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Robust Portfolio Choice under the Modified Constant Elasticity of Variance

Wei Li Fan [†] and Marcos Escobar Anel [*,†]

Department of Statistical and Actuarial Sciences, University of Western Ontario, London, ON N6A 5B7, Canada; wfan54@uwo.ca
* Correspondence: marcos.escobar@uwo.ca
† These authors contributed equally to this work.

Abstract: This study investigates ambiguity aversion within the framework of a utility-maximizing investor under a modified constant-elasticity-of-volatility (M-CEV) model for the underlying asset. We derive closed-form solutions of a non-affine type for the optimal allocation and value function via a Cauchy problem. This work generalizes previous results in non-ambiguous settings by extending existing work to Hyperbolic Absolute Risk Aversion utility (HARA), correcting some typos in the literature for Constant Relative Risk Aversion utility (CRRA). Helpful details and derivations are also included in the manuscript.

Keywords: M-CEV model; expected utility; HARA; ambiguity-aversion; Cauchy problem

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1. Introduction

Portfolio optimization in continuous time is a constantly evolving line of research, with both practitioners and academics actively seeking more realistic objective functions, improved models, and refined stylized facts to enhance decision-making. All of these efforts are essential to keep pace with the growing complexity of financial markets and social interactions.

In this study, we embrace the widely utilized framework of expected utility theory (EUT) as the objective function for investors. EUT provides a fruitful ground for analytical solutions and is, therefore, more easily interpretable, as pioneered in the seminal work by Merton [1]. This early achievement relied on simple geometric Brownian motion (GBM) to describe the underlying asset price with considerations only for risk aversion. Although many extensions have been considered in the literature, the vast majority involve more than one source of risk in explaining asset prices. Examples include stochastic volatility models and the addition of jumps, as evidenced in references [2,3]. However, incorporating multiple risk sources often leads to inaccuracies in the estimation and evaluation methodologies, thereby partially affecting their benefits. To preserve the simplicity of a single source of risk while generalizing the GBM, this study uses a new member of the family of CEV models (see the seminal work of Cox in [4]); the so-called modified constant elasticity of variance (M-CEV). This model, explored for pricing purposes in the reference [5], has been adapted to expected utility optimization in the recent work of Muravei in [6]. Additionally, refer to the references [7,8] for portfolio optimization results on two other types of CEV models: CEV and LVO-CEV.

The primary innovation in our work lies in considering ambiguity in the model, thereby introducing ambiguity aversion in the decision-making of the investor. The assumption of model ambiguity can be traced back to the experimental studies of Ellsberg [9] and more recently the work in [10], who demonstrate that individuals are averse to ambiguity (unknown probability) in addition to their well-known aversion to risk. A crucial framework for analytical solutions was presented in Maenhout [11], leading to a Hamilton-Jacobi-Bellman (HJBI) representation of the solution in the GBM case. Most extensions of this work consider multiple sources of risk (e.g., the reference [12] for stochastic volatility and jumps). This study is the first to extend the analysis to a member of the CEV family of models.

For clarity, we list the main contributions of our work as follows:

1. We solve the expected utility portfolio problem for a HARA (Hyperbolic Absolute Risk Aversion) investor in the absence of ambiguity aversion. This solution can be reinterpreted as a constant proportion portfolio insurance (CPPI), a strategy widely employed in the financial industry (see the reference [13]).
2. As a by-product of solving HARA, we identified and rectified a few typos in the literature regarding the solution for the embedded Constant Relative Risk Aversion (CRRA) case.
3. We find closed-form solutions for the optimal allocation, optimal reference model, value function, and optimal wealth process for an investor exhibiting both risk and ambiguity aversion within HARA utilities. Numerical and empirical studies with our findings will be conducted in a follow-up paper.

The remainder of this paper is organized as follows. Section 2 outlines the mathematical and financial settings and formulates the problem. In Section 3, we present the derived solutions and delve into details of various embedded cases that hold significance for readers. The appendix provides additional details that are organized into subsections related to the main proposition.

2. Problem Formulation

We consider the general price process S_t given by the following stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = \left(r + \bar{\lambda}\sigma S_t^{\psi}\right)\mathrm{d}t + \sigma S_t^{\beta}\mathrm{d}Z_t,\ s_0 > 0, \tag{1}$$

where r and $\bar{\lambda}$ are positive real numbers and s_0 is the initial asset price value. In this general setting, S_t^{β} represents the volatility of the risky asset return, and β is the elasticity of variance with respect to the stock price. We assume $\psi = 2\beta$ and $\beta \neq 0$, hence Equation (1) becomes the M-CEV model (see the manuscripts [6,14]).

We assume a market consisting of money market account B and risky asset S_t (such as a stock). These prices follow the following dynamics:

$$\frac{\mathrm{d}B_t}{B_t} = r\mathrm{d}t, \tag{2}$$

$$\frac{\mathrm{d}S_t}{S_t} = \left(r + \bar{\lambda}\sigma S_t^{2\beta}\right)\mathrm{d}t + \sigma S_t^{\beta}\mathrm{d}Z_t,\ s_0 > 0, \tag{3}$$

where r is a constant risk-free interest rate. As explained in the references [5,6,8], the M-CEV allows for a non-zero probability of the underlying touching zero (default) if $\beta > 1$, which makes it realistic for pricing and portfolio problems.

We refer to model (3) as the reference model. Our investors face uncertainty regarding the probability distribution associated with the reference model and consider a set of plausible alternative models when making investment decisions. Specifically, the investors are uncertain about the distribution of S. This formulation enables us to capture the uncertainty regarding the drift in stock prices and the prices of risk in the stock market.

Let $e := e_t^s$ denote an $\mathbb{R}$-valued $\mathcal{F}_t$-progressively measurable process. The Radon–Nikodym derivative process is defined as follows:

$$\zeta_t^e = \mathbb{E}\left[\frac{\mathrm{d}\mathbb{P}^e}{\mathrm{d}\mathbb{P}}|\mathcal{F}_t\right] = \exp\left(-\int_0^t \frac{(e_\tau^s)^2}{2}\mathrm{d}\tau + e_\tau \mathrm{d}Z\tau\right). \tag{4}$$

According to Girsanov's theorem, the process can be expressed as

$$\widetilde{Z}_t = \int_0^t e_\tau^s \mathrm{d}\tau + Z_t. \tag{5}$$

We denote the set of all $\mathcal{F}_t$-progressively measurable processes such that the process given by (4) is a well-defined Radon–Nikodym derivative process as $\mathcal{E}[0,T]$. This process, denoted by $\widetilde{Z}_t$, represents the Wiener process under probability measure $\mathbb{P}^e$. The reference model is generally regarded as the most accurate representation of available data for an agent. However, viable alternative models pose a challenge in terms of statistical differentiation from reference models. Alternative models were generated via the perturbation process e_t^s as follows:

$$\frac{dS_t}{S_t} = \left(r + \bar{\lambda}\sigma S_t^{2\beta} - e_t^s \sigma S_t^{\beta}\right)\mathrm{d}t + \sigma S_t^{\beta}\mathrm{d}Z_t. \tag{6}$$

In the model setup, it is important to acknowledge that this ambiguity stems from an investor's inability to capture the expected returns precisely in the probability laws governing the stock price process. This assumption aligns with the perspectives presented in seminar studies, such as the work by Merton [15], and more recently shown in Tables 1 and 2 in reference [8] on the large standard errors in estimating the parameter $\bar{\lambda}$.

