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A Signed Maximum Principle for Boundary Value Problems for Riemann–Liouville Fractional Differential Equations with Analogues of Neumann or Periodic Boundary Conditions

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Abstract: Sufficient conditions are obtained for a signed maximum principle for boundary value problems for Riemann–Liouville fractional differential equations with analogues of Neumann or periodic boundary conditions in neighborhoods of simple eigenvalues. The primary objective is to exhibit four specific boundary value problems for which the sufficient conditions can be verified. To show an application of the signed maximum principle, a method of upper and lower solutions coupled with monotone methods is developed to obtain sufficient conditions for the existence of a maximal solution and a minimal solution of a nonlinear boundary value problem. A specific example is provided to show that sufficient conditions for the nonlinear problem can be realized.

Keywords: fractional boundary value problem; signed maximum principle; fractional Neumann boundary conditions; fractional periodic boundary conditions

MSC: 34K37; 34A08; 34B27

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1. Introduction

Applications of the maximum principle in functional analysis are well known and we refer the interested reader to the authoritative account [1]. In recent years, the maximum principle has become an important tool in the study of boundary value problems for fractional differential equations. Early applications appear in [2,3] where explicit Green's functions, expressed in terms of power functions, were constructed; sign properties of the Green's function were analyzed so that fixed point theorems could be applied to give sufficient conditions for the existence of positive solutions. More recently, Green's functions, expressed in terms of Mittag-Leffler functions, have been constructed so that fixed-point theorems and the maximum principle can be applied. See, for example, Refs. [4–7].

Credit for the discovery of an anti-maximum principle is given to Clément and Peletier [8]. Although primarily interested in partial differential equations, they initially illustrated the anti-maximum principle with the boundary value problem, $y'' + \lambda y = f$, $y'(0) = 0, y'(1) = 0$, with $0 < \lambda < \frac{\pi^2}{4}$. They showed, if $0 < \lambda < \Lambda = \frac{\pi^2}{4}$ and if $f \in \mathcal{L}[0,1]$, then the boundary value problem is uniquely solvable and $f \geq 0$ implies $y \geq 0$ where y is the unique solution associated with f.

At $\lambda = 0$, the boundary value problem, $y'' + \lambda y = f$, $y'(0) = 0$, $y'(1) = 0$, is at resonance, and $\lambda = 0$ is a simple eigenvalue of the homogeneous problem. Moreover, for $\lambda < 0$, then $f \geq 0$ implies $y \geq 0$; that is, for $\lambda < 0$, the boundary value problem obeys a maximum principle. Thus, there has been a change in the sign property, maximum principle or anti-maximum principle, through the simple eigenvalue $\lambda = 0$. In more succinct terms, if $0 < |\lambda| < \Lambda = \frac{\pi^2}{4}$, and if $f \in \mathcal{L}[0,1]$, then the boundary value problem is uniquely solvable and $f \geq 0$ implies $\lambda y \geq 0$ where y is the unique solution associated with f. Since the publication of [8], the change in behavior from maximum to anti-maximum principles as a function of the parameter has received considerable attention. For partial differential equations, see [9–16]. For ordinary differential equations, see [17–21]. More recently, this change in behavior from maximum to anti-maximum principles has also been noticed and studied in fractional differential equations. For equations analyzing the fractional $p-$Laplacian, see [22,23]; for fractional differential equations of one independent variable, see [24].

In [9], the authors studied the nature of the maximum principle for boundary value problems for an abstract differential equation, $(\mathcal{A} + \lambda\mathcal{I})y = f$, defined on $[0,1]$ with $f \in \mathcal{L}[0,1]$, under a fundamental assumption that $\lambda = 0$ was a simple eigenvalue for the homogeneous problem. Under mild sufficient conditions, they proved the existence of $\Lambda > 0$, and a constant $K > 0$, independent of f, such that

$$\lambda y(t) \geq K|f|_1, \quad \lambda \in [-\Lambda, \Lambda] \setminus \{0\}, \quad 0 \leq t \leq 1, \tag{1}$$

where y is the unique solution of the boundary value problem associated with $(\mathcal{A} + \lambda\mathcal{I})y = f$ and $|f|_1 = \int_0^1 |f(s)|ds$. If (1) holds and $\lambda < 0$, then $f \geq 0$ implies $y \leq 0$; that is, the boundary value problem for (1) obeys a maximum principle. If (1) holds and $\lambda > 0$, then $f \geq 0$ implies $y \geq 0$; that is, the boundary value problem for (1) obeys an anti-maximum principle [8].

The methods of [9] were recently adapted to apply to a boundary value problem with a parameter for a Riemann–Liouville fractional differential equation [24]. During the review process for [24], those authors were asked by one referee if the methods of [9] could be successfully adapted to apply to analogues of Neumann or periodic boundary value problems for Riemann–Liouville fractional differential equations. In [24], the eigenspace generated by $\lambda = 0$ is contained in the space of continuous functions on $[0,1]$. The corresponding eigenspace for boundary value problems analogous to Neumann or periodic type boundary value problems will contain a singularity. Thus, the question is interesting. The purpose of this study is to address that question with a positive response.

In Section 2, we shall introduce preliminary notations and concepts from fractional calculus. We shall also introduce four boundary value problems for which the general theorem, stated in Section 3, applies. In Section 3, we introduce the notations adapted from [9] and state and prove the abstract theorem. The proof of the abstract theorem closely models the proofs of analogous theorems in [9,24]; with subtle differences in the technical details due to the specific function space, we shall produce a proof here for the self-containment of the manuscript. In Section 4, we shall apply the abstract theorem to each of the four examples introduced in Section 2. In Section 5, to illustrate an application of the abstract theorem, we develop a monotone method motivated by the abstract theorem and apply the monotone method to a nonlinear problem related to one of the examples introduced in Section 2. The monotone method closely models one that has been developed in [24] with subtle differences in the convergence argument. In Section 6, we illustrate the monotone method with a specific example. In this example, a Green's function is constructed using Mittag-Leffler functions. The purpose of introducing the Green's function is not to produce an explicit function on which to analyze sign properties, as is the case in say, [2] or [3]; the purpose is to obtain a verifiable bound on Λ so that if $0 < |\lambda| < \Lambda$, then $f \geq 0$ implies $\lambda y \geq 0$.

2. Preliminaries

In this section, we introduce notations from fractional calculus and state common properties that we shall employ throughout. For authoritative accounts on the development of fractional calculus, we refer to the monographs [25–27].

Assume $\gamma > 0$. For $y \in \mathcal{L}[0,1]$, the space of Lebesgue integrable functions, a Riemann–Liouville fractional integral of y of order γ, is defined by

$$I_0^\gamma y(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} y(s)ds, \quad 0 \le t \le 1,$$

where

$$\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}, \quad Re\, z > 0,$$

denotes the special gamma function. For $\gamma = 0$, I_0^0 is defined to be the identity operator. Let n denote a positive integer and assume $n-1 < \alpha \le n$. A Riemann–Liouville fractional derivative of y of order α is defined by $D_0^\alpha y(t) = D^n I_0^{n-\alpha} y(t)$, where $D^n = \frac{d^n}{dt^n}$, if this expression exists. In the case α is a positive integer, we may write $D_0^\alpha y(t) = D^\alpha y(t)$ or $I_0^\alpha y(t) = I^\alpha y(t)$ since the Riemann–Liouville derivative or integral agrees with the classical derivative or integral if α is a positive integer.

For the sake of self-containment, we state properties that we shall employ in this study. It is well known that the Riemann–Liouville fractional integrals commute; that is, if $\gamma_1, \gamma_2 > 0$, and $y \in \mathcal{L}[0,1]$, then

$$I_0^{\gamma_1} I_0^{\gamma_2} y(t) = I_0^{\gamma_1+\gamma_2} y(t) = I_0^{\gamma_2} I_0^{\gamma_1} y(t).$$

A power rule is valid for the Riemann–Liouville fractional integral; if $\delta > -1$ and $\gamma \ge 0$, then

$$I_0^\gamma t^\delta = I_0^\gamma (t-0)^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta+1+\gamma)} t^{\delta+\gamma}.$$

A power rule is valid for the Riemann–Liouville fractional derivative; if $\delta > -1$ and $\gamma \ge 0$, then

$$D_0^\gamma t^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta+1-\gamma)} t^{\delta-\gamma}.$$

If $n-1 < \alpha \le n$, and if $D_0^\alpha y(t)$ exists, then $D_0^{\alpha-1} y(t)$ exists and

$$D_0^\alpha y(t) = D^n I_0^{n-\alpha} y(t) = DD^{n-1} I_0^{(n-1)-(\alpha-1)} y(t) = DD_0^{\alpha-1} y(t).$$

Thus, it is clear that for each $j \in \{1, \ldots, n-1\}$, $D_0^{\alpha-j} y(t)$ exists and

$$D_0^\alpha y(t) = D^j D_0^{\alpha-j} y(t).$$

A Green's function will be constructed in Section 6. The two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad Re(\alpha) > 0, \quad \beta \in \mathbb{C}, \quad z \in \mathbb{C},$$

will be employed in those calculations. Many properties and identities for the two-parameter Mittag-Leffler are derived in [26].

