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The Role of Data on the Regularity of Solutions to Some Evolution Equations

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Abstract: In this paper, we study the influence of the initial data and the forcing terms on the regularity of solutions to a class of evolution equations including linear and semilinear parabolic equations as the model cases, together with the nonlinear p-Laplacian equation. We focus our study on the regularity (in terms of belonging to appropriate Lebesgue spaces) of the gradient of the solutions. We prove that there are cases where the regularity of the solutions as soon as $t > 0$ is not influenced at all by the initial data. We also derive estimates for the gradient of these solutions that are independent of the initial data and reveal, once again, that for this class of evolution problems, the real "actors of the regularity" are the forcing terms.

Keywords: regularity of solutions; p-Laplacian equation; nonlinear degenerate parabolic equations; smoothing effect; heat equation; gradient estimates

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1. Introduction

Let us consider the following evolution problem

$$\begin{cases} u_t - \operatorname{div}(a(x,t,\nabla u)) = f(x,t) & \text{in} \quad \Omega_T \equiv \Omega \times (0,T), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \end{cases} \tag{1}$$

where Ω is an open bounded set of $\mathbb{R}^N$ with $\partial\Omega$ regular (for example, satisfying the property of positive geometric density), $N \geq 3$, and $T > 0$.

Here, the function $a(x,t,\xi) : \Omega \times (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function (i.e., it is continuous with respect to ξ for almost every $(x,t) \in \Omega_T$, and measurable with respect to (x,t) for every $\xi \in \mathbb{R}^N$), satisfying, for a.e. $(x,t) \in \Omega_T$ and for every ξ and $\eta \in \mathbb{R}^N$, the following structure conditions

$$a(x,t,\xi)\xi \geq \alpha|\xi|^p\,, \tag{2}$$

$$[a(x,t,\xi) - a(x,t,\eta)][\xi - \eta] \geq \alpha|\xi - \eta|^p\,, \quad \alpha > 0\,, \quad 2 \leq p < N\,, \tag{3}$$

and

$$|a(x,t,\xi)| \leq \beta[|\xi|^{p-1} + h(x,t)], \quad \beta > 0, \quad h \in L^{p'}(\Omega_T) \qquad \frac{1}{p} + \frac{1}{p'} = 1\,. \tag{4}$$

The typical model cases included in this class of problems are as follows

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in} \quad \Omega_T\,, \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega\,, \end{cases}$$

i.e., the heat equation with a forcing term f, with all its linear slight modifications as follows

$$\begin{cases} u_t - \operatorname{div}(A(x,t)\nabla u) = f(x,t) & \text{in} \quad \Omega_T\,, \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega\,, \end{cases} \tag{5}$$

where A is a bounded matrix satisfying the coercivity condition

$$(A(x,t)\xi,\xi) \geq \alpha|\xi|^2\,,$$

together with the nonlinear version (1) with $p = 2$ and the p-Laplacian equation

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x,t) & \text{in} \quad \Omega_T\,, \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega\,. \end{cases} \tag{6}$$

Because of the numerous physical applications of this class of problems, there is extensive literature on these evolution problems, starting from the sixties up until today, and it is difficult to provide an exhaustive bibliography. The initial works have become milestones, like the papers of Nash, Moser, Ladyženskaja, Solonnikov, Ural'ceva, J.L. Lions, Aronson, and Serrin (see [1–5] and the references therein). Many interesting results can be found in more recent papers, for example [6–22] (see also the references therein).

The aim of this paper is to study the influence of the initial datum u_0 and the reaction term f on the regularity of the solution gradient, as soon as $t > 0$.

We recall that in the absence of the forcing term f, it is well known that, even if the initial datum is only a summable function, there exists a solution of (1) that becomes immediately bounded (see [20] if $u_0 \in L^2(\Omega)$ and [23] in the general case $u_0 \in L^1(\Omega)$). Moreover, this solution satisfies the following decay estimate

$$\|u(t)\|_{L^\infty(\Omega)} \leq c\,\frac{\|u_0\|_{L^1(\Omega)}^{\frac{p}{N(p-2)+p}}}{t^{\frac{N}{N(p-2)+p}}} \tag{7}$$

(see [23] and the references therein). Notice that the previous bound is exactly the same bound that is satisfied in the p-Laplacian case (6) (see [9,10,22,24–26] if $N = 1$, [23] and the references therein).

It is of note that if u_0 is in $L^1(\Omega)$, then the solutions of (1), even when $f \equiv 0$, are not unique. Despite this, it is unique the solution that immediately becomes bounded (see Theorem 1.7 in [27]) and this solution is also the unique solution constructed by approximation.