Let π represent the investment strategy applied from time t to time T in model (3). We define space $\mathcal{U}[0,T]$ as the set of admissible strategies π that satisfy the following conditions:

1. π is an $\mathcal{F}_t$-progressively measurable process.
2. Under π, the wealth X_t of the investor remains non-negative for $t \in [0,T]$.
3. The integrability conditions necessary for the expectation operator in (7) to be well-defined are satisfied.

We consider an investor with a preference for Hyperbolic Absolute Risk Aversion (HARA) utility on terminal wealth with risk-aversion level γ. The utility function for terminal wealth X_T is defined as:

$$u(X_T) = \frac{(X_T - F)^{\gamma}}{\gamma}.$$

The goal is to examine an ambiguous agent with HARA utility who aims to construct an investment strategy for the time interval $[0,T]$ that maximizes the expected utility for terminal wealth X_T. In line with this objective, we define the reward functional realized by the investor when selecting an alternative model specified by e as follows:

$$w^e(x,t;\pi,c) = \mathbb{E}_{x,t}^{\mathbb{P}^e}\left[\frac{(X_T - F)^{\gamma}}{\gamma}\right], \tag{7}$$

where $-\infty < \gamma < 1$, $\gamma \neq 0$, and denotes the constant relative aversion parameter. Then, the indirect utility function is given by:

$$J(x,t) = \sup_{\pi\in\mathcal{U}} \inf_{e\in\mathcal{E}[t,T]} \left(w^e(x,t;\pi) + \mathbb{E}^{\mathbb{P}^e}\left[\int_t^T \frac{(e_\tau^s)^2}{2\Psi(\tau, X_\tau)}\mathrm{d}\tau\right]\right), \tag{8}$$

where space $\mathcal{U}$ consists of admissible controls $\{\pi_t\}_{t\in[0,T]}$ with $\pi_t \in \mathbb{R}$ and satisfies the standard conditions.

According to reference [16], investors consider alternative models that are statistically challenging to distinguish from reference models. To address this issue, the value function incorporates a penalty term designed to discourage significant deviations from the reference model. This penalty term, which appears as the last expectation term in the problem (8), is computed based on relative entropy. The perturbations e_t^s in the penalty term were scaled by $\Psi(\tau, X_\tau)$. Function Ψ reflects the level of ambiguity or uncertainty associated with the models and is used to adjust the penalties accordingly. Higher values of Ψ indicate smaller penalties for deviating from the reference model, signifying greater uncertainty on the part of the investor.

By incorporating the penalty term, the investor considers both model ambiguity and the associated diffusion risk. This approach allows investors to make decisions that weigh the potential benefits of alternative models while accounting for the risks and uncertainties involved.

For the sake of analytical tractability, we maintain the assumption proposed in reference [11], that the ambiguity aversion parameter ϕ is related to the value function $J(x, t)$ and the risk aversion level γ by the expression

$$\Psi = \frac{\phi}{\gamma J(x,t)}, \tag{9}$$

where $\phi > 0$ denotes the level of ambiguity aversion. This assumption allows us to incorporate the degree of ambiguity into the model and analyze its impact on investors' decision-making processes.

In summary, the value function considers investors' preferences for terminal wealth while also considering the uncertainty associated with alternative models. The penalty term within the value function discourages significant deviation from the reference model. Scaling factor Ψ is a key component that reflects investors' level of ambiguity aversion. Higher values of Ψ indicate greater uncertainty and ambiguity regarding the alternative models, which corresponds to smaller penalties for deviations from the reference model. By incorporating this penalty term and adjusting it based on the level of ambiguity, investors can make decisions that balance their preferences with the associated uncertainties in the alternative models.

3. Optimal Investment Strategies

In this section, we address the problem described in (8) by employing the stochastic control approach for the M-CEV model with ambiguity. Our objective is to derive closed-form solutions to the investment problem. The solution will provide insights into the impact of ambiguity aversion levels on investors' decision-making processes.

3.1. The Hamilton-Jacobi-Bellman-Isaacs (HJBI) Equation

Assuming that π_t represents the fraction of wealth invested in stock S_t, the remaining portion of wealth $(1 - \pi_t)$ is invested in the risk-free money account with a constant interest rate r. For mathematical benefits, our focus is on the investor's position in the stock, denoted by ψ_t (i.e., the number of units of the asset held). Then, investor wealth X_t is governed by the following stochastic differential equation (SDE) derived from a self-financing condition:

$$\begin{aligned} dX_t &= X_t\left[\pi_t \frac{\mathrm{d}S_t}{S_t} + (1-\pi_t) r \mathrm{d}t\right] \\ &= X_t\left(r + \pi_t \bar{\lambda} \sigma S_t^{2\beta} - \pi_t e_t^s \sigma S_t^{\beta}\right)\mathrm{d}t + \pi_t \sigma S_t^{\beta} X_t \mathrm{d}Z_t. \end{aligned} \tag{10}$$

Since $\psi_t = \pi_t \frac{X_t}{S_t}$, we can express Equation (10) as follows:

$$\begin{aligned} dX_t &= \psi_t \mathrm{d}S_t + r(X_t - \psi_t S_t)\mathrm{d}t \\ &= \Big[\psi_t r S_t + \psi_t \bar{\lambda}\sigma S_t^{2\beta+1} - \psi_t e_t^s \sigma S_t^{\beta+1} + r(X_t - \psi_t S_t)\Big]\mathrm{d}t + \psi_t \sigma S_t^{\beta+1}\mathrm{d}Z_t \\ &= \Big(rX_t + \psi_t \bar{\lambda}\sigma S_t^{2\beta+1} - \psi_t e_t^s \sigma S_t^{\beta+1}\Big)\mathrm{d}t + \psi_t \sigma S_t^{\beta+1}\mathrm{d}Z_t. \end{aligned} \tag{11}$$

Consequently, the expression for the value function (8) should be modified to:

$$J(X,S,t) = \sup_{\psi \in \mathcal{U}} \inf_{e \in \mathcal{E}[t,T]} \left(w^e(X,S,t;\psi) + \mathbb{E}^{\mathbb{P}^e}\left[\int_t^T \frac{(e_\tau^s)^2}{2\Psi(\tau, X_\tau, S_\tau)}\mathrm{d}\tau\right]\right), \tag{12}$$

where $J(X,S,t)$ satisfies a HJBI equation.

