+\gamma}|\nabla^\epsilon\nabla_\mathcal{R} u_\epsilon|\varrho^{p+\gamma}\mathrm{d}x\mathrm{d}t \\ &+ 2(p+\gamma)\int_{t_1}^{t_2}\int_\Omega |\nabla^\epsilon u_\epsilon||\nabla_\mathcal{R} u_\epsilon|^{p-1+\gamma}|\nabla^\epsilon\nabla_\mathcal{R} u_\epsilon||\nabla^\epsilon\varrho|\varrho^{p-1+\gamma}\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Thus,

$$\begin{aligned} U \le &4(p+\gamma)\int_{t_1}^{t_2}\int_\Omega |\nabla^\epsilon u_\epsilon||\nabla_\mathcal{R} u_\epsilon|^{p-2+\gamma}|\nabla^\epsilon\nabla_\mathcal{R} u_\epsilon|\varrho^{p+\gamma}\mathrm{d}x\mathrm{d}t \\ &+ 4(p+\gamma)\int_{t_1}^{t_2}\int_\Omega |\nabla^\epsilon u_\epsilon||\nabla_\mathcal{R} u_\epsilon|^{p-1+\gamma}|\nabla^\epsilon\nabla_\mathcal{R} u_\epsilon||\nabla^\epsilon\varrho|\varrho^{p-1+\gamma}\mathrm{d}x\mathrm{d}t = L_1 + L_2. \end{aligned} \tag{33}$$

Below, we estimate L_1 and L_2. For L_1, applying Hölder's inequality, we have the following:

$$L_1 \le 4(p+\gamma)\chi^{\frac{1}{2}} V^{\frac{4-p}{2(p+\gamma)}} U^{\frac{2p-4+\gamma}{2(p+\gamma)}}, \tag{34}$$

where

$$\chi := \int_{t_1}^{t_2}\int_\Omega |\nabla^\epsilon u_\epsilon|^{p-2}|\nabla_\mathcal{R} u_\epsilon|^\gamma |\nabla^\epsilon\nabla_\mathcal{R} u_\epsilon|^2\varrho^{4+\gamma}\mathrm{d}x\mathrm{d}t. \tag{35}$$

For L_2, applying Hölder's inequality, we have the following:

$$L_2 \le 4(p+\gamma)\|\nabla^\epsilon\varrho\|_{L^\infty} V^{\frac{1}{p+\gamma}} U^{\frac{p-1+\gamma}{p+\gamma}}. \tag{36}$$

Thus,

$$U \le 4(p+\gamma)\chi^{\frac{1}{2}} V^{\frac{4-p}{2(p+\gamma)}} U^{\frac{2p-4+\gamma}{2(p+\gamma)}} + 4(p+\gamma)\|\nabla^\epsilon\varrho\|_{L^\infty} V^{\frac{1}{p+\gamma}} U^{\frac{p-1+\gamma}{p+\gamma}}. \tag{37}$$

From this, by Young's inequality, we obtain (31).

Second, we prove (32). Applying Lemma 3 to re-estimate $\mathcal{M}$ defined in (35), then we apply Hölder's inequality to obtain the following:

$$\chi \leq C(\gamma+1)^2\|\nabla^\epsilon\varrho\|_{L^\infty}^2 V^{\frac{p-2}{p+\gamma}} U^{\frac{\gamma+2}{p+\gamma}} + C\|\varrho\partial_t\varrho\|_{L^\infty}|\mathrm{spt}(\varrho)|^{\frac{p-2}{p+\gamma}} U^{\frac{\beta+2}{p+\gamma}} + C(\gamma+1)^2\|\varrho\|_{L^\infty}^4 V^{\frac{p}{p+\gamma}} U^{\frac{\gamma}{p+\gamma}},$$

where $C = C(v, \Upsilon) > 0$. This, with (34), yields the following:

$$L_1 \leq C(p+\gamma)^2\|\nabla^\epsilon\varrho\|_{L^\infty} V^{\frac{1}{p+\gamma}} U^{\frac{p-1+\gamma}{p+\gamma}} + C(p+\gamma)\|\varrho\partial_t\varrho\|_{L^\infty}^{\frac{1}{2}}|\mathrm{spt}(\varrho)|^{\frac{p-2}{2(p+\gamma)}} V^{\frac{4-p}{2(p+\gamma)}} U^{\frac{p-1+\gamma}{p+\gamma}} + C(p+\gamma)^2\|\varrho\|_{L^\infty}^2 V^{\frac{2}{p+\gamma}} U^{\frac{p-2+\gamma}{p+\gamma}}.$$