This immediate regularization of solution u in $L^\infty(\Omega)$, as soon as $t > 0$, produces an immediate regularization of its gradient. In detail, it results in $\nabla u \in (L^p(\Omega \times (t,T)))^N$ for every $0 < t < T$ (see [27]).

Here, we want to understand what happens in the presence of a forcing term $f \not\equiv 0$.

If the forcing term is not sufficiently regular, even in the presence of bounded initial data, we cannot expect to have bounded solutions for (1). As a matter of fact, from the classical theory, we know that the sharp condition on f to have bounded solutions is the following condition

$$f \in L^m(0,T;L^s(\Omega)) \qquad \text{with} \qquad \frac{1}{m} + \frac{N}{ps} < 1\,. \tag{8}$$

Moreover, it is well known that unbounded solutions of (1) exist with gradient belonging to $(L^p(\Omega_T))^N$ (and hence to $(L^p(\Omega \times (t,T)))^N$ for every $0 < t < T$) even if f doesn't satisfied (8).

In addition, even if u_0 is regular (for example the null function $u_0 \equiv 0$), solutions of (1) exist with a gradient that does not belong to $(L^p(\Omega_T))^N$ if f is not sufficiently summable.

Hence, the aim of this paper is to understand the role of the initial datum u_0 and of the forcing term f on the regularity of the gradient for the solutions of (1).

To gain a better understanding of what happens, we decided to study the worst case in terms of the regularity of u_0, i.e., to directly consider the case of only summable initial data.

As discussed above, in this case, the solutions of (1) were not unique without further requirements of the solutions. Hence, we focus our study on the unique solution of (1), which is constructed by approximation, not only because when $f \equiv 0$ it coincides with the unique solution that becomes immediately bounded, but above all because it is the most natural way to build solutions.

The main result proven in this paper is that if f is a function satisfying

$$f \in L^m(0,T;L^s(\Omega)) \qquad \text{with} \qquad s \geq 1\,,\, m \geq 1\,, \tag{9}$$

even if on the initial datum u_0, we assume only that

$$u_0 \in L^1(\Omega) \tag{10}$$

then the unique solution u of (1) constructed by approximation immediately increases its regularity reaching the same summability properties of the gradient of the unique solution u^0 obtained by approximation of the following problem

$$\begin{cases} (u^0)_t - \operatorname{div}(a(x,t,\nabla u^0)) = f(x,t) & \text{in} \quad \Omega_T, \\ u^0 = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u^0(x,0) = 0 & \text{on} \quad \Omega. \end{cases} \tag{11}$$

Thus, for every $t \in (0,T)$, it results

$$\nabla u^0 \in (L^q(\Omega \times (t,T)))^N \quad \Longrightarrow \quad \nabla u \in (L^q(\Omega \times (t,T)))^N\,. \tag{12}$$

Moreover, we derive estimates on ∇u in $(L^q(\Omega \times (t,T)))^N$, which become universal estimates when $p > 2$. To our knowledge, this regularizing phenomenon and these estimates are not known in the literature.

We recall that in [28], (if $p > 2$), and in [29], (if $p = 2$), we prove that under the previous assumptions (9) and (10) on the data of f and u_0, the unique solution u of (1) that is constructed by approximation immediately increases its regularity, reaching the same summability properties of the unique solution of u^0 obtained by the approximation of (11), i.e., for every $t \in (0,T)$, it results

$$u^0 \in L^\delta(t,T;L^\nu(\Omega)) \quad \Longrightarrow \quad u \in L^\delta(t,T;L^\nu(\Omega))\,.$$

Hence, this paper completes the previous known results, showing that the gradient of u immediately inherits the same regularity as the gradient of u^0.

Another interesting property of the solutions of (1) that we will prove here is that if v is the unique solution obtained by the approximation of the following problem

$$\begin{cases} v_t - \operatorname{div}(a(x,t,\nabla v)) = f(x,t) & \text{in} \quad \Omega_T\,, \\ v = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ v(x,0) = v_0(x) & \text{on} \quad \Omega, \end{cases} \tag{13}$$

(i.e., v solves (1), but with a different initial datum v_0) then, even if the data f, u_0 and v_0 are only summable functions, it results

$$\nabla(u-v) \in (L^p(\Omega \times (t,T))^N\,, \qquad \text{for every} \quad 0 < t < T\,. \tag{14}$$

It is important to note that (14) is truly surprising given that neither of the two functions ∇u and ∇v belongs to $(L^p(\Omega \times (t,T)))^N$.

The paper is organized as follows: in the next section, we state all of our results in detail and, then, in the following Section 3, we provide all the proofs.

2. Statement of Results

Before stating our results we recall our notion of solution and global solution of (1).