Define

$$\theta_t := \pi_t \sigma S_t^{\beta}, \tag{13}$$

where θ_t represents the portfolio exposure to the fundamental risk factor Z_t, and π_t is the portfolio weight that determines the allocation of wealth in the M-CEV model. Expression (13) illustrates how the investor's desired exposure to the risk factor is calculated, depending on the portfolio weight π_t and the volatility of the asset return σ, which is scaled by S_t^{β}. The portfolio weight determines the portion of investor wealth allocated to the risky asset, thereby influencing the overall exposure to the risk factor. As before, we can represent (13) as:

$$\begin{aligned} \theta_t &:= \frac{\psi_t S_t}{X_t}\sigma S_t^{\beta} \\ &= \psi_t X_t^{-1} \sigma S_t^{\beta+1}. \end{aligned} \tag{14}$$

This means that we can also state that the number of units of the asset dictates the allocation of the investor's wealth to the risky asset, thus impacting overall exposure. To simplify our analysis, we shift our focus to these exposures rather than investor position ψ in stock. As a result, the transformation of model (11) is given by:

$$\begin{aligned} dX_t &= \Big(rX_t + \psi_t \bar{\lambda}\sigma S_t^{2\beta+1} - \psi_t e_t^s \sigma S_t^{\beta+1}\Big)\mathrm{d}t + \psi_t \sigma S_t^{\beta+1}\mathrm{d}Z_t \\ &= X_t\Big(r + \bar{\lambda}\theta_t S_t^{\beta} - e_t^s \theta_t\Big)\mathrm{d}t + \theta_t X_t \mathrm{d}Z_t. \end{aligned} \tag{15}$$

Accordingly, the value function (12) satisfies the Hamilton (Jacobi)-Isaacs equation:

$$\begin{aligned} \sup_{\theta} \inf_{e^s} \Big\{ & J_t + x\Big(r + \bar{\lambda}S^{\beta}\theta - e^s\theta\Big)J_x + \frac{1}{2}x^2\theta^2 J_{xx} \\ & + \Big(rS + \bar{\lambda}\sigma S^{2\beta+1} - e_t^s \sigma S^{\beta+1}\Big)J_s + \frac{1}{2}\sigma^2 S^{2\beta+2}J_{ss} + x\sigma S^{\beta+1}\theta J_{xs} + \frac{(e^s)^2}{2\Psi}\Big\} = 0, \end{aligned} \tag{16}$$

where J_t, J_x, J_s, J_{xx}, J_{ss}, and J_{xs} denote the partial derivatives of the first and second order with respect to time, stock and wealth, respectively. Moreover, $\Psi = \frac{\phi}{\gamma J}$ (see (9)), where ϕ is a positive ambiguity-aversion parameter.

We first find the infimum, denoted as e^{s_*}, and then differentiate (16) to obtain the optimal exposure as:

$$\Big\{ \theta^* = \frac{e^{s_*}J_x - \bar{\lambda}S^{\beta}J_x - \sigma S^{\beta+1}J_{xs}}{xJ_{xx}}.$$

The solution to (16) is provided in the next section.

3.2. Closed-Form Solutions for Hara Utility

To obtain the optimal strategy, we solve HJBI Equation (16) (for additional details, refer to Appendix A). The main proposition is presented below.

Proposition 1. *Assume that $\beta < 0$, $\phi > 0$, and $\phi \neq \gamma - 1$. The solution to Problem (16) is as follows:*

1. *The indirect utility function of an ambiguity and risk averse investor is given by*

$$J(x,S,t) = \frac{\left(x - Fe^{-r(T-t)}\right)^{\gamma}}{\gamma} g^{1/\alpha}(S,T-t), \tag{17}$$

where $\alpha = \frac{\gamma-\phi}{\gamma(\phi-\gamma+1)}$, and we have the following properties:

(a) *Function $g(S,T-t)$ is determined by solving the Cauchy problem:*

$$\begin{cases} \mathcal{L}g \equiv -g_t + \frac{1}{2}\sigma^2 S^{2\beta+2} g_{ss} + \alpha S\left(r\frac{\gamma(\phi-\gamma+1)}{\gamma-\phi} + \frac{\bar{\lambda}\gamma\sigma S^{2\beta}}{\gamma-\phi}\right)g_s + \frac{\alpha\gamma\left(\bar{\lambda}S^{\beta}\right)^2}{2(\phi-\gamma+1)}g + \alpha\gamma r g = 0, \\ g(S,0) = 1 \text{ when } t = T. \end{cases} \tag{18}$$

(b) *Solution $g(S,T-t)$ can be represented as:*

$$g(S,T-t) = e^{R\tilde{\tau}+zB(\tilde{\tau})} D^{\delta}(\tilde{\tau}) \frac{\Gamma(\eta-\delta+1/2)}{\Gamma(1+2\eta)} e^{-\frac{z}{2}A(\tilde{\tau})} (zA(\tilde{\tau}))^{\delta} M_{\delta,\eta}(zA(\tilde{\tau})), \tag{19}$$

where $z = \omega S^{-2\beta}$, $\tilde{\tau} = \sigma^2\beta^2\omega(T-t)$, $\Gamma(x)$ is the gamma function, $M_{\delta,\eta}(x)$ is the Whittaker function with parameters

$$\delta = -\frac{1}{2} - \frac{1}{2\beta}\left(\frac{1}{2} - \frac{\alpha\bar{\lambda}\gamma}{\sigma(\gamma-\phi)}\right), \quad \eta = \sqrt{\left(\delta + \frac{1}{2}\right)^2 + \frac{\alpha\gamma\bar{\lambda}^2}{4\sigma^2\beta^2(\gamma-\phi-1)}}.$$

Meanwhile, the remaining constants and functions are given by

$$\omega = \frac{r}{\sigma^2|\beta|}, \quad Q = \frac{r}{\sigma^2\beta\omega}, \quad R = \frac{r\alpha\gamma}{\sigma^2\beta^2\omega} - 2Q\delta, \tag{20}$$

$$A(\tilde{\tau}) = \frac{1}{2\sinh^2\tilde{\tau}(\coth\tilde{\tau}+Q)}, \quad B(\tilde{\tau}) = \frac{Q^2-1}{2(\coth\tilde{\tau}+Q)}, \quad D(\tilde{\tau}) = \frac{Q^2-1}{4A(\tilde{\tau})B(\tilde{\tau})}. \tag{21}$$