In [24], a boundary value problem,

$$D_0^\alpha y(t) + \beta D_0^{\alpha-1} y(t) = f(t), \quad 0 < t \le 1, \quad 1 < \alpha \le 2,$$

$$y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1),$$

was studied. This is an example of a boundary value problem at resonance since $< t^{\alpha-1} >$, the linear span of $t^{\alpha-1}$, denotes the solution space of the homogeneous problem, $D_0^\alpha y = 0$, with the given homogeneous boundary conditions; moreover, $\beta = 0$ is a simple eigenvalue of the homogeneous problem. There, an abstract theorem was proved that gave the existence of $\mathcal{B} > 0$, and a constant $K > 0$, independent of f, such that

$$\beta D_0^{\alpha-1}y(t) \geq K|f|_1, \quad \beta \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}, \quad 0 \leq t \leq 1, \tag{2}$$

where y is the unique solution associated with f. Thus, $f \geq 0$ implies $\beta D_0^{\alpha-1}y \geq 0$. It was also proved in [24] that $\beta D_0^{\alpha-1}y(t) \geq 0$, $y(0) = 0$, implies $\beta y \geq 0$. Thus, with control of the sign of both $\beta D_0^{\alpha-1}y$ and y, a monotone method was developed to obtain sufficient conditions for a solution of the nonlinear problem,

$$D_0^{\alpha}y(t) + \beta D_0^{\alpha-1}y(t) = f(t, y(t), D_0^{\alpha-1}y(t)), \quad 0 < t \leq 1, \quad 1 < \alpha \leq 2,$$

$$y(0) = 0, \quad D_0^{\alpha-1}y(0) = D_0^{\alpha-1}y(1).$$

Since the purpose of this study is to modify the methods developed in [9] to apply to Neumann-like or periodic-like boundary conditions, we shall focus on a differential equation,

$$D_0^{\alpha}y(t) + \lambda y(t) = f(t), \quad 0 < t \leq 1, \quad n-1 < \alpha \leq n,$$

where $n \geq 2$ is an integer.

Consider the fractional differential equation To study the Neumann-like boundary conditions, assume $1 < \alpha \leq 2$. Consider the fractional differential equation

$$D_0^{\alpha}y(t) + \lambda y(t) = f(t), \quad 0 < t \leq 1, \quad 1 < \alpha \leq 2, \tag{3}$$

We shall refer to the boundary conditions

$$D_0^{\alpha-1}y(0) = 0, \quad D_0^{\alpha-1}y(1) = 0, \tag{4}$$

as Neumann boundary conditions. The first exhibited boundary value problem is the boundary value problem, (3), (4).

To study periodic-like boundary conditions we shall consider a fractional differential equation

$$D_0^{\alpha}y(t) + aD_0^{\alpha-1}y(t) + \lambda y(t) = f(t), \quad 0 < t \leq 1, \quad 1 < \alpha \leq 2, \tag{5}$$

or

$$D_0^{\alpha}y(t) + \lambda y(t) = f(t), \quad 0 < t \leq 1, \quad n-1 < \alpha \leq n. \tag{6}$$

In the second exhibited example, we study the boundary value problem, (5), with boundary conditions

$$I_0^{n-\alpha}y(0) = I_0^{n-\alpha}y(1), \quad D_0^{\alpha-1}y(0) = D_0^{\alpha-1}y(1),$$

in the third exhibited example, we study the boundary value problem, (6), with the boundary conditions

$$I_0^{n-\alpha}y(0) = I_0^{n-\alpha}y(1), \quad D_0^{\alpha-j}y(0) = D_0^{\alpha-j}y(1), \quad j = 1, \ldots, n-1, \tag{7}$$

and in the final exhibited boundary value problem we study the boundary value problem, (6), with the boundary conditions

$$\lim_{t \to 0^+} t^{n-\alpha}y(t) = y(1), \quad D_0^{\alpha-j}y(0) = D_0^{\alpha-j}y(1), \quad j = 1, \ldots, n-1. \tag{8}$$

3. The Abstract Theorem

Let $C[0,1]$ denote the Banach space of continuous functions defined on $[0,1]$ with the supremum norm, $|\cdot|_0$, and let $\mathcal{L}[0,1]$ denote the space of Lebesgue integrable functions with the usual $\mathcal{L}_1$ norm. Let $n \geq 2$ denote an integer. Assume $n-1 < \alpha \leq n$. Employing notation introduced in [28], define

$$C_{\alpha-n}[0,1] = \{y : (0,1] \to \mathbb{R} : y(t) \text{ is continuous for } t \in (0,1], \text{ and } \lim_{t\to 0^+} t^{n-\alpha} y(t) \text{ exists } \}.$$

It is clear that $y \in C_{\alpha-n}[0,1]$ if, and only if, there exists $z \in C[0,1]$ such that $y(t) = t^{\alpha-n} z(t)$ for $t \in (0,1]$. Define $|y|_{\alpha-n} = |z|_0$ and $C_{\alpha-n}[0,1]$ with norm $|\cdot|_{\alpha-n}$ is a Banach space.

The following definition is motivated by Definition 1 found in [9].

Definition 1. *Assume $\mathcal{A}$ is a linear operator with Dom $(\mathcal{A}) \subset C_{\alpha-n}$ and Im $(\mathcal{A}) \subset \mathcal{L}[0,1]$. For $\lambda \in \mathbb{R} \setminus \{0\}$, the operator $\mathcal{A} + \lambda\mathcal{I}$, where $\mathcal{I}$ denotes the identity operator, satisfies a signed maximum principle in λy if for each $f \in \mathcal{L}[0,1]$, the equation*

$$(\mathcal{A} + \lambda\mathcal{I})y = f, \quad y \in \text{ Dom } (\mathcal{A}),$$

has unique solution y, and $f \geq 0$, implies $\lambda y(t) \geq 0$, $0 < t \leq 1$. The operator $\mathcal{A} + \lambda\mathcal{I}$ satisfies a strong signed maximum principle in λy if $f \geq 0$, and $f(t) \neq 0$ a.e. implies $\lambda y(t) > 0$, $0 < t \leq 1$.

Remark 1. *In [9], the authors employed the phrase, maximum principle. We have taken the liberty to employ the phrase signed maximum principle to distinguish further from classical usage of maximum principle or anti-maximum principle.*

Remark 2. *The phrases "maximum principle" or "anti-maximum principle" are used loosely and we mean the following. Maximum principle means $f \geq 0$ implies $y \leq 0$. This is precisely the case for the classical second order ordinary differential equation with Dirichlet boundary conditions. Anti-maximum principle means $f \geq 0$ implies $y \geq 0$. This is the case observed in [8] for $\alpha = 2$, where the phrase anti-maximum principle was coined.*

For $f \in \mathcal{L}[0,1]$ (or $f \in C_{\alpha-n}[0,1]$), let $|f|_1 = \int_0^1 |f(s)|ds$ and define $\overline{f} = \int_0^1 f(t)dt$. Define

$$\tilde{C} \subset C_{\alpha-n}[0,1] = \{y \in C_{\alpha-n}[0,1] : \overline{y} = 0\}, \quad \tilde{\mathcal{L}} \subset \mathcal{L}[0,1] = \{f \in \mathcal{L}[0,1] : \overline{f} = 0\}.$$

Assume $\mathcal{A} :$ Dom $(\mathcal{A}) \to \mathcal{L}[0,1]$ denotes a linear operator satisfying

$$\text{Dom } (\mathcal{A}) \subset C_{\alpha-n}[0,1], \quad \text{Ker } (\mathcal{A}) = < t^{\alpha-n} >, \quad \text{Im } (\mathcal{A}) = \tilde{\mathcal{L}}, \tag{9}$$

where $< t^{\alpha-n} >$ denotes the linear span of $t^{\alpha-n}$. Assume further that for $\tilde{f} \in \tilde{\mathcal{L}}$, the problem $\mathcal{A}y = \tilde{f}$ is uniquely solvable with solution $\tilde{y} \in \text{Dom}(\mathcal{A})$ and such that $\int_0^1 \tilde{y}(t)dt = \overline{\tilde{y}} = 0$. In particular, define

$$\text{Dom } (\tilde{A}) = \{\tilde{y} \in \text{ Dom } (\mathcal{A}) : \overline{\tilde{y}} = 0\} \subset \tilde{C}, \tag{10}$$

and then

$$\mathcal{A}|_{\text{ Dom } (\tilde{A})} : \text{ Dom } (\tilde{A}) \to \tilde{\mathcal{L}}$$

is invertible. Moreover, if $\mathcal{A}\tilde{y} = \tilde{f}$ for $\tilde{f} \in \tilde{\mathcal{L}}$, $\tilde{y} \in \text{Dom}(\tilde{A})$, assume there exists a constant $M > 0$ depending only on $\mathcal{A}$ such that

$$|\tilde{y}|_{\alpha-n} \leq M|\tilde{f}|_1. \tag{11}$$

For $f \in \mathcal{L}$, define

$$\tilde{f} = f - (\alpha - n + 1)\overline{f} t^{\alpha-n},$$

which implies $\tilde{f} \in \tilde{\mathcal{L}}$, and for $y \in$ Dom $(\mathcal{A})$ define

$$\tilde{y} = y - (\alpha - n + 1)\overline{y} t^{\alpha-n},$$

which implies $\tilde{y} \in \text{Dom }(\tilde{\mathcal{A}})$.