Definition 1. *Assume that the data f and u_0 satisfy*

$$f \in L^1(\Omega_T)\,, \qquad u_0 \in L^1(\Omega)\,. \tag{15}$$

A measurable function $u \in L^\infty(0,T;L^1(\Omega)) \cap L^1(0,T;W_0^{1,1}(\Omega))$ is a solution of (1)*, if $a(x,t,\nabla u) \in (L^1(\Omega_T))^N$ and*

$$\iint_{\Omega_T}\left[-u\frac{\partial\varphi}{\partial t} + a(x,t,\nabla u)\nabla\varphi\right]dxdt = \int_\Omega u_0\varphi(0)dx + \iint_{\Omega_T} f\varphi, \tag{16}$$

for every $\varphi \in W^{1,\infty}(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W_0^{1,\infty}(\Omega))$ satisfying $\varphi(T) = 0$.

We observe that under the structure assumptions (2)–(4), if the data satisfy (15) there exists at least one solution u of (1) (see [7]). Moreover, this solution satisfies

$$\nabla u \in (L^q(\Omega_T))^N\,, \qquad \text{for every} \quad q < \frac{N(p-1)+p}{N+1}\,, \tag{17}$$

and is a solution constructed by approximation. Here, a solution constructed by approximation means the following.

Definition 2. *A solution u of* (1)*, (according to the above definition 1), is obtained by the approximation (sola) if it is the a.e. limit in Ω_T of the solutions $u_n \in L^\infty(\Omega_T) \cap C([0,T];L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ (denoted as "approximating solutions") for the following (approximating) problems*

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x,t,\nabla u_n)) = f_n(x,t) & \text{in} \quad \Omega_T, \\ u_n = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u_n(x,0) = u_{0,n} & \text{on} \quad \Omega\,, \end{cases} \tag{18}$$

where f_n and $u_{0,n}$ satisfy

$$f_n \in L^\infty(\Omega_T)\,, \quad f_n \to f \quad \text{in} \quad L^1(\Omega_T)\,, \tag{19}$$

$$u_{0,n} \in L^\infty(\Omega)\,, \quad u_{0,n} \to u_0 \quad \text{in} \quad L^1(\Omega)\,. \tag{20}$$

The solutions of (1) are not unique. Anyway, under further requirements of the solutions, for example that u is a solution constructed by approximation (according to the above definition 2), such a solution becomes unique, i.e., only one solution of (1) exists, which is constructed by approximation (see [16]). Indeed, some of our results concern the global solutions of (1); hence, we recall what we mean here by global solutions.

Definition 3. *Assume that the data f and u_0 satisfy* (15) *for every $T > 0$. We say that u is a global solution of* (1)*, or equivalently, that u is a solution of*

$$\begin{cases} u_t - \operatorname{div}(a(x,t,\nabla u)) = f(x,t) & \text{in} \quad \Omega_\infty \equiv \Omega \times (0,+\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \end{cases} \tag{21}$$

if u is a solution of (1) *for every $T > 0$. Finally, u is a global solution constructed by approximation of* (1)*, (or equivalently u is a solution constructed by approximation of* (21)*), if u is a solution constructed by approximation of* (1) *for every $T > 0$.*

We recall that if (2)–(4) and (15) are satisfied for every $T > 0$, then there exists one and only one global solution constructed by approximation of (1) (see [30]).

Theorem 1. *Let (2)–(4) hold true. Assume $f \in L^1(\Omega_T)$, $u_0 \in L^1(\Omega)$, and that u and u^0 are the unique solutions constructed by approximation of (1) and (11), respectively. If ∇u^0 belongs to $(L^q(t,T;L^m(\Omega)))^N$, for some $t \in (0,T)$, $q \in [1,p]$ and $m \in [1,p]$, then it results*

$$\nabla u \in (L^q(t,T;L^m(\Omega)))^N\,, \tag{22}$$

and the following estimate holds true

$$|||\nabla u|||_{L^q(t,T;L^m(\Omega))} \le c_0 \frac{\|u_0\|_{L^1(\Omega)}^{\frac{2}{N(p-2)+p}}(T-t)^{\frac{1}{q}-\frac{1}{p}}}{t^{\frac{2N}{p[N(p-2)+p]}}} + |||\nabla u^0|||_{L^q(t,T;L^m(\Omega))} \tag{23}$$

where $c_0 = C_0^{\frac{1}{p}}|\Omega|^{\frac{1}{m}-\frac{1}{p}}$ depends only on N, p, m, α, and $|\Omega|$ (see Formula (54) for the definition of C_0).