Combining the above inequality, (33) and (36), we obtain the following:

$$U \leq C(p+\gamma)^2\|\nabla^\epsilon\varrho\|_{L^\infty} V^{\frac{1}{p+\gamma}} U^{\frac{p-1+\gamma}{p+\gamma}} + C(p+\gamma)\|\varrho\partial_t\varrho\|_{L^\infty}^{\frac{1}{2}}|\mathrm{spt}(\varrho)|^{\frac{p-2}{2(p+\gamma)}} V^{\frac{4-p}{2(p+\gamma)}} U^{\frac{p-1+\gamma}{p+\gamma}} + C(p+\gamma)^2\|\varrho\|_{L^\infty}^2 V^{\frac{2}{p+\gamma}} U^{\frac{p-2+\gamma}{p+\gamma}}.$$

Dividing both sides of the above inequality by $U^{\frac{p-2+\gamma}{p+\gamma}}$ simultaneously, we apply Young's inequality to obtain the following:

$$U^{\frac{2}{p+\gamma}} \leq C(p+\gamma)^4\|\nabla^\epsilon\varrho\|_{L^\infty}^2 V^{\frac{2}{p+\gamma}} + C(p+\gamma)^2\|\varrho\partial_t\varrho\|_{L^\infty}|\mathrm{spt}(\varrho)|^{\frac{p-2}{p+\gamma}} V^{\frac{4-p}{p+\gamma}} + C(p+\gamma)^2\|\varrho\|_{L^\infty}^2 V^{\frac{2}{p+\gamma}},$$

which implies (32). □

3.2. A crucial Caccioppoli-Type Estimate

Based on Lemmas 4 and 5, we obtain the crucial Caccioppoli-type estimate for $\nabla^\epsilon u_\epsilon$ involving $\nabla^\epsilon\nabla^\epsilon u_\epsilon$.

Lemma 6. *Suppose u_ϵ is a weak solution to (7). Then, when $p \in [2,4]$, for every $\gamma \geq 0$ and $\varrho \in C^1([0,T], C_0^\infty(\Omega))$, we have the following:*

$$\begin{aligned}&\sup_{t_1<t<t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma+2}{2}}\varrho^2\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2+\gamma}{2}}|\nabla^\epsilon\nabla^\epsilon u_\epsilon|^2\varrho^2\mathrm{d}x\mathrm{d}t\\ &\leq C(p+\gamma)^9(\|\varrho\|_{L^\infty}^2 + \|\nabla^\epsilon\varrho\|_{L^\infty}^2 + \|\varrho\nabla_{\mathcal{R}}\varrho\|_{L^\infty})\int\int_{\mathrm{spt}(\varrho)}(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}}\mathrm{d}x\mathrm{d}t\\ &\quad + C(p+\gamma)^7\|\varrho\partial_t\varrho\|_{L^\infty}|\mathrm{spt}(\varrho)|^{\frac{p-2}{p+\gamma}}\left(\int\int_{\mathrm{spt}(\varrho)}(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}}\mathrm{d}x\mathrm{d}t\right)^{\frac{\gamma+2}{p+\gamma}},\end{aligned} \tag{38}$$

where $C = C(p, v, \Upsilon) > 0$.

Proof. To obtain (38), we need to re-estimate each integral term on the right-hand side of (26), separately.