Since $\text{Ker}(\mathcal{A}) =< t^{\alpha-n} >$, with the decompositions $\tilde{f} = f - (\alpha - n + 1)\overline{f}t^{\alpha-n}$ and $\tilde{y} = y - (\alpha - n + 1)\overline{y}t^{\alpha-n}$, it follows that

$$\mathcal{A}y + \lambda y = f, \quad y \in \text{ Dom }(\mathcal{A}), \tag{12}$$

which decouples as follows:

$$\mathcal{A}\tilde{y} + \lambda\tilde{y} = (\mathcal{A} + \lambda\mathcal{I})\tilde{y} = \tilde{f}, \tag{13}$$

$$\lambda(\alpha - n + 1)\overline{y}t^{\alpha-n} = (\alpha - n + 1)\overline{f}t^{\alpha-n}. \tag{14}$$

Denote the inverse of $(\mathcal{A} + \lambda\mathcal{I})$, if it exists, by $\mathcal{R}_\lambda$ and denote the inverse of

$$\mathcal{A}|_{\text{ Dom }(\tilde{\mathcal{A}})}$$

by $\mathcal{R}_0$. So, $\mathcal{R}_0 : \tilde{\mathcal{L}} \to \tilde{C}$ and

$$\tilde{y} = \mathcal{R}_0\tilde{f} \text{ if, and only if, } \mathcal{A}\tilde{y} = \tilde{f}. \tag{15}$$

Note that (15) implies that since $\tilde{y} \in \text{Dom}(\tilde{\mathcal{A}})$,

$$\tilde{y} = \mathcal{R}_0\mathcal{A}\tilde{y}. \tag{16}$$

Note that (11) implies that $\mathcal{R}_0 : \tilde{\mathcal{L}} \to \tilde{C}$ is continuous, and hence, $\mathcal{R}_0 : \tilde{\mathcal{L}} \to \tilde{C}$ is a bounded linear operator with $||\mathcal{R}_0||_{\tilde{\mathcal{L}} \to \tilde{C}} \leq M$. To note the continuity, if $\mathcal{R}_0(\tilde{f}_n) = \tilde{y}_n, \mathcal{R}_0(\tilde{f}) = \tilde{y}$, and $|\tilde{f}_n - \tilde{f}|_1 \to 0$, as $n \to \infty$, then $|\tilde{y}_n - \tilde{y}|_{\alpha-n} \leq M|\tilde{f}_n - \tilde{f}|_1 \to 0$, as $n \to \infty$.

Since $\tilde{C} \subset \tilde{\mathcal{L}}$, we can also consider $\mathcal{R}_0 : \tilde{C} \to \tilde{C}$. Equation (11) also implies that $\mathcal{R}_0 : \tilde{\mathcal{C}} \to \tilde{C}$ is continuous and hence, bounded. To see this, assume $|\tilde{f}_n - \tilde{f}|_{\alpha-n} \to 0$, as $n \to \infty$. Then, $t^{2-\alpha}|\tilde{f}_n - \tilde{f}| \to 0$ uniformly as $n \to \infty$. For each $\epsilon > 0$, $|\tilde{f}_n - \tilde{f}|(t) < \epsilon t^{\alpha-2}$ and $|\tilde{f}_n - \tilde{f}|_1 < \frac{\epsilon}{\alpha-1}$, eventually; in particular, $|\tilde{f}_n - \tilde{f}|_1 \to 0$, as $n \to \infty$, which implies $|\tilde{y}_n - \tilde{y}|_{\alpha-n} \to 0$, as $n \to \infty$.

Theorem 1. *Assume $\mathcal{A} :$ Dom $(\mathcal{A}) \to \mathcal{L}[0,1]$ denotes a linear operator satisfying (9). Define $\tilde{\mathcal{A}}$ by (10) and assume*

$$\mathcal{A}|_{\textit{ Dom }(\tilde{\mathcal{A}})} : \textit{ Dom }(\tilde{\mathcal{A}}) \to \tilde{\mathcal{L}}$$

is invertible. Finally, if $\mathcal{A}\tilde{y} = \tilde{f}$ for $\tilde{f} \in \tilde{\mathcal{L}}$, $\tilde{y} \in$ Dom$(\tilde{\mathcal{A}})$, assume there exists a constant $M > 0$ depending only on $\mathcal{A}$ such that (11) is satisfied. Then there exists $\Lambda_1 > 0$ such that if $0 < |\lambda| \leq \Lambda_1$, then $\mathcal{R}_\lambda : \tilde{C} \to \tilde{C}$, the inverse of $(\mathcal{A} + \lambda\mathcal{I})$, exists. Moreover, if $\tilde{f} \in \tilde{L}$, if $\Lambda_1||\mathcal{R}_0||_{\tilde{C} \to \tilde{C}} < 1$, where $\mathcal{R}_0$ denotes the inverse of $\mathcal{A}|_{\textit{ Dom }(\tilde{\mathcal{A}})}$, and if $0 < |\lambda| \leq \Lambda_1$, then

$$|\mathcal{R}_\lambda\tilde{f}|_{\alpha-n} \leq \frac{||\mathcal{R}_0||_{\tilde{\mathcal{L}} \to \tilde{C}}}{1 - \Lambda_1||\mathcal{R}_0||_{\tilde{C} \to \tilde{C}}}|\tilde{f}|_1. \tag{17}$$

Further, there exists $\Lambda \in (0, \Lambda_1)$ such that if $0 < |\lambda| \leq \Lambda$, then the operator $(\mathcal{A} + \lambda\mathcal{I})$ satisfies a strong signed maximum principle in λy.

Proof. Employ (16) and apply $\mathcal{R}_0$ to (13) to obtain

$$\tilde{y} + \lambda\mathcal{R}_0\tilde{y} = \mathcal{R}_0\tilde{f}.$$

It has been established that (11) implies that each of $\mathcal{R}_0 : \tilde{\mathcal{L}} \to \tilde{C}$ and $\mathcal{R}_0 : \tilde{\mathcal{C}} \to \tilde{C}$ are bounded linear operators. Since $|\lambda|||\mathcal{R}_0||_{\tilde{C} \to \tilde{C}} < 1$, it follows that $(\mathcal{I} + \lambda\mathcal{R}_0) : \tilde{C} \to \tilde{C}$ is invertible and

$$\tilde{y} = (\mathcal{I} + \lambda\mathcal{R}_0)^{-1}\mathcal{R}_0\tilde{f}.$$

Assume $0 < \Lambda_1 < \frac{1}{||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}$ and assume $|\lambda| \leq \Lambda_1$. Then, $\mathcal{R}_\lambda = (\mathcal{I} + \lambda\mathcal{R}_0)^{-1}\mathcal{R}_0$ exists. Since $\Lambda_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} < 1$ and $0 < |\lambda| \leq \Lambda_1$, it follows that

$$|\tilde{y}|_{\alpha-n} - |\lambda\mathcal{R}_0\tilde{y}|_{\alpha-1} = \Big||\tilde{y}|_{\alpha-n} - |\lambda\mathcal{R}_0\tilde{y}|_{\alpha-1}\Big|$$

and so the triangle inequality implies

$$\begin{aligned}|\tilde{y}|_{\alpha-n} - \Lambda_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}|\tilde{y}|_{\alpha-n} &\leq |\tilde{y}|_{\alpha-n} - |\lambda|||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}|\tilde{y}|_{\alpha-n} \\ &\leq |(\mathcal{I} + \lambda\mathcal{R}_0)\tilde{y}|_{\alpha-n} = |\mathcal{R}_0\tilde{f}|_{\alpha-n} \leq ||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}|\tilde{f}|_1.\end{aligned}$$

Thus, (17) is proved since $\mathcal{R}_0\tilde{f} = \tilde{y} \in C_{\alpha-n}[0,1]$.

Now assume $f \in \mathcal{L}[0,1]$ and assume $f \geq 0$ a.e. Then, $\overline{f} = |f|_1$. Let $0 < |\lambda| \leq \Lambda_1 < \frac{1}{||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}$, write $f = (\alpha - n + 1)\overline{f}t^{\alpha-n} + \tilde{f}$, and consider

$$\lambda y = \lambda\mathcal{R}_\lambda f = \lambda\mathcal{R}_\lambda\Big((\alpha - n + 1)\overline{f}t^{\alpha-n} + \tilde{f}\Big).$$

Note that $\lambda\mathcal{R}_\lambda(\alpha - n + 1)\overline{f}t^{\alpha-n} = (\alpha - n + 1)\overline{f}t^{\alpha-n}$ since $(\mathcal{A} + \lambda\mathcal{I})(\alpha - n + 1)\overline{f}t^{\alpha-n} = \lambda(\alpha - (n-1))\overline{f}t^{\alpha-n}$. Thus,

$$\begin{aligned}\lambda y = \lambda\mathcal{R}_\lambda f &= \lambda\mathcal{R}_\lambda\Big((\alpha - n + 1)\overline{f}t^{\alpha-n} + \tilde{f}\Big) \\ &= (\alpha - n + 1)\overline{f}t^{\alpha-n} + \lambda\mathcal{R}_\lambda\tilde{f} \geq (\alpha - n + 1)|f|_1 - |\lambda||\mathcal{R}_\lambda\tilde{f}|_{\alpha-n}.\end{aligned}$$

Continuing to assume that $0 < |\lambda| \leq \Lambda_1$, it now follows from (17) that

$$\lambda y \geq (\alpha - (n-1))|f|_1 - |\lambda|\Big(\frac{||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - \Lambda_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\Big)|\tilde{f}|_1.$$

Since $\tilde{f} = f - (\alpha - n + 1)\overline{f}t^{\alpha-n}$, and $|\tilde{f}|_1 \leq |f|_1 + \overline{f} = 2|f|_1$, the theorem is proved with

$$\Lambda < \min\Big\{\Lambda_1, (\alpha - n + 1)\Big(\frac{1 - \Lambda_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}{2||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}\Big)\Big\}.$$

In particular, if $0 < |\lambda| \leq \Lambda$, then

$$\lambda y(t) \geq K|f|_1 = (\alpha - n + 1)\Big(1 - \Lambda\Big(\frac{2||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - \Lambda_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\Big)\Big)|f|_1.$$

□

4. Four Examples

To apply Theorem 1, there are two primary tasks. First, if $\tilde{f} \in \tilde{\mathcal{L}}$, we must show there exists a unique solution $\tilde{y} \in \mathrm{Dom}\,(\mathcal{A})$ of $\mathcal{A}y = \tilde{f}$ satisfying $\overline{\tilde{y}} = 0$. In the case of ordinary differential equations or partial differential equations, one can often appeal to a Fredholm alternative to complete this task. For the Riemann–Liouville fractional differential equation, we only know to construct $\tilde{y}$ explicitly, and show uniqueness to complete this task. Second, we must show the existence of a constant $M > 0$ such that $|\tilde{y}|_{\alpha-n} \leq M|\tilde{f}|_1$. This will be a straightforward task since we will have constructed $\tilde{y}$ explicitly.