Moreover, if $p > 2$, the following universal estimate holds true

$$|||\nabla u|||_{L^q(t,T;L^m(\Omega))} \le C^* \frac{(T-t)^{\frac{1}{q}-\frac{1}{p}}}{t^{\frac{2}{p(p-2)}}} + |||\nabla u^0|||_{L^q(t,T;L^m(\Omega))} \tag{24}$$

where $C^* = (C_*)^{\frac{1}{p}}|\Omega|^{\frac{1}{m}-\frac{1}{p}}$ depends only on N, p, m, α and $|\Omega|$ (see Formula (50) for the definition of C_*).

Finally, if $p = 2$, the following exponential estimate holds

$$|||\nabla u|||_{L^q(t,T;L^m(\Omega))} \le c_1 \frac{\|u_0\|_{L^1(\Omega)}(T-t)^{\frac{1}{q}-\frac{1}{2}}}{t^{\frac{N}{2}}e^{\sigma t}} + |||\nabla u^0|||_{L^q(t,T;L^m(\Omega))} \tag{25}$$

where $c_1 = \sqrt{C_1}|\Omega|^{\frac{1}{m}-\frac{1}{p}}$ depends only on N, p, α, m and $|\Omega|$ (see Formula (52) for the definition of C_1) and $\sigma = c(N)\alpha|\Omega|^{-\frac{2}{N}}$.

Remark 1. *Theorem 1 reveals that, as soon as $t > 0$, the summability of the initial datum doesn't influence the summability of the gradient of u since it has the same summability of the gradient of the solution u^0 whose initial datum is the null function.*

Notice that estimate (24) on ∇u in $(L^q(t,T;L^m(\Omega)))^N$ is a universal estimate as it does not depend on u_0. Hence, when $p > 2$, the initial datum u_0 affects neither the regularity nor the estimates of the gradient of the solution u.

Remark 2. *We point out that it results*

$$\frac{2N}{p[N(p-2)+p]} < \frac{2}{p(p-2)} \quad \text{if} \quad p > 2\,,$$

and

$$\frac{2N}{p[N(p-2)+p]} = \frac{N}{2} \quad \text{if} \quad p = 2\,.$$

Hence, estimate (23) becomes interesting when $\nabla u \notin (L^q(0,T;L^m(\Omega)))^N$, as it allows for affirming that for $t \to 0$, the blow-up of the norm of ∇u in $(L^q(t,T;L^m(\Omega)))^N$ is controlled by the power $t^{-\frac{2N}{p[N(p-2)+p]}}$. Finally, estimate (25) becomes significant when t is large, as the exponential function $e^{\sigma t}$ surpasses every power of t when t is large.

Corollary 1. *Let (2)–(4) hold true. Assume $f \in L^\sigma(\Omega_T)$, $\sigma \ge 1$, $u_0 \in L^1(\Omega)$, and that u is the unique solution constructed by approximation of (1). If it results*

$$\sigma \ge \left(p\,\frac{N+2}{N}\right)'\,, \tag{26}$$

then for every $0 < t < T$ *we have*

$$\nabla u \in (L^p(\Omega \times (t,T)))^N. \tag{27}$$

If otherwise it results

$$1 < \sigma < \left(p\,\frac{N+2}{N}\right)', \tag{28}$$

then for every $0 < t < T$ *we have*

$$\nabla u \in (L^m(\Omega \times (t,T)))^N, \qquad m = \frac{[N(p-1)+p]\sigma}{N+2-\sigma}. \tag{29}$$

Remark 3. *Notice that it results*

$$\lim_{\sigma\to 1} \frac{[N(p-1)+p]\sigma}{N+2-\sigma} = \frac{N(p-1)+p}{N+1},$$

and hence the result of the previous limit is exactly the value that appears in (17) when $\sigma = 1$.

Indeed, the regularity (27) is achieved in the following more general case.

Corollary 2. *Let (2)–(4) hold true. Assume* $u_0 \in L^1(\Omega)$ *and* $f \in L^r(0,T;L^\sigma(\Omega))$, *with* r *and* σ *satisfying*

$$p < \frac{N}{\sigma} + \frac{p}{r} \le \frac{N}{r}\left(1 - \frac{p}{2}\right) + \frac{Np+2p-N}{2}, \tag{30}$$

$$1 \le r < p' \qquad \sigma \ge 1. \tag{31}$$

If u *is the unique solution constructed by approximation of (1), then, for every* $0 < t < T$, *we have*

$$\nabla u \in (L^p(\Omega \times (t,T)))^N. \tag{32}$$

Remark 4. *Notice that if* $r = \sigma$, *it results*

$$\frac{N}{\sigma} + \frac{p}{r} \le \frac{N}{r}\left(1 - \frac{p}{2}\right) + \frac{Np+2p-N}{2} \iff \sigma \ge \left(p\,\frac{N+2}{N}\right)'.$$