First, we bind the second integral term on the hand side of (26). Applying Hölder's inequality, then by (32) in Lemma 5, we obtain the following:

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega (\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p-2+\gamma}{2}}|\nabla_{\mathcal{R}} u_\epsilon|^2\varrho^2 \mathrm{d}x\mathrm{d}t \\
\leq &\left(\int\int_{\mathrm{spt}(\varrho)}(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}}\mathrm{d}x\mathrm{d}t\right)^{\frac{p-2+\gamma}{p+\gamma}}\left(\int_{t_1}^{t_2}\int_\Omega |\nabla_{\mathcal{R}} u_\epsilon|^{p+\gamma}\varrho^{p+\gamma}\mathrm{d}x\mathrm{d}t\right)^{\frac{2}{p+\gamma}} \\
\leq &C(p+\gamma)^4(\|\varrho\|^2_{L^\infty}+\|\nabla^\epsilon\varrho\|^2_{L^\infty})\int\int_{\mathrm{spt}(\varrho)}(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}}\mathrm{d}x\mathrm{d}t \\
&+C(p+\gamma)^2\|\varrho\partial_t\varrho\|_{L^\infty}|\mathrm{spt}(\varrho)|^{\frac{p-2}{p+\gamma}}\left(\int\int_{\mathrm{spt}(\varrho)}(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}}\mathrm{d}x\mathrm{d}t\right)^{\frac{\gamma+2}{p+\gamma}},
\end{aligned}$$

where $C = C(p, v, \Upsilon) > 0$.

Second, we bind the final integral term on the hand side of (26). We apply Hölder's inequality to obtain the following:

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega (\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\gamma+2}{2}}|\partial_t\varrho|\varrho \mathrm{d}x\mathrm{d}t \\
\leq &\|\varrho\partial_t\varrho\|_{L^\infty}|\mathrm{spt}(\varrho)|^{\frac{p-2}{p+\gamma}}\left(\int\int_{\mathrm{spt}(\varrho)}(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{2}}\mathrm{d}x\mathrm{d}t\right)^{\frac{\gamma+2}{p+\gamma}}.
\end{aligned}$$

Combining the above estimates and (26), we obtain the following: (38). □

4. Proof of Theorem 2

In this section, we apply the crucial Caccioppoli-type estimate to prove Theorem 2.

The Proof of Theorem 2. For every non-negative cut-off function $\varrho \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of $\mathcal{Q}$, satisfying $|\varrho| \leq 1$ in $\mathcal{Q}$, and for any $\gamma \geq 0$, we denote the following:

$$w := (\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{p+\gamma}{4}}\varrho^2.$$

Then, (38) is rewritten as follows:

$$\begin{aligned}
&\sup_{t_1<t<t_2}\int_\Omega w^{\frac{2(\gamma+2)}{p+\gamma}}\mathrm{d}x+\int_{t_1}^{t_2}\int_\Omega |\nabla^\epsilon w|^2\mathrm{d}x\mathrm{d}t \\
\leq &C(p+\gamma)^9(\|\varrho\|^2_{L^\infty}+\|\nabla^\epsilon\varrho\|^2_{L^\infty}+\|\varrho\nabla_{\mathcal{R}}\varrho\|_{L^\infty})\int\int_{\mathrm{spt}(\varrho)} w^2\mathrm{d}x\mathrm{d}t \\
&+C(p+\gamma)^7\|\varrho\partial_t\varrho\|_{L^\infty}|\mathrm{spt}(\varrho)|^{\frac{p-2}{p+\gamma}}\left(\int\int_{\mathrm{spt}(\varrho)} w^2\mathrm{d}x\mathrm{d}t\right)^{\frac{\gamma+2}{p+\gamma}}.
\end{aligned} \tag{39}$$