Example 1. *Let $1 < \alpha \leq 2$, and consider the linear boundary value problem, with a Riemann–Liouville analogue of Neumann boundary conditions,* (3), (4)*; that is, consider,*

$$D_0^\alpha y(t) + \lambda y(t) = f(t), \quad 0 < t \leq 1,$$

$$D_0^{\alpha-1}y(0) = 0, \quad D_0^{\alpha-1}y(1) = 0.$$

For the boundary value problem (3), (4), $\mathcal{A} = D_0^\alpha$, *and* $Ker(\mathcal{A}) =< t^{\alpha-2} >$. *We show that the operator* $\mathcal{A}$ *satisfies the hypotheses of Theorem 1.*

One can show directly that $Im(\mathcal{A}) = \tilde{\mathcal{L}}$. *If* $f \in Im(\mathcal{A})$, *then there exists a solution* y *of*

$$D_0^\alpha y(t) = f(t), \quad 0 < t \le 1, \quad D_0^{\alpha-1} y(0) = 0, \quad D_0^{\alpha-1} y(1) = 0,$$

which implies

$$0 = D_0^{\alpha-1} y(1) - D_0^{\alpha-1} y(0) = \int_0^1 D_0^\alpha y(t) dt = \int_0^1 f(t) dt,$$

and $f \in \tilde{\mathcal{L}}$. *Likewise, if* $f \in \tilde{\mathcal{L}}$, *then*

$$\begin{aligned} \tilde{y}(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds - \frac{(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha f(s) ds \\ &= I_0^\alpha f(t) - (\alpha-1) I_0^{\alpha+1} f(1) t^{\alpha-2} \in \; Dom\,(\mathcal{A}) \end{aligned} \tag{18}$$

is a solution of

$$D_0^\alpha y(t) = f(t), \quad 0 < t \le 1, \quad D_0^{\alpha-1} y(0) = 0, \quad D_0^{\alpha-1} y(1) = 0,$$

and $\overline{\tilde{y}} = 0$. *To verify that* $\tilde{y}$ *satisfies these properties, note that any solution of* $D_0^\alpha y(t) = f(t)$, $0 < t \le 1$, *has the form,* $I_0^\alpha f(t) + c_2 t^{\alpha-2} + c_1 t^{\alpha-1}$. *Thus,* $D_0^\alpha \tilde{y}(t) = f(t), 0 < t \le 1$. *To see that the boundary conditions are satisfied, write*

$$D_0^{\alpha-1} I_0^\alpha f(t) = D_0^{\alpha-1} I_0^{\alpha-1} I_0^1 f(t) = I_0^1 f(t) = \int_0^t f(s) ds,$$

and note that $D_0^{\alpha-1} t^{\alpha-2} = 0$. *Thus,* $D_0^{\alpha-1} I_0^\alpha f|_{t=0} = 0$, *and* $D_0^{\alpha-1} I_0^\alpha f|_{t=1} = 0$ *since* $f \in \tilde{\mathcal{L}}$; *in particular, the boundary conditions are satisfied. To see that* $\overline{\tilde{y}} = 0$, *note that*

$$I I_0^\alpha f(t) = I_0^{\alpha+1} f(t)$$

and so,

$$\overline{\tilde{y}} = \overline{I_0^\alpha f} - I_0^{\alpha+1} f(1) = I_0^{\alpha+1} f(1) - I_0^{\alpha+1} f(1) = 0.$$

To argue that $\mathcal{A}y = \tilde{f}$ *is uniquely solvable with solution* $\tilde{y} \in Dom\,(\tilde{\mathcal{A}})$, (18) *implies the solvability. For uniqueness, if* y_1 *and* y_2 *are two such solutions, then* $(y_1 - y_2)(t) = ct^{\alpha-2}$ *and* $\overline{y_1 - y_2} = 0$ *implies* $c = 0$.

Finally, (18) *implies* (11) *is satisfied with* $M = \frac{1}{\Gamma(\alpha)} + \frac{\alpha-1}{\Gamma(\alpha+1)} = \frac{2\alpha-1}{\Gamma(\alpha+1)}$.

Theorem 1 applies and there exists $\Lambda > 0$ *such that if* $0 < |\lambda| \le \Lambda$, *then* $(\mathcal{A} + \lambda\mathcal{I})$ *satisfies a signed maximum principle in* y; *that is,* $f \ge 0$ *implies* $\lambda y \ge 0$.

Example 2. *For the second example, let* $1 < \alpha \le 2$, *and let* $a \in \mathbb{R}$. *Consider the linear boundary value problem, with a Riemann–Liouville analogue of periodic boundary conditions,* (5), (7); *that is, consider,*

$$D_0^\alpha y(t) + a D_0^{\alpha-1} y(t) + \lambda y(t) = f(t), \quad 0 < t \le 1,$$
$$I_0^{2-\alpha} y(0) = I_0^{2-\alpha} y(1), \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1).$$

Now, $\mathcal{A} = D_0^\alpha + a D_0^{\alpha-1}$, *and* $Ker(\mathcal{A}) =< t^{\alpha-2} >$. *We show that the operator* $\mathcal{A}$ *satisfies the hypotheses of Theorem 1.*

We show directly that $Im\,(\mathcal{A}) = \tilde{\mathcal{L}}$. *If* $f \in Im\,(\mathcal{A})$, *then*

$$\begin{aligned} I f(t) &= I\Big(D_0^\alpha y(t) + a D_0^{\alpha-1} y(t)\Big) \\ &= \Big(D_0^{\alpha-1} y(t) - D_0^{\alpha-1} y(0)\Big) + a\Big(I_0^{2-\alpha} y(t) - I_0^{2-\alpha} y(0)\Big); \end{aligned}$$

thus, $If(1) = 0$ since y satisfies the periodic boundary conditions. In particular, $f \in \tilde{\mathcal{L}}$.

Now assume $f \in \tilde{\mathcal{L}}$. We first construct a general solution of

$$D_0^\alpha y(t) + aD_0^{\alpha-1}y(t) = f(t), \quad 0 < t \leq 1,$$

$$I_0^{2-\alpha}y(0) = I_0^{2-\alpha}y(1), \quad D_0^{\alpha-1}y(0) = D_0^{\alpha-1}y(1).$$

Since $D_0^\alpha y = DD_0^{\alpha-1}y$, apply an integrating factor, e^{at}, and

$$D(e^{at}D_0^{\alpha-1}y(t)) = e^{at}f(t),$$

which implies

$$D_0^{\alpha-1}y(t) = D_0^{\alpha-1}y(0)e^{-at} + \int_0^t e^{-a(t-s)}f(s)ds.$$

Then,

$$\begin{aligned} y(t) &= ct^{(\alpha-1)-1} + I_0^{\alpha-1}\Big(D_0^{\alpha-1}y(0)e^{-at} + \int_0^t e^{-a(t-s)}f(s)ds\Big) \\ &= ct^{\alpha-2} + D_0^{\alpha-1}y(0)I_0^{\alpha-1}e^{-at} + I_0^{\alpha-1}\Big(\int_0^t e^{-a(t-s)}f(s)ds\Big). \end{aligned}$$

Apply the periodic boundary conditions. Then,

$$D_0^{\alpha-1}y(t) = D_0^{\alpha-1}y(0)e^{-at} + \int_0^t e^{-a(t-s)}f(s)ds,$$

and the boundary condition $D_0^{\alpha-1}y(0) = D_0^{\alpha-1}y(1)$ implies

$$D_0^{\alpha-1}y(0) = \frac{1}{1-e^{-a}}\int_0^1 e^{-a(1-s)}f(s)ds$$

is uniquely determined. Now,

$$\begin{aligned} I_0^{2-\alpha}y(t) &= c\Gamma(\alpha-1) + D_0^{\alpha-1}y(0)Ie^{-at} + I\Big(\int_0^t e^{-a(t-s)}f(s)ds\Big) \\ &= c\Gamma(\alpha-1) + D_0^{\alpha-1}y(0)\int_0^t e^{-as}ds + \int_0^t\Big(\int_0^s e^{-a(s-r)}f(r)dr\Big)ds \\ &= c\Gamma(\alpha-1) + D_0^{\alpha-1}y(0)\frac{(1-e^{-at})}{a} - \int_0^t \frac{(e^{-a(t-s)}-1)}{a}f(s)ds. \end{aligned}$$