Moreover, being $r < p'$, *it follows that* $f \notin L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Remark 5. *We recall that when* u_0 *belongs to* $L^1(\Omega)$, *the regularity* $(L^p(\Omega \times (t,T)))^N$ *of* ∇u *was already proven in [27] when* $f \equiv 0$, *in [29] when* $p = 2$ *and* $f \in L^2(0,T;H^{-1}(\Omega))$, *and in [28] when* $p > 2$ *and* $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Theorem 2. *Let (2)–(4) hold true. Assume* $f \in L^1(\Omega_T)$, $u_0 \in L^1(\Omega)$, *and* $v_0 \in L^1(\Omega)$. *If* u *and* v *are the unique solutions constructed by approximation of (1) and (13), respectively, then for every* $t \in (0,T)$, *it results*

$$\nabla(u-v) \in (L^p(\Omega \times (t,T)))^N, \tag{33}$$

and the following estimate holds true

$$\int_t^T \int_\Omega |\nabla(u-v)|^p \le C_0\, \frac{\|u_0 - v_0\|_{L^1(\Omega)}^{\frac{2p}{N(p-2)+p}}}{t^{\frac{2N}{N(p-2)+p}}}, \tag{34}$$

where C_0 *depends only on* N, p, α, *and* $|\Omega|$ *(see Formula (54)).*

Moreover, if $p > 2$ for every $t \in (0, T)$ it results

$$\int_t^T \int_\Omega |\nabla(u - v)|^p \le \frac{C_*}{t^{\frac{2}{p-2}}}, \tag{35}$$

where the constant C_* depends only on N, p, α, and $|\Omega|$ (see Formula (50)).

Otherwise, if $p = 2$, for every $t \in (0, T)$, we have the following bound

$$\int_t^T \int_\Omega |\nabla(u - v)|^2 \le C_1 \frac{\|u_0 - v_0\|^2_{L^1(\Omega)}}{t^N e^{\sigma_0 t}}, \tag{36}$$

where C_1 depends only on N, p, α, and $|\Omega|$ (see Formula (52)) and $\sigma_0 = C(N)\alpha|\Omega|^{-\frac{2}{N}}$.

Finally, if assumptions (2)–(4) and $f \in L^1(\Omega_T)$ are satisfied for every $T > 0$, and u and v are the unique global solutions constructed by approximation of (1) and (13), respectively, then for every $t > 0$, it results

$$\nabla(u - v) \in (L^p(\Omega \times (t, +\infty)))^N, \tag{37}$$

and

$$\int_t^{+\infty} \int_\Omega |\nabla(u - v)|^p \le C_0 \frac{\|u_0 - v_0\|^{\frac{2p}{N(p-2)+p}}_{L^1(\Omega)}}{t^{\frac{2N}{N(p-2)+p}}}, \tag{38}$$

where C_0 is the same as in (34).

Moreover, if $p > 2$ the following estimate holds true for every $t > 0$

$$\int_t^{+\infty} \int_\Omega |\nabla(u - v)|^p \le \frac{C_*}{t^{\frac{2}{p-2}}}, \tag{39}$$

where the constant C_* is as in (35), while if $p = 2$, it results (for every $t > 0$)

$$\int_t^{+\infty} \int_\Omega |\nabla(u - v)|^2 \le C_1 \frac{\|u_0 - v_0\|^2_{L^1(\Omega)}}{t^N e^{\sigma_0 t}}, \tag{40}$$

where C_1 and σ_0 are the same as in (36).

Remark 6. *The previous result is rather surprising as it reveals that when the data of f and u_0 are only summable functions, even if both the gradients of the solutions u and v are not in $(L^p(\Omega \times (t, +\infty)))^N$, their difference $\nabla(u - v)$ belongs to $(L^p(\Omega \times (t, +\infty)))^N$. In addition, for $p > 2$, it is also possible to obtain an estimate for $\nabla(u - v)$, which is independent of both the initial data of u_0 and v_0.*

3. Proofs of the Results

Since in the proof of Theorem 1 we use the results of Theorem 2, we start proving Theorem 2.