We denote $q := 2 + \frac{4(\gamma+2)}{N(p+\gamma)}$, where $N = 10$ is the homogeneous dimension of SU(3). Applying Hölder's inequality, we apply the Sobolev inequality to obtain the following:

$$\begin{aligned}
\int_{t_1}^{t_2}\int_\Omega w^q\mathrm{d}x\mathrm{d}t \leq &\int_{t_1}^{t_2}\left(\int_\Omega w^{\frac{2(\gamma+2)}{p+\gamma}}\mathrm{d}x\right)^{\frac{2}{N}}\left(\int_\Omega w^{\frac{2N}{N-2}}\mathrm{d}x\right)^{\frac{N-2}{N}}\mathrm{d}t \\
\leq &C\left(\sup_{t_1<t<t_2}\int_\Omega w^{\frac{2(\gamma+2)}{p+\gamma}}\mathrm{d}x\right)^{\frac{2}{N}}\int_{t_1}^{t_2}\int_\Omega |\nabla^\epsilon w|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

which, together with (39), yields the following:

$$\left(\int_{t_1}^{t_2}\int_\Omega w^q \mathrm{d}x\mathrm{d}t\right)^{\frac{N}{N+2}} \leq C(p+\gamma)^9(\|\varrho\|_{L^\infty}^2+\|\nabla^\epsilon\varrho\|_{L^\infty}^2+\|\varrho\nabla_{\mathcal{R}}\varrho\|_{L^\infty})\int\int_{\mathrm{spt}(\varrho)} w^2\mathrm{d}x\mathrm{d}t$$
$$+C(p+\gamma)^7\|\varrho\partial_t\varrho\|_{L^\infty}|\mathrm{spt}(\varrho)|^{\frac{p-2}{p+\gamma}}\left(\int\int_{\mathrm{spt}(\varrho)} w^2\mathrm{d}x\mathrm{d}t\right)^{\frac{\gamma+2}{p+\gamma}}, \quad (40)$$

where $C=C(p,v,\Upsilon)>0$.

For any $\mu,r>0$, we define the parabolic cylinder $\mathcal{Q}_{\mu,r}:=B_\epsilon(x_0,r)\times(t_0-\mu r^2,t_0)$. Given any $\mathcal{Q}_{\mu,2r}\subset\mathcal{Q}_{\mu,2r_0}\subset\mathcal{Q}$, we denote $r_i=(1+2^{-i})r$ and $\gamma_i=2(\kappa^i-1)$ with $\kappa=\frac{N+2}{N}$ such that

$$p+\gamma_{i+1}=(p+\gamma_i)\left(1+\frac{2(\gamma_i+2)}{N(p+\gamma_i)}\right),\quad i=0,1,2,\ldots;$$

we write $\mathcal{Q}_i=\mathcal{Q}_{\mu,r_i}$ with $\mathcal{Q}_0=\mathcal{Q}_{\mu,2r}$ and $\mathcal{Q}_\infty=\mathcal{Q}_{\mu,r}$, then choose a standard parabolic cut-off function $\varrho_i\in C^\infty(\mathcal{Q}_i)$ satisfying the following:

$$\begin{cases}\varrho_i=1 \quad \text{in } \mathcal{Q}_{i+1},\\ |\nabla^\epsilon\varrho_i|\leq\frac{2^{i+8}}{r},\ |\nabla_{\mathcal{R}}\varrho_i|\leq\frac{2^{2i+8}}{r^2},\ |\partial_t\varrho_i|\leq\frac{2^{2i+8}}{\mu r^2} \quad \text{in } \mathcal{Q}_i.\end{cases}$$

By (40) with $\varrho=\varrho_i$ and $\gamma=\gamma_i$, writing $\vartheta_i=p+\gamma_i=p-2+2\kappa^i$, we obtain the following:

$$\left(\int\int_{\mathcal{Q}_{i+1}}(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\vartheta_{i+1}}{2}}\mathrm{d}x\mathrm{d}t\right)^{\frac{N}{N+2}}$$
$$\leq C2^{2i}\vartheta_i^9(r^{-2}+1)\left[\left(\int\int_{\mathcal{Q}_i}(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\vartheta_i}{2}}\mathrm{d}x\mathrm{d}t\right)^{\frac{p-2}{\vartheta_i}}+\mu^{-1}(\mu r^{N+2})^{\frac{p-2}{\vartheta_i}}\right]$$
$$\times\left(\int\int_{\mathcal{Q}_i}(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\vartheta_i}{2}}\mathrm{d}x\mathrm{d}t\right)^{\frac{\vartheta_i-p+2}{\vartheta_i}}, \quad (41)$$

where $C=C(p,v,\Upsilon)>0$. To simplify writing, we denote

$$\chi_i=\left(\int\int_{\mathcal{Q}_i}(\sigma+|\nabla^\epsilon u_\epsilon|^2)^{\frac{\vartheta_i}{2}}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{\vartheta_i}}.$$