Thus, $I_0^{2-\alpha}y(0) = c\Gamma(\alpha-1)$ and

$$\begin{aligned} I_0^{2-\alpha}y(1) &= c\Gamma(\alpha-1) + D_0^{\alpha-1}y(0)\frac{(1-e^{-a})}{a} - \int_0^1 \frac{(e^{-a(1-s)}-1)}{a}f(s)ds \\ &= c\Gamma(\alpha-1) + \int_0^1 \frac{e^{-a(1-s)}}{a}f(s)ds - \int_0^1 \frac{e^{-a(1-s)}}{a}f(s)ds - \frac{1}{a}\int_0^1 f(s)ds \\ &= c\Gamma(\alpha-1). \end{aligned}$$

At this point in the construction, c is still undetermined and

$$y(t) = ct^{\alpha-2} + D_0^{\alpha-1}y(0)I_0^{\alpha-1}e^{-at} + I_0^{\alpha-1}\Big(\int_0^t e^{-a(t-s)}f(s)ds\Big)$$

is a general solution of

$$D_0^\alpha y(t) + aD_0^{\alpha-1}y(t) = f(t), \quad 0 < t \leq 1,$$

$$I_0^{2-\alpha}y(0) = I_0^{2-\alpha}y(1), \quad D_0^{\alpha-1}y(0) = D_0^{\alpha-1}y(1).$$

To obtain the parameter c uniquely, Theorem 1 requires that $\overline{y} = 0$. *Thus,*

$$0 = \frac{c}{\alpha-1} + D_0^{\alpha-1}y(0)\overline{I_0^{\alpha-1}e^{-at}} + \overline{I_0^{\alpha-1}\int_0^t e^{-a(t-s)}f(s)ds}$$

and

$$c = (1-\alpha)D_0^{\alpha-1}y(0)\overline{I_0^{\alpha-1}e^{-at}} + \overline{I_0^{\alpha-1}\int_0^t e^{-a(t-s)}f(s)ds}$$

is uniquely determined.

Note that

$$D_0^{\alpha-1}y(0) = \frac{1}{1-e^{-a}}\int_0^1 e^{-a(1-s)}f(s)ds \text{ implies } |D_0^{\alpha-1}y(0)| \leq \frac{1}{1-e^{-a}}|f|_1.$$

Thus,

$$y(t) = ct^{\alpha-2} + D_0^{\alpha-1}y(0)I_0^{\alpha-1}e^{-at} + I_0^{\alpha-1}\Big(\int_0^t e^{-a(t-s)}f(s)ds\Big)$$

implies (11) is satisfied.

This concludes the second example.

Before proceeding to the third example, we observe that Theorem 1 does not apparently apply to a Neumann boundary value problem (5), (4) in the case $1 < \alpha \leq 2$, $a \neq 0$. *Assume* $f \in \tilde{\mathcal{L}}$ *and begin the construction of a general solution. As before, one obtains*

$$D_0^{\alpha-1}y(t) = D_0^{\alpha-1}y(0)e^{-at} + \int_0^t e^{-a(t-s)}f(s)ds = \int_0^t e^{-a(t-s)}f(s)ds.$$

Take for example, $f(t) = t - \frac{1}{2} \in \tilde{\mathcal{L}}$. *Then,* $D_0^{\alpha-1}y(1) \neq 0$.

Example 3. *For the third example, let* $n \geq 2$, *let* $n-1 < \alpha \leq n$, *and consider the linear boundary value problem, with a Riemann–Liouville analogue of periodic boundary conditions, (6), (7); that is, consider,*

$$D_0^{\alpha}y(t) + \lambda y(t) = f(t), \quad 0 \leq t \leq 1,$$

$$I_0^{n-\alpha}y(0) = I_0^{n-\alpha}y(1), \quad D_0^{\alpha-j}y(0) = D_0^{\alpha-j}y(1), \quad j = 1, \ldots, n-1.$$

For the boundary value problem (6), (7), $\mathcal{A} = D_0^{\alpha}$ *and* $Ker(\mathcal{A}) =< t^{\alpha-n} >$. *Again, we show Im* $(\mathcal{A}) = \tilde{\mathcal{L}}$. *First, note that if the boundary value problem (6), (7) is solvable, then the boundary condition* $D_0^{\alpha-1}y(0) = D_0^{\alpha-1}y(1)$ *implies* $f \in \tilde{\mathcal{L}}$ *since* $If(t) = ID_0^{\alpha}y(t) = D_0^{\alpha-1}y(t) - D_0^{\alpha-1}y(0)$. *Thus,* $\int_0^1 f(t)dt = D_0^{\alpha-1}y(0) - D_0^{\alpha-1}y(0) = 0$.

Now assume $f \in \tilde{\mathcal{L}}$. *If* $\tilde{y} \in Dom\ (\tilde{\mathcal{A}})$, *then*

$$\tilde{y}(t) = I_0^{\alpha}f(t) + \sum_{j=1}^{n} c_{\alpha-j}t^{\alpha-j}.$$

We show the coefficients $c_{\alpha-j}$ *are uniquely determined. The condition* $D_0^{\alpha-1}\tilde{y}(0) = D_0^{\alpha-1}\tilde{y}(1)$ *implies*

$$If(0) + c_{\alpha-1}\Gamma(\alpha) = If(1) + c_{\alpha-1}\Gamma(\alpha)$$

which implies $c_{\alpha-1}$ *is undetermined at this point in the construction. Let* $k \in \{2, \ldots, n\}$. *Then,*

$$D_0^{\alpha-k}\tilde{y}(t) = I_0^k f(t) + \sum_{j=1}^{k} c_{\alpha-j}\frac{\Gamma(\alpha+1-j)}{\Gamma(k+1-j)}t^{k-j}. \tag{19}$$

Apply the boundary conditions $D_0^{\alpha-j}\tilde{y}(0) = D_0^{\alpha-j}\tilde{y}(1)$ *in the order* $j = 2, \ldots, n$. *At* $j = 2$,

$$I^2 f(0) + c_{\alpha-2}\Gamma(\alpha-1) = D_0^{\alpha-2}\tilde{y}(0) = D_0^{\alpha-2}\tilde{y}(1) = I^2 f(1) + c_{\alpha-2}\Gamma(\alpha-1) + c_{\alpha-1}\frac{\Gamma(\alpha)}{\Gamma(2)}.$$

Thus, $c_{\alpha-1} = -\frac{\Gamma(2)}{\Gamma(\alpha)} I^2 f(1)$ *is uniquely determined. Employ* (19) *inductively and for* $j = k$,

$$\begin{aligned} I^k f(0) + c_{\alpha-k}\Gamma(\alpha+1-k)) = D_0^{\alpha-k}\tilde{y}(0) &= D_0^{\alpha-k}\tilde{y}(1) \\ &= I^k f(1) + c_{\alpha-k}\Gamma(\alpha+1-k) + \sum_{j=1}^{k-1} c_{\alpha-j}\frac{\Gamma(\alpha+1-j)}{\Gamma(k+1-j)}. \end{aligned}$$

Inductively, $c_{\alpha-j}$, $j = 1, \ldots k-2$ *have been uniquely determined and so,*

$$c_{\alpha-(k-1)} = -\frac{\Gamma(2)}{\Gamma(\alpha-(k-2))}\left(I^k f(1) + \sum_{j=1}^{k-2} c_{\alpha-j}\frac{\Gamma(\alpha+1-j)}{\Gamma(k+1-j)}\right) \quad (20)$$

is uniquely determined. To summarize, the boundary conditions $D_0^{\alpha-j}y(0) = D_0^{\alpha-j}y(1), j = 1, \ldots, n-1$, *uniquely determine the coefficients,* $c_{\alpha-1}, \ldots, c_{\alpha-(n-2)}$.

To determine the coefficient, $c_{\alpha-(n-1)}$, *employ the boundary condition* $I_0^{n-\alpha}\tilde{y}(0) = I_0^{n-\alpha}\tilde{y}(1)$. *Since*

$$I_0^{n-\alpha}\tilde{y}(t) = I_0^n f(t) + \sum_{j=1}^{n} c_{\alpha-j}\frac{\Gamma(\alpha+1-j)}{\Gamma(n+1-j)} t^{n-j},$$

it follows that

$$c_{\alpha-(n-1)} = -\frac{\Gamma(2)}{\Gamma(\alpha-(n-2))}\left(I^n f(1) + \sum_{j=1}^{n-2} c_{\alpha-j}\frac{\Gamma(\alpha+1-j)}{\Gamma(n+1-j)}\right) \quad (21)$$

is uniquely determined.

Finally, the application of Theorem 1 *requires that* $\overline{\tilde{y}} = 0$. *Thus,*

$$0 = \overline{I_0^\alpha f} + \frac{c_{\alpha-n}}{\alpha+1-n} + \sum_{j=1}^{n-1}\frac{c_{\alpha-j}}{\alpha+1-j}.$$

Hence, $c_{\alpha-n}$ *is uniquely determined and the proof that* $f \in \tilde{\mathcal{L}}$ *implies* $\tilde{y} \in Dom\ (\tilde{\mathcal{A}})$ *is uniquely determined is complete.*

To see that M *in* (11) *can be computed, recall that*

$$\tilde{y}(t) = I_0^\alpha f(t) + \sum_{j=1}^{n} c_{\alpha-j} t^{\alpha-j}.$$

and employ (20) *and* (21). *Note that* $c_{\alpha-1}$ *is a multiple of* $I^2 f(1)$, *which implies that* $c_{\alpha-k}$ *is a linear combination of* $I^k f(1), \ldots, I^2 f(1)$, *for* $k = 1, \ldots, n$. *Thus,* M *is computable. Thus, Theorem* 1 *applies and there exists* $\Lambda > 0$ *such that if* $0 < |\lambda| \leq \Lambda$, *then* $(\mathcal{A} + \lambda\mathcal{I})$ *satisfies the strong signed maximum principle in* y.