Proof of Theorem 2. Let u and v be the unique solutions constructed by approximation of (1) and (13), respectively. Hence, u is the a.e. limit in Ω_T of the sequence of solutions $u_n \in L^\infty(\Omega_T) \cap C([0, T]; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ of the following problems

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x, t, \nabla u_n)) = f_n(x, t) & \text{in} \quad \Omega_T, \\ u_n = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ u_n(x, 0) = u_{0,n}(x) & \text{on} \quad \Omega, \end{cases} \tag{41}$$

where the data $u_{0,n}(x) \in L^\infty(\Omega)$ and $f_n(x, t) \in L^\infty(\Omega_T)$ satisfy

$$u_{0,n}(x) \to u_0 \quad \text{in} \quad L^1(\Omega), \tag{42}$$

$$f_n(x,t) \to f(x,t) \quad \text{in} \;\; L^1(\Omega_T)\,. \tag{43}$$

Thanks to the fact that by assumption v is the unique solution constructed by approximation of (13), we can choose in the following construction of the sequence v_n that converges a.e. in Ω_T to v the same approximation f_n of f in (43)

$$\begin{cases} (v_n)_t - \operatorname{div}(a(x,t,\nabla v_n)) = f_n(x,t) & \text{in} \quad \Omega_T, \\ v_n = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ v_n(x,0) = v_{0,n}(x) & \text{on} \quad \Omega, \end{cases} \tag{44}$$

where the data $v_{0,n}(x) \in L^\infty(\Omega)$ satisfy

$$v_{0,n}(x) \to v_0 \quad \text{in} \;\; L^1(\Omega)\,, \tag{45}$$

Choosing $u_n - v_n$ as the test function in the approximating problems (41) and (44), and subtracting the equations obtained in this way, we obtain for every $0 < t_1 < t_2 \le T$

$$\frac{1}{2}\int_\Omega |(u_n - v_n)(t_2)|^2 - \frac{1}{2}\int_\Omega |(u_n - v_n)(t_1)|^2 + \\ \int_{t_1}^{t_2}\!\!\int_\Omega [a(x,t,\nabla u_n) - a(x,t,\nabla v_n)]\nabla(u_n - v_n) = 0\,.$$

From the previous estimate and (3), it follows

$$\frac{1}{2}\int_\Omega |(u_n - v_n)(t_2)|^2 + \alpha\int_{t_1}^{t_2}\!\!\int_\Omega |\nabla(u_n - v_n)|^p \le \\ \frac{1}{2}\int_\Omega |(u_n - v_n)(t_1)|^2\,. \tag{46}$$

Hence, to derive an estimate on the gradient of the function of $u_n - v_n$, it is sufficient to estimate the right-hand side of (46). To this aim, we recall that using $G_k(u_n - v_n)$ as a test function instead of $u_n - v_n$ and proceeding in the same way as above. we obtain for every $0 < t_1 < t_2 \le T$

$$\frac{1}{2}\int_\Omega |G_k(u_n - v_n)(t_2)|^2 + \alpha\int_{t_1}^{t_2}\!\!\int_\Omega |\nabla G_k(u_n - v_n)|^p \le \\ \frac{1}{2}\int_\Omega |G_k(u_n - v_n)(t_1)|^2\,. \tag{47}$$

Here, $G_k(s)$ is the following function

$$G_k(s) = (|s| - k)_+ \operatorname{sign}(s)\,.$$

Moreover, if $p > 2$, in [28], using suitable test functions, it is proven that $u_n - v_n$ also satisfies the following integral inequality for every $0 < t_0 < t < T$

$$\int_\Omega |G_k(u_n - v_n)|(t) \le \int_\Omega |G_k(u_n - v_n)|(t_0)\,.$$

The previous bound together with (47) allows to apply Theorem 2.2 in [23] (see the Appendix A) and to conclude that the following estimate holds true for every $t \in (0,T)$

$$\|u_n(t) - v_n(t)\|_{L^\infty(\Omega)} \le \frac{C_\sharp}{t^{\frac{1}{p-2}}}\,, \tag{48}$$

where the positive constant $C_\sharp$ depends only on N, p, and α (for further details, see the proof of estimate (4.10) in [28]). Thus, by (46) (applied with $t_1 = t$ and $t_2 = T$) and (48), we deduce that for every $0 < t < T$

$$\int_t^T \int_\Omega |\nabla(u_n - v_n)|^p \le \frac{1}{2\alpha} \int_\Omega |(u_n - v_n)(t)|^2 \le$$
$$\frac{|\Omega|}{2\alpha} \|u_n(t) - v_n(t)\|^2_{L^\infty(\Omega)} \le \frac{C_*}{t^{\frac{2}{p-2}}}, \tag{49}$$

where

$$C_* \equiv \frac{|\Omega|}{2\alpha} C_\sharp^2, \tag{50}$$

is a positive constant depending only on N, p, α, and $|\Omega|$. Recalling that by construction the sequence $u_n - v_n$ converges a.e. to $u - v$ in Ω_T by (49) we conclude that the following universal bound holds for every $0 < t < T$

$$\int_t^T \int_\Omega |\nabla(u - v)|^p \le \frac{C_*}{t^{\frac{2}{p-2}}},$$

and, consequently, (33) and (35) hold true if $p > 2$.