2. *The optimal exposure to the risk factor Z_t is*

$$\theta^* = \frac{\left(x-Fe^{-r(T-t)}\right)S^{\beta}}{x}\left[\frac{\bar{\lambda}}{\phi-\gamma+1} + \sigma S\left(B(\tilde{\tau}) + \frac{\delta+\eta+\frac{1}{2}}{z}\frac{M_{\delta+1,\eta}\left(zA(\tilde{\tau})\right)}{M_{\delta,\eta}\left(zA(\tilde{\tau})\right)}\right)\frac{dz}{dS}\right], \tag{22}$$

which is equivalent to writing

$$\theta^* = \frac{(x-Fe^{-r(T-t)})S^{\beta}}{x}\left[\frac{\bar{\lambda}}{\phi-\gamma+1} - 2\sigma\omega\beta S^{-2\beta}B(\tilde{\tau}) - 2\sigma\beta\left(\delta+\eta+\frac{1}{2}\right)\frac{M_{\delta+1,\eta}\left(\omega A(\tilde{\tau})S^{-2\beta}\right)}{M_{\delta,\eta}\left(\omega A(\tilde{\tau})S^{-2\beta}\right)}\right]. \tag{23}$$

3. *The worst-case measure is determined by:*

$$e^{s_*} = \frac{\phi\bar{\lambda}S^{\beta}}{\phi-\gamma+1} + \frac{\phi\sigma S^{\beta+1}}{\gamma-\phi}\left[B(\tilde{\tau}) + \frac{\delta+\eta+\frac{1}{2}}{z}\frac{M_{\delta+1,\eta}(zA(\tilde{\tau}))}{M_{\delta,\eta}(zA(\tilde{\tau}))}\right]\frac{dz}{dS}, \tag{24}$$

which can also be written as follows:

$$e^{s_*} = \frac{\phi\bar{\lambda}S^{\beta}}{\phi-\gamma+1} + \frac{\phi\sigma S^{\beta}}{\phi-\gamma}\left[-2\omega\beta S^{-2\beta}B(\tilde{\tau}) - 2\beta(\delta+\eta+\frac{1}{2})\frac{M_{\delta+1,\eta}\left(\omega A(\tilde{\tau})S^{-2\beta}\right)}{M_{\delta,\eta}\left(\omega A(\tilde{\tau})S^{-2\beta}\right)}\right]. \tag{25}$$

Proof. The proof is divided into several steps, all of which are presented in Appendix A.

Appendix A.1 derives the solution to the HJBI equation up to the Cauchy representation of g.

Appendix A.2 details the solution to the Cauchy equation. In particular, Appendix A.2.1 provides the scaling transformations needed to obtain an equation for a new function h. Appendix A.2.2 finds the PDE for h. Appendix A.2.3 applies a Laplace transform to solve the PDE for h. Appendix A.2.4 combines all results to obtain g. Lastly, Appendix A.2.5 computes a ratio involving derivatives of g needed for the next step.

Appendix A.3 uses the previous results to derive the optimal exposure and worst-case measure, denoted as θ^* and e^{s_*}, respectively, which are dedicated to the Whittaker function. □

As indicated by reference [6], in the absence of ambiguity and for CRRA, it is crucial to emphasize that assuming a risk-free interest rate r of zero leads to the following simplifications:

$$\begin{aligned} \omega = z = \tilde{\tau} = 0, \\ Q = R = 0. \end{aligned}$$

Meanwhile, the limit values from the expressions (21) are provided by:

$$\begin{aligned} & \lim_{\omega \longrightarrow 0} D(\tilde{\tau}) = 1, \\ & \lim_{\omega \longrightarrow 0} zB(\tilde{\tau}) = 0, \\ & \lim_{\omega \longrightarrow 0} R\tilde{\tau} = 0, \\ \varphi(S,t) \quad & := \lim_{\omega \longrightarrow 0} zA(\tilde{\tau}) = \tfrac{1}{2}\sigma^{-2}\beta^{-2}S^{-2\beta}(T-t)^{-1}. \end{aligned}$$

Consequently, function $g(S,t)$ in Proposition 1 can be simplified, resulting in the following corollary.

Corollary 1. *Assume that the risk-free interest rate of $r = 0$, $\beta < 0$, $\phi > 0$, and $\phi \neq \gamma - 1$.*

1. *The indirect utility function is given by:*

$$J(x,S,t) = \frac{(x-F)^{\gamma}}{\gamma} g^{1/\alpha}(S,T-t), \quad \alpha = \frac{\gamma-\phi}{\gamma(\phi-\gamma+1)}, \tag{26}$$

   and we have the following properties:

   (a) *Function $g(S,T-t)$ solves the Cauchy problem*

$$\begin{cases} \mathcal{L}g \equiv -g_t + \frac{1}{2}\sigma^2 S^{2\beta+2} g_{ss} + \frac{\alpha\bar{\lambda}\gamma\sigma S^{2\beta+1}}{\gamma-\phi} g_s + \frac{\alpha\gamma\left(\bar{\lambda}S^{\beta}\right)^2}{2(\phi-\gamma+1)} g = 0, \\ g(S,0) = 1. \end{cases} \tag{27}$$

   (b) *Solution $g(S,T-t)$ is given by*

$$g(S,T-t) = \frac{\Gamma(\eta-\delta+1/2)}{\Gamma(1+2\eta)} e^{-\frac{1}{2}\varphi(S,t)} \varphi^{\delta}(S,t) M_{\delta,\eta}(\varphi(S,t)), \tag{28}$$

   where $\Gamma(x)$ is the gamma function, $M_{\delta,\eta}(x)$ is the Whittaker function with parameters

$$\delta = -\frac{1}{2} - \frac{1}{2\beta}\left(\frac{1}{2} - \frac{\alpha\bar{\lambda}\gamma}{\sigma(\gamma-\phi)}\right), \qquad \eta = \sqrt{\left(\delta+\frac{1}{2}\right)^2 + \frac{\alpha\gamma\bar{\lambda}^2}{4\sigma^2\beta^2(\gamma-\phi-1)}},$$

   and there is a limit value

$$\varphi(S,t) = \lim_{\omega \longrightarrow 0} zA(\tilde{\tau}) = \frac{1}{2}\sigma^{-2}\beta^{-2}S^{-2\beta}(T-t)^{-1}. \tag{29}$$

2. *The optimal exposure is determined by:*

$$\theta^* = \frac{(x-F)S^\beta}{x}\left[\frac{\bar{\lambda}}{\phi-\gamma+1} - 2\sigma\beta(\delta+\eta+\frac{1}{2})\frac{M_{\delta+1,\eta}\big(\varphi(S,t)\big)}{M_{\delta,\eta}\big(\varphi(S,t)\big)}\right]. \tag{30}$$

3. *The worst-case measure can be obtained as:*

$$e^{s_*} = \frac{\phi\bar{\lambda}S^\beta}{\phi-\gamma+1} - \frac{2\phi\sigma\beta S^\beta(\delta+\eta+\frac{1}{2})}{\gamma-\phi}\frac{M_{\delta+1,\eta}\big(\varphi(S,t)\big)}{M_{\delta,\eta}\big(\varphi(S,t)\big)}. \tag{31}$$

Proof. The proof is available in Appendix A.4. □

For further simplicity, we set F equal to 0; this would effectively be a CRRA utility setting. Assuming a risk-free interest rate of $r = 0$, we obtain an even simpler case.