Example 4. *Theorem* 1 *can also apply to the boundary value problem with boundary conditions analogous to periodic boundary conditions,* (6), (8); *that is, consider,*

$$D_0^\alpha y(t) + \lambda y(t) = f(t), \quad 0 \leq t \leq 1,$$

$$\lim_{t \to 0^+} t^{n-\alpha} y(t) = y(1), \quad D_0^{\alpha-j}y(0) = D_0^{\alpha-j}y(1), \quad j = 1, \ldots, n-1.$$

The unique determination of $c_{\alpha-k}, k = 1, \ldots n-2$ *proceeds precisely as in Example* (3)*. Apply the boundary condition* $\lim_{t\to 0^+} t^{n-\alpha} y(t) = y(1)$ *to* $\tilde{y}(t) = I_0^\alpha f(t) + \sum_{j=1}^{n} c_{\alpha-j} t^{\alpha-j}$ *to obtain*

$$c_{\alpha-n} = I^n f(1) + \sum_{j=1}^{n} c_{\alpha-j}$$

and $c_{\alpha-(n-1)} = -\left(I^n f(1) + \sum_{j=1}^{n-2} c_{\alpha-j}\right)$ *is uniquely determined. Then, as in Example 3,* $c_{\alpha-n}$ *is uniquely determined by the requirement that* $\bar{\bar{y}} = 0$.

Thus, Theorem 1 applies and there exists $\Lambda > 0$ *such that if* $0 < |\lambda| \le \Lambda$ *then* $(\mathcal{A} + \lambda\mathcal{I})$ *satisfies the strong signed maximum principle in* y.

5. A Monotone Method

The application of monotone methods in the presence of a maximum principle or in the presence of an anti-maximum principle to construct approximate solutions of initial value or boundary value type problems enjoys a long history. The purpose of this section is to employ (1) to quickly recognize the presence of the maximum principle or the anti-maximum principle. There are recent applications of monotone methods to periodic-like boundary value problems for Riemann–Liouville fractional differential equations; see, for example, [6,7]. In each of those application, $0 < \alpha \le 1$, and the anti-maximum principle is observed by the explicit construction of a corresponding Green's function in terms of Mittag-Leffler functions.

Assume $f : (0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and consider the boundary value problem

$$D_0^\alpha y(t) = f(t, y(t)), \quad 0 < t \le 1, \quad 1 < \alpha \le 2, \tag{22}$$

$$D_0^{\alpha-1} y(0) = 0, \quad D_0^{\alpha-1} y(1) = 0. \tag{23}$$

Assume that

$$y(t) \in C_{\alpha-2}[0, 1] \text{ implies } f(t, y(t)) \in C_{\alpha-2}[0, 1], \tag{24}$$

and assume further that f satisfies the following monotonicity property,

$$f(t, y_1) < f(t, y_2) \text{ for } (t, y) \in (0, 1] \times \mathbb{R}, \quad y_1 > y_2. \tag{25}$$

Thus, f is monotone decreasing in the second component.

Apply Theorem 1 and find $\Lambda > 0$ such that if $0 < \lambda \le \Lambda$, then $(\mathcal{A} + \lambda\mathcal{I})$ satisfies a strong signed maximum principle in λy. Apply a shift [29] to (22) and consider the equivalent boundary value problem,

$$D_0^\alpha y(t) + \lambda y(t) = f(t, y(t)) + \lambda y(t), \quad 0 < t \le 1,$$

with boundary conditions (23) where $-\Lambda \le \lambda < 0$ and $\Lambda > 0$ is shown to exist in Theorem 1. Note that if $g(t, y) = f(t, y) + \lambda y$ and f satisfies (24) and (25), then g satisfies (24) and g satisfies (25) if $\lambda < 0$.

Assume the existence of solutions, $w_1, v_1 \in C_{\alpha-2}[0, 1]$, of the following boundary value problems for differential inequalities

$$\begin{aligned} D_0^\alpha w_1(t) &\ge f(t, w_1(t)), \quad 0 < t \le 1, & D_0^\alpha v_1(t) &\le f(t, v_1(t)), \quad 0 < t \le 1, \\ D_0^{\alpha-1} w_1(0) = 0, &\quad D_0^{\alpha-1} w_1(1) = 0, & D_0^{\alpha-1} v_1(0) = 0, &\quad D_0^{\alpha-1} v_1(1) = 0. \end{aligned} \tag{26}$$

Assume further that

$$v_1(t) - w_1(t) \ge 0, \quad 0 < t \le 1. \tag{27}$$

Since $\lambda < 0$, define a partial order $\succeq_{\lambda<0}$ on $C_{\alpha-2}[0, 1]$ by

$$u \succeq_{\lambda<0} 0 \iff u(t) \le 0, 0 < t \le 1.$$

Then, the assumption (27) implies $w_1 \succeq_{\lambda<0} v_1$.

Define iteratively the sequences $\{v_k\}_{k=1}^{\infty}$, $\{w_k\}_{k=1}^{\infty}$, where

$$D_0^{\alpha} v_{k+1}(t) + \lambda v_{k+1}(t) = f(t, v_k(t)) + \lambda v_k(t), \quad 0 < t \leq 1, \tag{28}$$
$$D_0^{\alpha-1} v_{k+1}(0) = 0, \quad D_0^{\alpha-1} v_{k+1}(1) = 0,$$

and

$$D_0^{\alpha} w_{k+1}(t) + \lambda w_{k+1}(t) = f(t, w_k(t)) + \lambda w_k(t), \quad 0 < t \leq 1, \tag{29}$$
$$D_0^{\alpha-1} w_{k+1}(0) = 0, \quad D_0^{\alpha-1} w_{k+1}(1) = 0.$$

Inductively, Theorem 1 implies the existence of each v_{k+1}, w_{k+1} since $|\lambda| \leq \Lambda$ implies the inverse of $(\mathcal{A} + \lambda \mathcal{I})$ exists, and, for example, $f(t, v_k(t)) + \lambda v_k(t) \in C_{\alpha-2}[0,1]$.

Theorem 2. *Assume $f : (0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, assume that f satisfies (24), and assume f satisfies the monotonicity properties (25). Assume the existence of functions $v_1, w_1 \in C_{\alpha-2}[0,1]$ satisfying (26) and (27). Define the sequences of iterates $\{v_k\}_{k=1}^{\infty}$, $\{w_k\}_{k=1}^{\infty}$ by (28) and (29), respectively. Then, for each positive integer k,*

$$w_k \succeq_{\lambda<0} w_{k+1} \succeq_{\lambda<0} v_{k+1} \succeq_{\lambda<0} v_k. \tag{30}$$

Moreover, $\{v_k\}_{k=1}^{\infty}$ converges in $C_{\alpha-2}$ to a solution $v \in C_{\alpha-2}[0,1]$ of the boundary value problem (22), (23) and $\{w_k\}_{k=1}^{\infty}$ converges in $C_{\alpha-2}[0,1]$ to a solution $w \in C_{\alpha-2}[0,1]$ of the boundary value problem (22), (23) satisfying

$$w_k \succeq_{\lambda<0} w_{k+1} \succeq_{\lambda<0} w \succeq_{\lambda<0} v \succeq_{\lambda<0} v_{k+1} \succeq_{\lambda<0} v_k. \tag{31}$$

Proof. Since v_1 satisfies a differential inequality given in (27), then for $0 < t \leq 1$,

$$D_0^{\alpha} v_2(t) + \lambda v_2(t) = f(t, v_1(t)) + \lambda v_1(t) \geq D_0^{\alpha} v_1(t) + \lambda v_1(t).$$

Set $u = v_2 - v_1$ and u satisfies a boundary value problem for a differential inequality,

$$D_0^{\alpha} u(t) + \lambda u(t) \geq 0, \quad 0 < t \leq 1, \quad D_0^{\alpha-1} u(0) = 0, \quad D_0^{\alpha-1} u(1) = 0.$$

The signed maximum principle applies and $u \succeq_{\lambda<0} 0$; in particular, $v_2 \succeq_{\lambda<0} v_1$. Similarly, $w_1 \succeq_{\lambda<0} w_2$. Now set $u = w_2 - v_2$ and

$$D_0^{\alpha} u(t) + \lambda u(t) = \Big(f(t, w_1(t)) - f(t, v_1(t)) \Big) + \lambda (w_1(t) - v_1(t)), \quad 0 < t \leq 1,$$
$$D_0^{\alpha-1} u(0) = 0, \quad D_0^{\alpha-1} u(1) = 0.$$

Since f satisfies (25) and $w_1 \succeq_{\lambda<0} v_1$, then

$$D_0^{\alpha} u(t) + \lambda u(t) \geq 0, \quad 0 \leq t \leq 1,$$

and again the signed maximum principle applies and $u \succeq_{\lambda<0} 0$. In particular, $w_2 \succeq_{\lambda<0} v_2$. Thus, (30) is proved for $k = 1$.