Then (41) becomes

$$\chi_{i+1}^{\frac{\vartheta_{i+1}}{\kappa}}\leq C\mu^{\frac{2}{N+2}}2^{2i}\vartheta_i^9(\chi_i^{p-2}+\mu^{-1})\chi_i^{\vartheta_i-p+2},$$

where $C=C(p,v,\Upsilon,r_0)=C(p,v,\Upsilon)(1+r_0^2)>0$. From this, letting $\bar\chi_i=\max(\chi_i,\mu^{\frac{1}{2-p}})$, we obtain the following:

$$\bar\chi_{i+1}^{\frac{\vartheta_{i+1}}{\kappa}}\leq C\mu^{\frac{2}{N+2}}2^{2i}\vartheta_i^9\bar\chi_i^{\vartheta_i}. \quad (42)$$

Without loss of generality, we may assume $C=C(p,v,\Upsilon,r_0)\geq 1$. Iterating (42), we have the following:

$$\bar\chi_{i+1}\leq\left(\prod_{j=0}^{i}K_j^{\frac{\kappa^{i+1-j}}{\vartheta_{i+1}}}\right)\bar\chi_0^{\frac{\vartheta_0\kappa^{i+1}}{\vartheta_{i+1}}},$$

where $K_j=C\mu^{\frac{2}{N+2}}2^{2j}\vartheta_j^9$, $\vartheta_i=p-2+2\kappa^i$ and $\kappa=\frac{N+2}{N}$. From this, letting $i\to\infty$, we obtain the following:

$$\bar\chi_\infty:=\limsup_{i\to\infty}\bar\chi_i\leq C\mu^{\frac{1}{2}}\bar\chi_0^{\frac{p}{2}}, \quad (43)$$

where $C=C(p,v,\Upsilon,r_0)>0$. Since $\sup_{\mathcal{Q}_{\mu,r}}|\nabla^\epsilon u_\epsilon|\leq\bar\chi_\infty$, combining (43), we obtain (9). □

5. Higher Integrability of $\partial_t u$

In this section, based on Theorem 1, when $2 \le p \le 4$, we prove the higher integrability of $\partial_t u$. Setting $\sigma \to 0$ in the following theorem, we gain $\partial_t u \in L^q_{\mathrm{loc}}$ for any $1 \le q < \infty$.

Theorem 3. *Suppose u_σ is a weak solution to (5) in $\Omega \times (0,T)$. Then, when $2 \le p \le 4$, we have $\partial_t u_\sigma \in L^q_{\mathrm{loc}}(\Omega \times (0,T))$ for any $q \in [1,\infty)$. Moreover, when $p \in [2,4]$, for every $\gamma \ge 0$ and every $\varrho \in C^1([0,T], C_0^\infty(\Omega))$, we have*

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega |\partial_t u_\sigma|^{\gamma+2}\varrho^{\gamma+2}\mathrm{d}x\mathrm{d}t \\
&\le C^{\gamma+2}(\gamma+2)^{\gamma+2}|\mathrm{spt}(\varrho)|\left(\chi^{2p-2}\|\nabla_{\mathcal{H}}\varrho\|^2_{L^\infty} + \chi^p\|\varrho\partial_t\varrho\|_{L^\infty}\right)^{\frac{\gamma+2}{2}},
\end{aligned}
\tag{44}
$$

where $C = C(p, v, \Upsilon) > 0$ and $\chi = \sup_{\mathrm{spt}(\varrho)}(\sigma + |\nabla_{\mathcal{H}} u_\sigma|^2)^{\frac{1}{2}}$.