Corollary 2. *Consider risk-free interest rates of $r = 0$, $\beta < 0$, $\phi > 0$, and $\phi \neq \gamma - 1$. We have the following properties.*

1. *The indirect utility function of an ambiguity and risk averse investor is given by:*

$$J(x,S,t) = \frac{x^\gamma}{\gamma}g^{1/\alpha}(S,T-t), \ \ \alpha = \frac{\gamma-\phi}{\gamma(\phi-\gamma+1)}, \tag{32}$$

where the function $g(S,t)$ is identical to those in parts (a) and (b) of Corollary 1.

2. *The optimal exposure to the risk factor Z_t is determined by:*

$$\theta^* = \frac{\bar{\lambda}S^\beta}{\phi-\gamma+1} - 2\sigma\beta S^\beta(\delta+\eta+\frac{1}{2})\frac{M_{\delta+1,\eta}\big(\varphi(S,t)\big)}{M_{\delta,\eta}\big(\varphi(S,t)\big)}. \tag{33}$$

Proof. The proofs are analogous to those presented in Corollary 1. □

Note that the Cauchy problem and worst-case scenario are the same as those in Corollary 1

3.3. Corrections to the Crra Case

Our work extends the results of the previous study in presented in reference [14] and in the formal publications [6]. However, discrepancies were identified in the Cauchy problem (2.13), as well as in parameters (3.5) and (3.6). This is related to Theorems 1 and 2 in the manuscript [6]. In this section, we address and rectify these inaccuracies by amalgamating the corrections for the aforementioned issues.

Corollary 3. *(Correction) For the M-CEV model, the value function $J(x,S,t)$ is given by:*

$$J(x,S,t) = \frac{x^\gamma}{\gamma}f^{\frac{1}{\delta}}(S,t), \ \ \delta = \frac{1}{1-\gamma}.$$

1. *The function f solves the Cauchy problem,*

$$\begin{cases} \mathcal{L}f(S,t) \equiv f_t + \frac{a^2S^{2\beta+2}}{2}f_{ss} + \delta S\big(\alpha-\gamma r + ca^2S^{2\beta}\big)f_s + \frac{\delta(\delta-1)}{2a^2}\big[(\alpha-r)S^{-\beta}+ca^2S^\beta\big]^2 f + r\gamma\delta f = 0, \\ f(S,T) = 1. \end{cases} \tag{34}$$

2. *The solution of boundary-value problem (34) is the function*

$$f(S,t) = e^{R\tau+zB(\tau)}D^\lambda(\tau)\frac{\Gamma(\eta-\lambda+\frac{1}{2})}{\Gamma(1+2\eta)}e^{-\frac{zA(\tau)}{2}}(zA(\tau))^\lambda M_{\lambda,\eta}(zA(\tau)), \tag{35}$$

where $z = \Lambda S^{-2\beta}$, $\tau = a^2\beta^2\Lambda(T-t)$, $\Gamma(z)$ *is the gamma function,* $M_{\lambda,\eta}(z)$ *is the Whittaker function with parameters*

$$\lambda = -\frac{1}{2} - \frac{1}{2\beta}\left(\frac{1}{2} - \delta c\right), \quad \eta = \sqrt{\left(\lambda + \frac{1}{2}\right)^2 + \frac{\delta(1-\delta)c^2}{4\beta^2}}.$$

The remaining constants and functions are given by:

$$\begin{array}{l} \Lambda = \frac{\sqrt{\delta}}{a^2|\beta|}\sqrt{\alpha^2 - \gamma r^2}, \quad Q = \frac{\delta(\alpha-\gamma r)}{\Lambda a^2 \beta}, \quad R = \frac{r\gamma\delta}{a^2\beta^2\Lambda} - 2Q\lambda - \frac{\delta(1-\delta)(\alpha-r)c}{\Lambda a^2\beta^2}, \\ A(\tau) = \frac{1}{2\sinh^2\tau(\coth\tau + Q)}, \quad B(\tau) = \frac{Q^2-1}{2(\coth\tau+Q)}, \quad D(\tau) = \frac{Q^2-1}{4A(\tau)B(\tau)}. \end{array}$$

3. *The optimal investor strategy is:*

$$\pi^*(X,S,t) = X\left[\delta\frac{\alpha - r + ca^2S^{2\beta}}{a^2S^{2\beta+1}} + \left(B(\tau) + \frac{\left(\lambda+\eta+\frac{1}{2}\right)M_{\lambda+1,\eta}\left(A(\tau)z\right)}{zM_{\lambda,\eta}\left(A(\tau)z\right)}\right)\frac{dz}{dS}\right]. \tag{36}$$

Based on the remark in reference [6], and assuming that $\alpha = r\sqrt{\gamma}$, it is easy to see that $\alpha^2 - \gamma r^2 = 0$. Consequently, $\Lambda = \frac{\sqrt{\delta}}{a^2|\beta|}\sqrt{\alpha^2 - \gamma r^2} = 0$. Therefore, the corrected outcomes are as follows:

Corollary 4. *If* $\alpha = r\sqrt{\gamma}$, *Formulas (35) and (36) are simplified. In this case, the following properties exist:*

1.
$$\Lambda = \tau = z = 0,$$

and the limit values are

$$\begin{array}{l} \lim_{\Lambda\longrightarrow 0} D(\tau) = 1, \quad \theta(S,t) = \lim_{\Lambda\longrightarrow 0} zA(\tau) = 0.5a^{-2}\beta^{-2}S^{-2\beta}(T-t)^{-1}, \\ \Omega(S,t) = \lim_{\Lambda\longrightarrow 0} zB(\tau) = \frac{1}{2S^{2\beta}}\frac{\delta^2(\alpha-\gamma r)^2(T-t)}{a^2\left[1+\beta\delta(\alpha-\gamma r)(T-t)\right]}, \\ \Psi(t) = \lim_{\Lambda\longrightarrow 0} R\tau = r\gamma\delta(T-t) - 2\delta\beta\lambda(\alpha-\gamma r)(T-t) - \frac{\delta(1-\delta)c(\alpha-r)(T-t)}{a^2}. \end{array}$$

2. *The solution* $f(S,t)$ *is given by:*

$$f(S,t) = \frac{\Gamma(\eta - \lambda + \frac{1}{2})}{\Gamma(1+2\eta)}e^{\Psi(t)+\Omega(S,t)-\frac{1}{2}\theta(S,t)}\theta^{\lambda}(S,t)M_{\lambda,\eta}(\theta(S,t)). \tag{37}$$

3. *The optimal policy is:*

$$\pi^*(X,S,t) = X\left[\delta\frac{\alpha - r + ca^2S^{2\beta}}{a^2S^{2\beta+1}} - \frac{2\beta\left(\lambda+\eta+\frac{1}{2}\right)}{S}\frac{M_{\lambda+1,\eta}\left(\theta(S,t)\right)}{M_{\lambda,\eta}\left(\theta(S,t)\right)}\right].$$

4. Discussion

Our findings reveal closed-form solutions to all elements of interest to a financial investor: optimal investment allocation on the risky asset, optimal alternative model due to ambiguity, optimal wealth process evolution, and finally, the expression for the value function (i.e., the value of the objective function at the optimal). This is very rare outside exponentially linear structures, which are also known as affine solutions.