Otherwise, if $p = 2$, with the same procedure as the degenerate case $p > 2$ described above, it is proven in [29] that the following estimate holds true for every $0 < t < T$

$$\|u_n(t) - v_n(t)\|_{L^\infty(\Omega)} \le c \frac{\|u_{0,n} - v_{0,n}\|_{L^1(\Omega)}}{t^{\frac{N}{2}} e^{\sigma t}}, \tag{51}$$

where the positive constant c depends only on N, and α and $\sigma = c(N)\alpha|\Omega|^{-\frac{2}{N}}$ (for further details, see the proof of estimate (4.22) in [29]). By (46) and (51) (applied again with $t_1 = t$ and $t_2 = T$), we deduce that if $p = 2$. for every $0 < t < T$, it results

$$\int_t^T \int_\Omega |\nabla(u_n - v_n)|^2 \le C_1 \frac{\|u_{0,n} - v_{0,n}\|^2_{L^1(\Omega)}}{t^N e^{2\sigma t}},$$

where

$$C_1 \equiv \frac{c^2|\Omega|}{2\alpha}. \tag{52}$$

Consequently, we can conclude that

$$\int_t^T \int_\Omega |\nabla(u - v)|^2 \le C_1 \frac{\|u_0 - v_0\|^2_{L^1(\Omega)}}{t^N e^{\sigma_0 t}},$$

where $\sigma_0 = C(N)\alpha|\Omega|^{-\frac{2}{N}}$ ($C(N) = 2c(N)$) and (33) and (36) also follow if $p = 2$.

Indeed, in [28], applying Theorem 2.1 in [23] instead of Theorem 2.2 (see the Appendix A), it is also proven that the following estimate holds true for every $p \ge 2$ and $t \in (0, T)$

$$\|u_n(t) - v_n(t)\|_{L^\infty(\Omega)} \le c_1 \frac{\|u_{0,n} - v_{0,n}\|^{\frac{p}{N(p-2)+p}}_{L^1(\Omega)}}{t^{\frac{N}{N(p-2)+p}}}, \tag{53}$$

where c_1 is a positive constant depending only on N, p, and α (for all details, see the proof of (4.12) in [28]). Hence, by (46) and (53) (applied as above with $t_1 = t$ and $t_2 = T$), we deduce that (34) also holds true with C_0, defined as follows

$$C_0 \equiv \frac{c_1^2|\Omega|}{2\alpha}. \tag{54}$$

Finally, if u and v are the unique solutions constructed by approximation of (1) and (13), respectively, all the other statements follow from estimates (34)–(36). □

Proof of Theorem 1. Let u and u^0 be the unique solutions constructed by approximation of, respectively, (1) and (11). By Theorem 2 (applied with $v = u^0$) we know that $\nabla(u - u^0) \in (L^p(\Omega \times (t, T)))^N$. Hence, by the triangular inequality

$$|\nabla u| \leq |\nabla(u - u^0)| + |\nabla u^0| \tag{55}$$

it follows that if ∇u^0 belongs to $(L^q(t, T; L^m(\Omega)))^N$, for some q and m in $[1, p]$, then also ∇u belongs to $(L^q(t, T; L^m(\Omega)))^N$.

To conclude the proof we show now that estimates (23)–(25) are satisfied. To this aim, by (55) we deduce that

$$\begin{aligned} &\||\nabla u|\|_{L^q(t,T;L^m(\Omega))} \leq \||\nabla(u - u^0)|\|_{L^q(t,T;L^m(\Omega))} + \||\nabla u^0|\|_{L^q(t,T;L^m(\Omega))} \leq \\ &\||\nabla(u - u^0)|\|_{L^p(\Omega\times(t,T))}(T - t)^{\frac{1}{q}-\frac{1}{p}}|\Omega|^{\frac{1}{m}-\frac{1}{p}} + \||\nabla u^0|\|_{L^q(t,T;L^m(\Omega))}. \end{aligned} \tag{56}$$

Hence, the assertions follow thanks to estimates (34)–(36) of Theorem 2 applied with $v = u^0$. □

Proof of Corollary 1. From Theorem 2.2 in [11], we deduce that if f belongs to $L^\sigma(\Omega_T)$, with σ satisfying (26), then the unique solution u^0 constructed by approximation of (11) satisfies

$$\nabla u^0 \in (L^p(\Omega_T))^N.$$

Hence, from Theorem 1, it follows that if (26) holds, then the unique solution u constructed by approximation of (1) satisfies (26).