Proof. For any $\gamma \ge 0$, from (5), we have

$$
|\partial_t u_\sigma|^{\gamma+2} = |\partial_t u_\sigma|^\gamma \partial_t u_\sigma \sum_{i=1}^6 X_i(\mathcal{A}^\sigma(\nabla_{\mathcal{H}} u_\sigma)).
$$

From this, integrating by parts, we have

$$
\begin{aligned}
L = \int_{t_1}^{t_2}\int_\Omega |\partial_t u_\sigma|^{\gamma+2}\varrho^{\gamma+2}\mathrm{d}x\mathrm{d}t &= \int_{t_1}^{t_2}\int_\Omega |\partial_t u_\sigma|^\gamma \partial_t u_\sigma \sum_{i=1}^6 X_i(\mathcal{A}_i^\sigma(\nabla_{\mathcal{H}} u_\sigma))\varrho^{\gamma+2}\mathrm{d}x\mathrm{d}t \\
&= -(\gamma+2)\int_{t_1}^{t_2}\int_\Omega |\partial_t u_\sigma|^\gamma \partial_t u_\sigma \sum_{i=1}^6 \mathcal{A}_i^\sigma(\nabla_{\mathcal{H}} u_\sigma)\varrho^{\gamma+1}X_i\varrho\mathrm{d}x\mathrm{d}t \\
&\quad -(\gamma+1)\int_{t_1}^{t_2}\int_\Omega |\partial_t u_\sigma|^\gamma X_i\partial_t u_\sigma \sum_{i=1}^6 \mathcal{A}_i^\sigma(\nabla_{\mathcal{H}} u_\sigma)\varrho^{\gamma+2}\mathrm{d}x\mathrm{d}t = I_1 + I_2.
\end{aligned}
\tag{45}
$$

We apply condition (6) and Hölder's inequality to obtain the following:

$$
\begin{aligned}
|I_1| \le& C(\gamma+2)\int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla_{\mathcal{H}} u_\sigma|^2)^{\frac{p-1}{2}}|\partial_t u_\sigma|^{\gamma+1}\varrho^{\gamma+1}|\nabla_{\mathcal{H}}\varrho|\mathrm{d}x\mathrm{d}t \\
\le& C(\gamma+2)\left(\int_{t_1}^{t_2}\int_\Omega |\partial_t u_\sigma|^{\gamma+2}\varrho^{\gamma+2}\mathrm{d}x\mathrm{d}t\right)^{\frac{\gamma+1}{\gamma+2}} \\
&\times\left(\int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla_{\mathcal{H}} u_\sigma|^2)^{\frac{(p-1)(\gamma+2)}{2}}|\nabla_{\mathcal{H}}\varrho|^{\gamma+2}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{\gamma+2}} \\
\le& C(\gamma+2)\|\nabla_{\mathcal{H}}\varrho\|_{L^\infty}|\mathrm{spt}(\varrho)|^{\frac{1}{\gamma+2}}\chi^{p-1}L^{\frac{\gamma+1}{\gamma+2}};
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
|I_2| \le& C(\gamma+1)\int_{t_1}^{t_2}\int_\Omega (\sigma + |\nabla_{\mathcal{H}} u_\sigma|^2)^{\frac{p-1}{2}}|\partial_t u_\sigma|^{\gamma}|\nabla_{\mathcal{H}}\partial_t u_\sigma|\varrho^{\gamma+2}\mathrm{d}x\mathrm{d}t \\
\le& C(\gamma+1)\left(\int_{t_1}^{t_2}\int_\Omega |\partial_t u_\sigma|^{\gamma+2}\varrho^{\gamma+2}\mathrm{d}x\mathrm{d}t\right)^{\frac{\gamma}{2(\gamma+2)}} \\
&\times\left(\int\int_{\mathrm{spt}(\varrho)} (\sigma + |\nabla_{\mathcal{H}} u_\sigma|^2)^{\frac{p(\gamma+2)}{4}}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{\gamma+2}} J^{\frac{1}{2}} \\
\le& C(\gamma+1)|\mathrm{spt}(\varrho)|^{\frac{1}{\gamma+2}}\chi^{\frac{p}{2}}L^{\frac{\gamma}{2(\gamma+2)}}J^{\frac{1}{2}},
\end{aligned}
\tag{47}
$$

where $\chi = \sup_{\mathrm{spt}(\varrho)}(\sigma + |\nabla_{\mathcal{H}} u_\sigma|^2)^{\frac{1}{2}}$ and

$$J = \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla_{\mathcal{H}} u_\sigma|^2)^{\frac{p-2}{2}} |\partial_t u_\sigma|^\gamma |\nabla_{\mathcal{H}} \partial_t u_\sigma|^2 \varrho^{\gamma+4} \mathrm{d}x\mathrm{d}t.$$