Our choice of model, the M-CEV, has the advantage of keeping only one source of risk, and therefore, a lower parametric space; this is ideal for practitioners, always searching for a combination of realism and simplicity.

Our methodology to find the solution involved many transformations of the original PDE problem (from the HJBI equation), resulting in analytical expressions for a non-trivial Cauchy problem. The details provided in our work can serve as a foundation for future research in dealing with complex PDEs and boundary conditions. Thanks to the detailed analysis, we could detect and correct problems in previously known solutions to embedded problems (M-CEV in the absence of ambiguity with a CRRA utility).

This work has the potential for many extensions and applications. First, other CEV models in the literature have provided solutions (in some cases, approximations) to the embedded problem of CRRA with no ambiguity. The methodology developed here can be used to solve such problems analytically, while extending to HARA and ambiguity. Once this is achieved, an empirical comparison would help shed light on the importance of the various models in the presence/absence of ambiguity as well as in the presence/absence of a floor on wealth (HARA versus CRRA).

5. Ethical Compliance

All procedures performed in studies involving human participants were in accordance with the ethical standards of the institutional and/or national research committee and with the 1964 Helsinki Declaration and its later amendments or comparable ethical standards.

Author Contributions: W.L.F.: Conceptualization, methodology, validation, formal analysis, investigation, writing—original draft preparation, writing—review and editing, project administration. M.E.A.: Conceptualization, methodology, validation, formal analysis, investigation, writing—original draft preparation, writing—review and editing, supervision, project administration. All authors have read and agreed to the published version of the manuscript.

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Appendix A. Proof of Proposition 1

Appendix A.1. Proof of Proposition 1 up to the Cauchy Equation

We now solve the HJBI Equation (16) to determine the optimal investment strategy in the M-CEV model. To facilitate the solution later, we parameterize the change in measure, defined as

$$\epsilon_t := \frac{e_t^s}{S_t^\beta}.$$

The modified SDEs (6) and (15) can be rewritten as

$$\frac{dS_t}{S_t} = \left[r + (\bar{\lambda} - \epsilon_t)\sigma S_t^{2\beta}\right]\mathrm{d}t + \sigma S_t^\beta \mathrm{d}Z_t, \tag{A1}$$

$$\begin{aligned} \mathrm{d}X_t &= X_t\left(r + \bar{\lambda}\theta_t S_t^\beta - \epsilon_t\theta_t S_t^\beta\right)\mathrm{d}t + \theta_t X_t \mathrm{d}Z_t \\ &= X_t\left(r + (\bar{\lambda} - \epsilon_t)\theta_t S_t^\beta\right)\mathrm{d}t + \theta_t X_t \mathrm{d}Z_t. \end{aligned} \tag{A2}$$

Accordingly, the HJBI Equation (16) becomes

$$\begin{aligned} &\sup_\theta \inf_\epsilon \Bigg\{ J_t + x\left(r + \bar{\lambda}S^\beta\theta - \epsilon S^\beta\theta\right)J_x + \frac{1}{2}x^2\theta^2 J_{xx} \\ &+ \left(rS + \bar{\lambda}\sigma S^{2\beta+1} - \epsilon\sigma S^{2\beta+1}\right)J_s + \frac{1}{2}\sigma^2 S^{2\beta+2}J_{ss} + x\sigma S^{\beta+1}\theta J_{xs} + \frac{(\epsilon S^\beta)^2}{2\Psi}\Bigg\} = 0. \end{aligned} \tag{A3}$$

First and foremost, we begin by solving the minimization problem as follows:

$$\begin{aligned} &- xS^\beta\theta J_x - \sigma S^{2\beta+1}J_s + \frac{\epsilon S^{2\beta}}{\Psi} = 0 \\ &\Longrightarrow \epsilon^* = \Psi\left(x\theta S^{-\beta}J_x + \sigma S J_s\right). \end{aligned} \tag{A4}$$

Substituting the value of ϵ^* into Equation (A3) yields

$$\begin{aligned}
&\sup_{\theta}\Big\{J_t+\Big(xr+x\bar{\lambda}S^{\beta}\theta-x\Psi\big(x\theta S^{-\beta}J_x+\sigma SJ_s\big)S^{\beta}\theta\Big)J_x+\frac{1}{2}x^2\theta^2J_{xx}\\
&+\Big[rS+\bar{\lambda}\sigma S^{2\beta+1}-\Psi\big(x\theta S^{-\beta}J_x+\sigma SJ_s\big)\sigma S^{2\beta+1}\Big]J_s+\frac{1}{2}\sigma^2S^{2\beta+2}J_{ss}+x\sigma S^{\beta+1}\theta J_{xs}\\
&+\frac{\Psi^2\big(x\theta S^{-\beta}J_x+\sigma SJ_s\big)^2S^{2\beta}}{2\Psi}\Big\}=0\\
\Longrightarrow&\sup_{\theta}\Big\{J_t+xrJ_x+x\bar{\lambda}S^{\beta}\theta J_x-\Psi x^2\theta^2J_x^2-\Psi x\sigma\theta S^{\beta+1}J_sJ_x+\frac{1}{2}x^2\theta^2J_{xx}\\
&+rSJ_s+\bar{\lambda}\sigma S^{2\beta+1}J_s-\Psi x\sigma\theta S^{\beta+1}J_xJ_s-\Psi\sigma^2S^{2\beta+2}J_s^2+\frac{1}{2}\sigma^2S^{2\beta+2}J_{ss}+x\sigma S^{\beta+1}\theta J_{xs}\\
&+\frac{\Psi}{2}\Big(x^2\theta^2J_x^2+\sigma^2S^{2\beta+2}J_s^2+2x\theta\sigma S^{\beta+1}J_xJ_s\Big)\Big\}=0. \qquad (A5)
\end{aligned}$$

Applying the Bellman principle, the value function satisfies the HJBI equation with the terminal condition $J(x,T)=\frac{(x-F)^{\gamma}}{\gamma}$. Since $\Psi=\frac{\phi}{\gamma J}$ where $\phi>0$ (as shown in Formula (9)), substituting this into Equation (A5) results in

$$\begin{aligned}
&\sup_{\theta}\Big\{J_t+xrJ_x+x\bar{\lambda}S^{\beta}\theta J_x-\frac{1}{2}\frac{\phi}{\gamma J}x^2\theta^2J_x^2-\frac{\phi}{\gamma J}x\sigma\theta S^{\beta+1}J_sJ_x+\frac{1}{2}x^2\theta^2J_{xx}\\
&+rSJ_s+\bar{\lambda}\sigma S^{2\beta+1}J_s-\frac{1}{2}\frac{\phi}{\gamma J}\sigma^2S^{2\beta+2}J_s^2+\frac{1}{2}\sigma^2S^{2\beta+2}J_{ss}+x\sigma S^{\beta+1}\theta J_{xs}\Big\}=0. \qquad (A6)
\end{aligned}$$