Moreover, from Theorem 2.3 in [11], we deduce that if f belongs to $L^\sigma(\Omega_T)$, with σ satisfying (28), then the unique solution u^0 constructed by approximation of (11) satisfies

$$\nabla u^0 \in (L^m(\Omega_T))^N,$$

with m as in (29). Hence, (29) follows from Theorem 1. □

Proof of Corollary 2. The proof is similar at all to that of Corollary 1 once observed that if $f \in L^r(0, T; L^\sigma(\Omega))$, with r and σ satisfying (30), by Theorem 1.3 in [8], it follows that the unique solution u^0 constructed by approximation of (11) satisfies

$$\nabla u^0 \in (L^p(\Omega_T))^N.$$

□

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Appendix A

For the convenience of the reader, we recall here the results of [23] used in the proof of Theorem 2. In more detail these are results that allow to obtain L^∞-estimate for a function u simply showing that it satisfies suitable integral estimates of "energy type".

Theorem A1 (Theorem 2.1 in [23])**.** *Assume that*

$$u \in C((0, T); L^r(\Omega)) \cap L^b(0, T; L^q(\Omega)) \cap C([0, T); L^{r_0}(\Omega)) \tag{A1}$$

where Ω *is an open set of* $\mathbb{R}^N$, $N \geq 1$, $0 < T \leq +\infty$ *and*

$$1 \leq r_0 < r < q \leq +\infty, \quad b_0 < b < q, \quad b_0 = \frac{(r - r_0)}{1 - \frac{r_0}{q}}. \tag{A2}$$

Suppose that u satisfies the following integral estimates for every $k > 0$

$$\int_\Omega |G_k(u)|^r(t_2)dx - \int_\Omega |G_k(u)|^r(t_1)dx + c_1 \int_{t_1}^{t_2} \|G_k(u)(\tau)\|^b_{L^q(\Omega)} d\tau \le 0 \quad \text{for every } \ 0 < t_1 < t_2 < T \tag{A3}$$

$$\|G_k(u)(t)\|_{L^{r_0}(\Omega)} \le c_2 \|G_k(u)(t_0)\|_{L^{r_0}(\Omega)} \quad \text{for every } \ 0 \le t_0 < t < T, \tag{A4}$$

where c_1 *and* c_2 *are positive constants independent on k and*

$$u_0 \equiv u(x,0) \in L^{r_0}(\Omega). \tag{A5}$$

Then there exists a positive constant C_1 *(see Formula (4.19) in [23]) depending only on* N, c_1, c_2, r, r_0, q *and* b *such that*

$$\|u(t)\|_{L^\infty(\Omega)} \le C_1 \frac{\|u_0\|^{h_0}_{L^{r_0}(\Omega)}}{t^{h_1}} \quad \text{for every } \ t \in (0,T), \tag{A6}$$

where

$$h_1 = \frac{1}{b - (r - r_0) - \frac{r_0 b}{q}}, \quad h_0 = h_1 \left(1 - \frac{b}{q}\right) r_0. \tag{A7}$$

We recall that here G_k is the same function defined in Section 3, i.e.,

$$G_k(s) = (|s| - k)_+ \mathrm{sign}(s).$$

If Ω has finite measure it is possible to prove that universal bounds hold if $b > r$ and that exponential estimates are satisfied by u if $b = r$. More in detail we have the following results.

Theorem A2 (Theorem 2.2 in [23])**.** *Let the assumptions of Theorem A1 hold true.*

If Ω *has finite measureand* $b > r$ *we have the following universal bound*

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{C_\sharp}{t^{h_2}} \quad \text{for every } \ t \in (0,T), \tag{A8}$$

where

$$h_2 = h_1 + \frac{h_0}{b - r} = \frac{1}{b - r}, \tag{A9}$$

and $C_\sharp$, *(see Formula (4.19) in [23]), is a constant depending only on* r, r_0, q, b, c_1, c_2 *and the measure of* Ω.

Moreover, if Ω *has finite measure and* $b = r$ *the following exponential bound holds*

$$\|u(t)\|_{L^\infty(\Omega)} \le C_2 \frac{\|u_0\|_{L^{r_0}(\Omega)}}{t^{h_1} e^{\sigma t}}, \quad \text{for every } \ t \in (0,T), \tag{A10}$$

where C_2 *is a positive constant depending only on* N, c_1, c_2, r, r_0, b *and* q, h_1 *is as in* (A7) *and*

$$\sigma = \frac{c_1 \kappa}{4(r - r_0)|\Omega|^{1 - \frac{b}{q}}}, \quad \kappa \quad \text{arbitrarily fixed in} \quad \left(0, 1 - \frac{r_0}{r}\right). \tag{A11}$$

We point out that in the previous results u is not assumed to satisfy any partial differential equation but simply suitable integral inequalities.

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