Below, we estimate J. Differentiating (5) with respect to t, we obtain the following:

$$\partial_t(\partial_t u_\sigma) = \sum_{i=1}^{6} X_i(\partial_t \mathcal{A}_i^\sigma(\nabla_{\mathcal{H}} u_\sigma)) = \sum_{i,j=1}^{6} X_i(\mathcal{A}_{i,\xi_j}^\sigma(\nabla_{\mathcal{H}} u_\sigma) X_j \partial_t u_\sigma). \tag{48}$$

Applying $\psi = |\partial_t u_\sigma|^\gamma \partial_t u_\sigma \varrho^{\gamma+4}$ to test (48), we integrate by parts to obtain the following:

$$\begin{aligned}
\mathcal{L} =& \int_{t_1}^{t_2} \int_\Omega \partial_t(\partial_t u_\sigma) |\partial_t u_\sigma|^\gamma \partial_t u_\sigma \varrho^{\gamma+4} \mathrm{d}x\mathrm{d}t \\
=& -(\gamma+1) \sum_{i,j=1}^{6} \int_{t_1}^{t_2} \int_\Omega \mathcal{A}_{i,\xi_j}^\sigma(\nabla_{\mathcal{H}} u_\sigma) X_j \partial_t u_\sigma |\partial_t u_\sigma|^\gamma X_i \partial_t u_\sigma \varrho^{\gamma+4} \mathrm{d}x\mathrm{d}t \\
& -(\gamma+4) \sum_{i,j=1}^{6} \int_{t_1}^{t_2} \int_\Omega \mathcal{A}_{i,\xi_j}^\sigma(\nabla_{\mathcal{H}} u_\sigma) X_j \partial_t u_\sigma |\partial_t u_\sigma|^\gamma \partial_t u_\sigma \varrho^{\gamma+3} X_i \varrho \mathrm{d}x\mathrm{d}t = -S_1 - S_2.
\end{aligned}$$

Thus,

$$S_1 = -\mathcal{L} - S_2.$$

For $\mathcal{L}$, we integrate by parts to obtain

$$\mathcal{L} = \frac{1}{\gamma+2} \int_{t_1}^{t_2} \int_\Omega \partial_t(|\partial_t u_\sigma|^{\gamma+2}) \varrho^{\gamma+4} \mathrm{d}x\mathrm{d}t = -\frac{\gamma+4}{\gamma+2} \int_{t_1}^{t_2} \int_\Omega |\partial_t u_\sigma|^{\gamma+2} \varrho^{\gamma+3} \partial_t \varrho \mathrm{d}x\mathrm{d}t,$$

which, together with condition (6), yields

$$|\mathcal{L}| \le C \int_{t_1}^{t_2} \int_\Omega |\partial_t u_\sigma|^{\gamma+2} \varrho^{\gamma+3} |\partial_t \varrho| \mathrm{d}x\mathrm{d}t.$$

For S_1, condition (6) implies

$$S_1 \ge v(\gamma+1)J.$$

For S_2, by condition (6), by Young's inequality, we have the following:

$$\begin{aligned}
|S_2| \le& C(\gamma+1) \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla_{\mathcal{H}} u_\sigma|^2)^{\frac{p-2}{2}} |\partial_t u_\sigma|^{\gamma+1} |\nabla_{\mathcal{H}} \partial_t u_\sigma| \varrho^{\gamma+3} |\nabla_{\mathcal{H}} \varrho| \mathrm{d}x\mathrm{d}t \\
\le& \frac{v(\gamma+1)}{2} J + C(\gamma+1) \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla_{\mathcal{H}} u_\sigma|^2)^{\frac{p-2}{2}} |\partial_t u_\sigma|^{\gamma+2} \varrho^{\gamma+2} |\nabla_{\mathcal{H}} \varrho|^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Combining these estimates, we obtain the estimate of J, as follows:

$$\begin{aligned}
J \le& C \int_{t_1}^{t_2} \int_\Omega (\sigma + |\nabla_{\mathcal{H}} u_\sigma|^2)^{\frac{p-2}{2}} |\partial_t u_\sigma|^{\gamma+2} \varrho^{\gamma+2} |\nabla_{\mathcal{H}} \varrho|^2 \mathrm{d}x\mathrm{d}t \\
& + \frac{C}{\gamma+1} \int_{t_1}^{t_2} \int_\Omega |\partial_t u_\sigma|^{\gamma+2} \varrho^{\gamma+3} |\partial_t \varrho| \mathrm{d}x\mathrm{d}t \\
\le& C\left(\chi^{p-2} \|\nabla_{\mathcal{H}} \varrho\|_{L^\infty}^2 + \frac{1}{\gamma+1} \|\varrho \partial_t \varrho\|_{L^\infty}\right) L.
\end{aligned} \tag{49}$$

Combining (47) and (49), we obtain the following:

$$|I_2| \le C(\gamma+1) |\mathrm{spt}(\varrho)|^{\frac{1}{\gamma+2}} \chi^{\frac{p}{2}} L^{\frac{\gamma+1}{\gamma+2}} (\chi^{p-2} \|\nabla_{\mathcal{H}} \varrho\|_{L^\infty}^2 + \|\varrho \partial_t \varrho\|_{L^\infty})^{\frac{1}{2}}. \tag{50}$$

Combining (45), (46) and (50), we obtain the following:

$$L \leq C(\gamma+2)\|\nabla_{\mathcal{H}}\varrho\|_{L^\infty}|\mathrm{spt}(\varrho)|^{\frac{1}{\gamma+2}}\chi^{p-1}L^{\frac{\gamma+1}{\gamma+2}} + C(\gamma+1)|\mathrm{spt}(\varrho)|^{\frac{1}{\gamma+2}}\chi^{\frac{p}{2}}L^{\frac{\gamma+1}{\gamma+2}}(\chi^{p-2}\|\nabla_{\mathcal{H}}\varrho\|^2_{L^\infty} + \|\varrho\partial_t\varrho\|_{L^\infty})^{\frac{1}{2}}.$$

From this, we obtain (44). □

6. Conclusions

In this article, we construct a crucial Caccioppoli-type inequality (38). Based on the inequality, when $p \in [2,4]$, we built up the $C^{0,1}_{\mathrm{loc}}$-regularity of weak solutions to the degenerate parabolic p-Laplacian equation on the group SU(3) granted with the horizontal vector fields $X_1, \ldots, X_6$. Compared to the Heisenberg group $\mathbb{H}^n$, our new result achieves the same range of p as [26]. Unfortunately, the $C^{0,1}$-regularity for the range $p \in (1,2) \cup (4,\infty)$ cannot be achieved with our current technology because our argument rests in a crucial way on Lemma 5 with the condition $p \in [2,4]$. Our approach can also be used for more general sub-Riemannian manifolds, for instance, a special class of the semi-simple Lie group proposed in [17] and Hörmander vector fields of step two in [19], to establish regularity for the parabolic p-Laplacian equation. Technically speaking, our method can also be extended to other types of partial differential equations, for example, the non-homogeneous equation $\partial_t u = -\sum_{i=1}^{6} X_i^* \mathcal{A}_i(\nabla_{\mathcal{H}} u) + \mathcal{B}(x,t,u,\nabla_{\mathcal{H}} u)$. The establishment of the regularity for the range $p \in (1,2) \cup (4,\infty)$ will be the focus and difficulty of our next work.

In conclusion, the results shown in this article are original. We believe that our results will be widely applied in the study of regularities for equations involving the p-Laplacian operator and other areas of applied science.

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