Hence, the first-order conditions lead to

$$\begin{aligned}
&x\bar{\lambda}S^{\beta}J_x-\frac{\phi}{\gamma J}x^2J_x^2\theta-\frac{\phi}{\gamma J}x\sigma S^{\beta+1}J_sJ_x+x^2J_{xx}\theta+x\sigma S^{\beta+1}J_{xs}=0\\
\Longleftrightarrow\ \theta&=\frac{x\bar{\lambda}S^{\beta}J_x-\frac{\phi}{\gamma J}x\sigma S^{\beta+1}J_sJ_x+x\sigma S^{\beta+1}J_{xs}}{\frac{\phi}{\gamma J}x^2J_x^2-x^2J_{xx}}\\
&=\frac{\gamma\bar{\lambda}S^{\beta}J_xJ-\phi\sigma S^{\beta+1}J_sJ_x+\gamma\sigma S^{\beta+1}J_{xs}J}{\phi xJ_x^2-\gamma xJ_{xx}J}. \qquad (A7)
\end{aligned}$$

Now, substituting back into Equation (A6) gives

$$\begin{aligned}
&J_t+xrJ_x+\frac{\gamma\bar{\lambda}^2S^{2\beta}J_xJ-\phi\bar{\lambda}\sigma S^{2\beta+1}J_sJ_x+\gamma\bar{\lambda}\sigma S^{2\beta+1}J_{xs}J}{\phi J_x^2-\gamma J_{xx}J}J_x\\
&-\frac{1}{2}\frac{\phi}{\gamma J}\left(\frac{\gamma\bar{\lambda}S^{\beta}J_xJ-\phi\sigma S^{\beta+1}J_sJ_x+\gamma\sigma S^{\beta+1}J_{xs}J}{\phi J_x^2-\gamma J_{xx}J}\right)^2J_x^2\\
&-\frac{\phi}{\gamma J}\frac{\gamma\bar{\lambda}\sigma S^{2\beta+1}J_xJ-\phi\sigma^2S^{2\beta+2}J_sJ_x+\gamma\sigma^2S^{2\beta+2}J_{xs}J}{\phi J_x^2-\gamma J_{xx}J}J_sJ_x\\
&+\frac{1}{2}\left(\frac{\gamma\bar{\lambda}S^{\beta}J_xJ-\phi\sigma S^{\beta+1}J_sJ_x+\gamma\sigma S^{\beta+1}J_{xs}J}{\phi J_x^2-\gamma J_{xx}J}\right)^2J_{xx}+rSJ_s+\bar{\lambda}\sigma S^{2\beta+1}J_s-\frac{1}{2}\frac{\phi}{\gamma J}\sigma^2S^{2\beta+2}J_s^2\\
&+\frac{1}{2}\sigma^2S^{2\beta+2}J_{ss}+\frac{\gamma\bar{\lambda}\sigma S^{2\beta+1}J_xJ-\phi\sigma^2S^{2\beta+2}J_sJ_x+\gamma\sigma^2S^{2\beta+2}J_{xs}J}{\phi J_x^2-\gamma J_{xx}J}J_{xs}=0. \qquad (A8)
\end{aligned}$$

To find the solution, we employ the separation ansatz

$$J(x,S,t)=\frac{\big(x-Fe^{-r(T-t)}\big)^{\gamma}}{\gamma}g^{1/\alpha}(S,T-t),\quad \alpha=\frac{\gamma-\phi}{\gamma(\phi-\gamma+1)}, \qquad (A9)$$

where $\phi \neq \gamma - 1$, and the function $g(S, T-t)$ satisfies the boundary condition $g(S,0) = 1$ when $t = T$.

Thus, the derivatives of the function $J(x,S,t)$ are computed as

$$\begin{aligned}
J_t &= -rFe^{-r(T-t)}\left(x - Fe^{-r(T-t)}\right)^{\gamma-1} g^{1/\alpha}(S, T-t) \\
&- \alpha^{-1}\frac{g_t(S,T-t)}{g(S,T-t)}\frac{\left(x - Fe^{-r(T-t)}\right)^{\gamma}}{\gamma} g^{1/\alpha}(S,T-t) \\
&= \left(\frac{-rFe^{-r(T-t)}\gamma}{x - Fe^{-r(T-t)}} - \alpha^{-1}\frac{g_t}{g}\right)J, \\
J_x &= \left(x - Fe^{-r(T-t)}\right)^{\gamma-1} g^{1/\alpha}(S,T-t) = \frac{\gamma}{x - Fe^{-r(T-t)}}J, \\
J_{xx} &= (\gamma-1)\left(x - Fe^{-r(T-t)}\right)^{\gamma-2} g^{1/\alpha}(S,T-t) = \frac{\gamma(\gamma-1)}{\left(x - Fe^{-r(T-t)}\right)^2}J, \\
J_s &= \alpha^{-1}\frac{g_s(S,T-t)}{g(S,T-t)}\frac{\left(x - Fe^{-r(T-t)}\right)^{\phi-\gamma}}{\phi-\gamma} g^{1/\alpha}(S,T-t) = \alpha^{-1}\frac{g_s}{g}J, \\
J_{ss} &= \alpha^{-1}\frac{g_s}{g}J_s + \alpha^{-1}\left(\frac{g_{ss}}{g} - \frac{g_s^2}{g^2}\right)J = \alpha^{-2}\frac{g_s^2}{g^2}J + \alpha^{-1}\left(\frac{g_{ss}}{g} - \frac{g_s^2}{g^2}\right)J \\
&= \left[(\alpha^{-2} - \alpha^{-1})\frac{g_s^2}{g^2} + \alpha^{-1}\frac{g_{ss}}{g}\right]J, \\
J_{xs} &= \frac{\gamma}{x - Fe^{-r(T-t)}}J_s = \alpha^{-1}\frac{g_s}{g}\left(\frac{\gamma}{x - Fe^{-r(T-t)}}\right)J.
\end{aligned}$$

Referring to (A7), we denote the A and B as shown below for ease of computation:

$$\begin{aligned}
&A := \gamma\bar{\lambda}S^{\beta}J_xJ - \phi\sigma S^{\beta+1}J_sJ_x + \gamma\sigma S^{\beta+1}J_{xs}J, \\
&B := \phi x J_x^2 - \gamma x J_{xx}J.
\end{aligned}$$