Before applying a straightforward induction to obtain (30), we must show $D_0^{\alpha} w_2(t) \geq f(t, w_2(t))$, and $D_0^{\alpha} v_2(t) \leq f(t, v_2(t))$, for $0 < t \leq 1$. Since $f(t, v_1(t)) \leq f(t, v_2(t))$, $\lambda < 0$ and $(v_1 - v_2)(t) \geq 0$, it follows that

$$D_0^{\alpha} v_2(t) = f(t, v_1(t)) + \lambda (v_1 - v_2)(t) \leq f(t, v_2(t)).$$

Similarly, $D_0^{\alpha} w_2(t) \geq f(t, w_2(t))$ and (30) is valid.

To obtain the existence of limiting solutions v and w satisfying (31), note that the sequence $\{v_k\}$ is monotone decreasing and bounded below by $\{w_1\}$. Thus, the sequence $\{v_k\}$ is converging pointwise to some $v(t)$ for each $t \in (0,1]$. Moreover, if

$$z_k(t) = t^{2-\alpha} v_k \in C[0,1], \quad z_k(0) = a_k,$$

the sequence $\{z_k\}$ is converging pointwise to some $z(t) = t^{2-\alpha}v(t)$, $z(0) = a_0$ where a_k converges monotonically to a_0. At this point in the argument, the convergence is pointwise. Since

$$D_0^\alpha v_{k+1}(t) = f(t, v_k(t)) + \lambda(v_k(t) - v_{k+1}(t)), \quad 0 < t \le 1,$$

if follows that $\{D_0^\alpha v_k\}$ is converging pointwise to $g(t) = f(t, v(t))$ for each $t \in (0,1]$. Since $D_0^{\alpha-1} v_k(0) = 0$,

$$v_k(t) = a_k t^{\alpha-2} + I_0^\alpha D_0^\alpha v_k(t), \quad 0 < t \le 1.$$

Thus, by the dominated convergence theorem

$$v(t) = a_0 t^{\alpha-2} + I_0^\alpha g(t), \quad 0 < t \le 1;$$

in particular,

$$D_0^\alpha v(t) = g(t) = f(t, v(t)), \quad 0 < t \le 1,$$

and v satisfies the fractional differential equation. To see that v satisfies the Neumann type boundary conditions, again observe

$$D_0^\alpha v_{k+1}(t) = f(t, v_k(t)) + \lambda(v_k(t) - v_{k+1}(t)), \quad 0 < t \le 1,$$

$$D_0^{\alpha-1} v_k(0) = 0, \quad D_0^{\alpha-1} v_k(1) = 0.$$

Since $0 = D_0^{\alpha-1} v_k(1) - D_0^{\alpha-1} v_k(0) = \int_0^1 D_0^\alpha v_{k+1}(s)ds$, it follows that

$$\int_0^1 \Big(f(s, v_k(s)) + \lambda(v_k(s) - v_{k+1}(s)) \Big) ds = 0.$$

Again, the dominated convergence theorem implies that $\int_0^1 f(s, v(s))ds = 0$. Thus,

$$D_0^{\alpha-1} v(t) = \int_0^t f(s, v(s))ds$$

which implies $D_0^{\alpha-1} v(0) = 0$ and $D_0^{\alpha-1} v(1) = \int_0^1 f(s, v(s))ds = 0$.

Note that since $w_1(t) \le v(t) \le v_1(t)$ on $(0,1]$ and $D_0^{\alpha-1} v(t) = \int_0^t f(s, v(s))ds$, then $D_0^{\alpha-1} v$ is uniformly continuous on any compact subinterval of $(0,1]$. Thus,

$$v(t) = a_0 t^{\alpha-2} + I_0^{\alpha-1} D_0^{\alpha-1} v(t), \quad 0 < t \le 1,$$

implies $v \in C_{\alpha-2}[0,1]$ and

$$w_k \succeq_{\lambda<0} v \succeq_{\lambda<0} v_k$$

for each k. Moreover, Dini's theorem now applies and the convergence of $t^{2-\alpha} v_k(t)$ is uniform.

Similar details apply to $\{w_k\}$ and the theorem is proved. □

Suppose now f satisfies the "anti"-inequalities to (25); that is, suppose f satisfies

$$f(t, y_1) > f(t, y_2) \text{ for } (t, y) \in (0,1] \times \mathbb{R}, \quad y_1 > y_2. \tag{32}$$

One can appeal to the signed maximum principle, apply a shift to (22), and consider the equivalent boundary value problem, $D_0^\alpha y(t) + \lambda y(t) = f(t, y(t)) + \lambda y(t)$, $0 < t \le 1$, where

$0 < \lambda \leq \Lambda$, and $\Lambda > 0$ is given by Theorem 1. Note, if f satisfies (32) and $\lambda > 0$, then $g(t,y) = f(t,y) + \lambda y$ satisfies (32).

Now, assume the existence of solutions, $w_1, v_1 \in C_{\alpha-2}[0,1]$, of the following differential inequalities

$$\begin{aligned} D_0^\alpha w_1(t) &\leq f(t, w_1(t)), \quad 0 < t \leq 1, & D_0^\alpha v_1(t) &\geq f(t, v_1(t)), \quad 0 < t \leq 1, \\ D_0^{\alpha-1} w_1(0) = 0, &\quad D_0^{\alpha-1} w_1(1) = 0, & D_0^{\alpha-1} v_1(0) = 0, &\quad D_0^{\alpha-1} v_1(1) = 0. \end{aligned} \tag{33}$$

Assume further that

$$(v_1(t) - w_1(t)) \geq 0, \quad 0 < t \leq 1. \tag{34}$$

Noting that $\lambda > 0$ defines a partial order $\succeq_{\lambda>0}$ on $C_{\alpha-2}[0,1]$ by

$$u \succeq_{\lambda>0} 0 \iff u(t) \geq 0, 0 < t \leq 1.$$

In particular, in (34), assume $v_1 \succeq_{\lambda>0} w_1$.

Theorem 3. *Assume $f : (0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, assume that f satisfies (24), and assume f satisfies the monotonicity properties, (32). Assume the existence of $w_1, v_1 \in C_{\alpha-2}[0,1]$ satisfying (33) and (34). Define the sequences of iterates $\{v_k\}_{k=1}^\infty$, $\{w_k\}_{k=1}^\infty$ by (28) and (29), respectively. Then, for each positive integer k,*

$$v_k \succeq_{\lambda>0} v_{k+1} \succeq_{\lambda>0} w_{k+1} \succeq_{\lambda>0} w_k.$$

Moreover, $\{v_k\}_{k=1}^\infty$ converges in $C_{\alpha-2}$ to a solution $v \in C_{\alpha-2}[0,1]$ of the boundary value problem (22), (23) and $\{w_k\}_{k=1}^\infty$ converges in $C_{\alpha-2}[0,1]$ to a solution $w \in C_{\alpha-2}[0,1]$ of the boundary value problem (22), (23) satisfying

$$v_k \succeq_{\lambda>0} v_{k+1} \succeq_{\lambda>0} v \succeq_{\lambda>0} w \succeq_{\lambda>0} w_{k+1} \succeq_{\lambda>0} w_k. \tag{35}$$

6. Example

We close the article with an example in which Theorem 3 applies and in which upper and lower solutions, v_1 and w_1, are explicitly produced. To do so, we construct an explicit Green's function to obtain an estimate on $\Lambda > 0$, and we exhibit verifiable conditions on f so that (24) is satisfied.

The two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad Re(\alpha) > 0, \quad \beta \in \mathbb{C}, \quad z \in \mathbb{C}$$

will be employed to construct an appropriate Green's function.

Assume $1 < \alpha < 2$, assume $\lambda \neq 0$, and consider a Neumann boundary value problem for nonhomogenous linear Equations (3) and (4). We restate the boundary value problem for convenience.

$$D_0^\alpha y(t) + \lambda y(t) = f(t), \quad 0 < t \leq 1, \quad 1 < \alpha < 2,$$

$$D_0^{\alpha-1} y(0) = 0, \quad D_0^{\alpha-1} y(1) = 0.$$

Thus, $y(t) = -\lambda I_0^\alpha y(t) + I_0^\alpha f(t) + ct^{\alpha-2}$ where c is still undetermined or

$$(I + \lambda I_0^\alpha) y(t) = I_0^\alpha f(t) + ct^{\alpha-2}.$$

Employ the Neumann series to see that if $(I+\lambda I_0^\alpha)y(t)=h(t)$, then

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty}(-\lambda)^n I_0^{\alpha n}h(t) = \Big(I+\sum_{n=1}^{\infty}(-\lambda)^n I_0^{\alpha n}\Big)h(t)\\
&= h(t)+\int_0^t\sum_{n=1}^{\infty}(-\lambda)^n\frac{(t-s)^{\alpha n-1}}{\Gamma(\alpha n)}h(s)ds\\
&= h(t)-\lambda\int_0^t(t-s)^{\alpha-1}\sum_{n=0}^{\infty}\frac{(-\lambda(t-s)^\alpha)^n}{\Gamma(\alpha n+\alpha)}h(s)ds\\
&= h(t)-\lambda\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)h(s)ds.
\end{aligned}$$

Thus,

$$y(t)=h(t)+(-\lambda)\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha))h(s)ds,$$

where $h(t)=I_0^\alpha f(t)+ct^{\alpha-2}$. Employ the identity

$$\int_a^b(t-a)^\beta(x-t)^{n-1}dt=\frac{\Gamma(\beta+1)\Gamma(n)}{\Gamma(\beta+1+n)}(x-a)^{n+\beta}$$

and note that

$$\begin{aligned}
&t^{\alpha-2}+(-\lambda)\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)s^{\alpha-2}ds\\
&=\Gamma(\alpha-1)\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+\sum_{n=0}^{\infty}(-\lambda)^{n+1}\int_0^t\frac{(t-s)^{\alpha n+\alpha-1}s^{\alpha-2}}{\Gamma(\alpha n+\alpha)}dt\\
&=\Gamma(\alpha-1)\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}+\sum_{n=0}^{\infty}(-\lambda)^{n+1}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha(n+1)+\alpha-1)}t^{\alpha(n+1)+\alpha-2}\\
&=\Gamma(\alpha-1)t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha).
\end{aligned}$$

Thus,

$$\begin{aligned}
y(t)=I_0^\alpha f(t)&+(-\lambda)\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha))I_0^\alpha f(s)ds\\
&+c\Gamma(\alpha-1)t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha).
\end{aligned}\tag{36}$$

To calculate $D_0^{\alpha-1}y(t)$, we have $D_0^{\alpha-1}I_0^\alpha f(t)=I^1f(t)$,

$$\begin{aligned}
D_0^{\alpha-1}t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha) &= t^{-1}E_{\alpha,0}(-\lambda t^\alpha)\\
&= t^{-1}\sum_{n=0}^{\infty}\frac{(-\lambda t^\alpha)^n}{\Gamma(\alpha n)} = t^{-1}\sum_{n=1}^{\infty}\frac{(-\lambda t^\alpha)^n}{\Gamma(\alpha n)}\\
&= (-\lambda)t^{\alpha-1}\sum_{n=0}^{\infty}\frac{(-\lambda t^\alpha)^n}{\Gamma(\alpha n+\alpha)} = (-\lambda)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha),
\end{aligned}$$

and

$$
\begin{aligned}
&D_0^{\alpha-1} t^{\alpha-2} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)) I_0^\alpha f(s) ds \\
&= D I_0^{2-\alpha} t^{\alpha-2} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)) I_0^\alpha f(s) ds \\
&= \int_0^t E_{\alpha,1}(-\lambda(t-s)^\alpha) I_0^\alpha f(s) ds \\
&= \int_0^t E_{\alpha,1}(-\lambda(t-r)^\alpha) \int_0^r \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds dr \\
&= \int_0^t \Big(\int_s^t \sum_{n=0}^{\infty} \frac{(-\lambda(t-r)^\alpha)^n}{\Gamma(\alpha n+1)} \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)} dr \Big) f(s) ds \\
&= \int_0^t (\sum_{n=0}^{\infty} \frac{(-\lambda)^n (t-s)^{\alpha n+\alpha}}{\Gamma(\alpha n+\alpha+1)} (t-s)^{\alpha n+\alpha}) f(s) ds \\
&= \int_0^t (t-s)^\alpha E_{\alpha,\alpha+1}(-\lambda(t-s)^\alpha) f(s) ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
D_0^{\alpha-1} y(1) = I^1 f(1) + (-\lambda) \int_0^1 (1-s)^\alpha E_{\alpha,\alpha+1}(-\lambda(1-s)^\alpha)) f(s) ds \\
- \lambda c \Gamma(\alpha-1) E_{\alpha,\alpha}(-\lambda).
\end{aligned}
$$

Employ the boundary condition $D_0^{\alpha-1} y(1) = 0$ and obtain

$$
c = \frac{\int_0^1 f(s) ds - \lambda \int_0^1 (1-s)^\alpha E_{\alpha,\alpha+1}(-\lambda(1-s)^\alpha) f(s) ds}{\lambda \Gamma(\alpha-1) E_{\alpha,\alpha}(-\lambda)},
$$

if $E_{\alpha,\alpha}(-\lambda) \neq 0$.

The solution y in (36) satisfies $\lambda y = -\lambda D_0^\alpha y + f$ or

$$
\begin{aligned}
y &= \frac{1}{\lambda} \Big(\lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s) ds + \lambda c \Gamma(\alpha-1) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha) \Big) \\
&= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s) ds + c \Gamma(\alpha-1) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha)
\end{aligned}
$$

Define

$$
\begin{aligned}
g(\alpha, \lambda; t, s) &= \frac{t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha) \Big(1 - \lambda(1-s)^\alpha E_{\alpha,\alpha+1}(-\lambda(1-s)^\alpha) \Big)}{\lambda E_{\alpha,\alpha}(-\lambda)} \\
&= \frac{t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha) E_{\alpha,1}(-\lambda(1-s)^\alpha) \Big)}{\lambda E_{\alpha,\alpha}(-\lambda)},
\end{aligned}
$$

where an identity $E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z E_{\alpha,\alpha+\beta}$ has been employed. Then,

$$
y(t) = \int_0^1 G(\alpha, \lambda; t, s) f(s) ds,
$$

where

$$
G(\alpha, \lambda; t, s) = \begin{cases} g(\alpha, \lambda; t, s), & 0 \le t \le s \le 1, \\ g(\alpha, \lambda; t, s) + (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha), & 0 \le s < t \le 1. \end{cases}
$$

One can see from this construction that a maximum principle will be valid for $\lambda \in (-\infty, 0)$. For the anti-maximum principle, it is shown in ([30], Corollary 3) that $E_{\alpha,\alpha}(-z)$ has the smallest in modulus root which is a positive root. From the identity,

$$I_0^{\alpha-1} E_{\alpha,1}(-\lambda t^\alpha) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha),$$

and integrating from 0 to 1, it is clear that $E_{\alpha,1}(-z)$ has the smallest positive root which is smaller than the smallest root of $E_{\alpha,\alpha}(-z)$. Then, the identity

$$I_0^{2-\alpha} t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha) = E_{\alpha,1}(-\lambda t^\alpha),$$

implies that $E_{\alpha,\alpha-1}(-z)$ has the smallest positive root which is smaller than the smallest positive root of $E_{\alpha,1}(-z)$. Thus, from the construction, an anti-maximum principle will be valid for $\lambda \in (0, \lambda_0)$, where λ_0 is the smallest positive real root of the Mittag-Leffler function, $E_{\alpha,\alpha-1}(-z)$.

Now, consider a boundary value problem for nonlinear fractional differential Equations (22) and (23). Assume $f : (0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, assume f satisfies the monotonicity property (25), and assume there exists $\lambda < 0$ such that $f(t,s) = g(t,s) - \lambda s$ and $g(t,s)$ is bounded and continuous on $(0,1] \times \mathbb{R}$. Then, f satisfies (24).

Corollary 1. *Assume $1 < \alpha < 2$. Assume $f : (0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and assume f satisfies the monotonicity property (25). Assume there exists $\lambda < 0$ such that $f(t,s) = g(t,s) - \lambda s$ and $g(t,s)$ is bounded and continuous on $(0,1] \times \mathbb{R}$. Then, there exists a solution of the boundary value problem*

$$D_0^\alpha y(t) = f(t, y(t)), \quad 0 < t \le 1,$$
$$D_0^{\alpha-1} y(0) = 0, \quad D_0^{\alpha-1} y(1) = 0.$$

Proof. As noted above, the boundedness condition on g implies that f satisfies (24). Let $(-\lambda)M$ denote an upper bound on $|g|$. Set $v_1(t) = Mt^{\alpha-2}$ and set $w_1(t) = -Mt^{\alpha-2}$. Thus, v_1 and w_1 satisfy the boundary conditions (4). Moreover,

$$D^\alpha v_1(t) + \lambda v_1(t) = \lambda M t^{\alpha-2} \le \lambda M \le -|g(t, Mt^{\alpha-2})| \le f(t, v_1(t)) + \lambda v_1(t),$$

or $D^\alpha v_1(t) \le f(t, v_1(t))$. Similarly, $D^\alpha w_1(t) \ge f(t, w_1(t))$ and Theorem 2 applies. □

Corollary 2. *Assume $1 < \alpha < 2$. Assume $f : (0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and assume f satisfies the monotonicity property (32). Let $\lambda_0 > 0$ denote the smallest positive real root of $E_{\alpha,\alpha-1}(-z)$. Assume there exists $\lambda \in (0, \lambda_0)$ such that $f(t,s) = g(t,s) - \lambda s$ and $g(t,s)$ is bounded and continuous on $(0,1] \times \mathbb{R}$. Then, there exists a solution of the boundary value problem*

$$D_0^\alpha y(t) = f(t, y(t)), \quad 0 < t \le 1,$$

$$D_0^{\alpha-1} y(0) = 0, \quad D_0^{\alpha-1} y(1) = 0.$$

Proof. Let λM denote an upper bound on $|g|$. Set $v_1(t) = Mt^{\alpha-2}$ and set $w_1(t) = -Mt^{\alpha-2}$. v_1 and w_1 satisfy (33) and Theorem 3 applies. □

7. Conclusions

In this paper, we study a λ dependent boundary value problem for a Riemann–Liouville fractional differential equation. Denoting the boundary value problem abstractly as $Ay + \lambda y = f$, $\lambda = 0$ is assumed to be a simple eigenvalue. Sufficient conditions are obtained to show the existence of $\Lambda > 0$ such that if $|\lambda| \in (0, \Lambda)$, then $(A + \lambda I)$ is invertible and $f \ge 0$ implies $\lambda y \ge 0$ where y denotes the unique solution of $(A + \lambda I)y = f$. Four examples are produced illustrating the abstract result. An application of monotone methods and the method of upper and lower solutions is produced for a nonlinear boundary value problem.

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