It follows that

$$\begin{aligned}
A &= \gamma\bar{\lambda}S^{\beta}J_xJ - \phi\sigma S^{\beta+1}J_sJ_x + \gamma\sigma S^{\beta+1}J_{xs}J \\
&= \left(\tfrac{\gamma}{x - Fe^{-r(T-t)}}\right)\left(\gamma\bar{\lambda} - \phi\sigma\alpha^{-1}S\tfrac{g_s}{g} + \gamma\sigma\alpha^{-1}S\tfrac{g_s}{g}\right)S^{\beta}J^2, \quad \text{and} \\
B &= \phi x J_x^2 - \gamma x J_{xx}J \\
&= \tfrac{\gamma^2\phi x - \gamma^2 x(\gamma-1)}{\left(x - Fe^{-r(T-t)}\right)^2}J^2,
\end{aligned}$$

which implies that the optimal exposure to the risk factor Z_t is given by

$$\begin{aligned}
\theta^* &= \frac{\gamma\bar{\lambda}S^{\beta}J_xJ - \phi\sigma S^{\beta+1}J_sJ_x + \gamma\sigma S^{\beta+1}J_{xs}J}{\phi x J_x^2 - \gamma x J_{xx}J} = \frac{A}{B} \\
&= \frac{\left(\tfrac{\gamma}{x - Fe^{-r(T-t)}}\right)\left(\gamma\bar{\lambda} - \phi\sigma\alpha^{-1}S\tfrac{g_s}{g} + \gamma\sigma\alpha^{-1}S\tfrac{g_s}{g}\right)S^{\beta}J^2}{\tfrac{\gamma^2\phi x - \gamma^2 x(\gamma-1)}{\left(x - Fe^{-r(T-t)}\right)^2}J^2} \\
&= \frac{\left[\gamma\bar{\lambda}S^{\beta} - (\phi-\gamma)\sigma\alpha^{-1}S^{\beta+1}\tfrac{g_s}{g}\right]}{\gamma(\phi-\gamma+1)}\frac{\left(x - Fe^{-r(T-t)}\right)}{x}.
\end{aligned} \tag{A10}$$

Next, we substitute (A9) and its corresponding partial differential equations into Equation (A8) to derive (only the key steps are shown).

$$
\begin{aligned}
&\left(\frac{-rFe^{-r(T-t)}\gamma}{x-Fe^{-r(T-t)}}-\alpha^{-1}\frac{g_t}{g}\right)J+\frac{\gamma xr}{\left(x-Fe^{-r(T-t)}\right)}J\\
&+\left[\frac{\gamma\bar{\lambda}^2S^{2\beta}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2-(\phi-\gamma)\bar{\lambda}\sigma\alpha^{-1}S^{2\beta+1}\frac{g_s}{g}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2}{\phi\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}J^2-\frac{\gamma^2(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}J^2}\right]\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J\\
&-\frac{\phi}{2\gamma J}\left[\frac{\gamma\bar{\lambda}S^{\beta}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2-(\phi-\gamma)\sigma\alpha^{-1}S^{\beta+1}\frac{g_s}{g}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2}{\phi\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}J^2-\frac{\gamma^2(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}J^2}\right]^2\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}J^2\\
&-\frac{\phi}{\gamma J}\left[\frac{\gamma\bar{\lambda}\sigma S^{2\beta+1}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2-(\phi-\gamma)\sigma^2\alpha^{-1}S^{2\beta+2}\frac{g_s}{g}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2}{\phi\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}J^2-\frac{\gamma^2(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}J^2}\right]\frac{\alpha^{-1}\frac{g_s}{g}\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2\\
&+\frac{1}{2}\left[\frac{\gamma\bar{\lambda}S^{\beta}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2-(\phi-\gamma)\sigma\alpha^{-1}S^{\beta+1}\frac{g_s}{g}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2}{\phi\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}J^2-\frac{\gamma^2(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}J^2}\right]^2\frac{\gamma(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}J\\
&+rS\alpha^{-1}\frac{g_s}{g}J+\bar{\lambda}\sigma\alpha^{-1}S^{2\beta+1}\frac{g_s}{g}J-\frac{1}{2}\frac{\phi}{\gamma J}\sigma^2\alpha^{-2}S^{2\beta+2}\frac{g_s^2}{g^2}J^2+\frac{1}{2}\sigma^2S^{2\beta+2}\left[\left(\alpha^{-2}-\alpha^{-1}\right)\frac{g_s^2}{g^2}+\alpha^{-1}\frac{g_{ss}}{g}\right]J\\
&+\left[\frac{\gamma\bar{\lambda}\sigma S^{2\beta+1}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2-(\phi-\gamma)\sigma^2\alpha^{-1}S^{2\beta+2}\frac{g_s}{g}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J^2}{\phi\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}J^2-\frac{\gamma^2(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}J^2}\right]\frac{\alpha^{-1}\frac{g_s}{g}\gamma}{\left(x-Fe^{-r(T-t)}\right)}J=0\\
\Longrightarrow\quad&\left(\frac{-rFe^{-r(T-t)}\gamma}{x-Fe^{-r(T-t)}}-\alpha^{-1}\frac{g_t}{g}\right)+\frac{\gamma xr}{\left(x-Fe^{-r(T-t)}\right)}\\
&+\left[\frac{\gamma\bar{\lambda}^2S^{2\beta}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}-(\phi-\gamma)\bar{\lambda}\sigma\alpha^{-1}S^{2\beta+1}\frac{g_s}{g}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}}{\phi\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}-\frac{\gamma^2(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}}\right]\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}\\
&-\frac{\phi}{2\gamma}\left[\frac{\gamma\bar{\lambda}S^{\beta}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}-(\phi-\gamma)\sigma\alpha^{-1}S^{\beta+1}\frac{g_s}{g}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}}{\phi\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}-\frac{\gamma^2(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}}\right]^2\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}\\
&-\frac{\phi}{\gamma}\left[\frac{\gamma\bar{\lambda}\sigma S^{2\beta+1}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}-(\phi-\gamma)\sigma^2\alpha^{-1}S^{2\beta+2}\frac{g_s}{g}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}}{\phi\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}-\frac{\gamma^2(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}}\right]\frac{\alpha^{-1}\frac{g_s}{g}\gamma}{\left(x-Fe^{-r(T-t)}\right)}\\
&+\frac{1}{2}\left[\frac{\gamma\bar{\lambda}S^{\beta}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}-(\phi-\gamma)\sigma\alpha^{-1}S^{\beta+1}\frac{g_s}{g}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}}{\phi\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}-\frac{\gamma^2(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}}\right]^2\frac{\gamma(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}\\
&+rS\alpha^{-1}\frac{g_s}{g}+\bar{\lambda}\sigma\alpha^{-1}S^{2\beta+1}\frac{g_s}{g}-\frac{1}{2}\frac{\phi}{\gamma}\sigma^2\alpha^{-2}S^{2\beta+2}\frac{g_s^2}{g^2}+\frac{1}{2}\sigma^2S^{2\beta+2}\left[\left(\alpha^{-2}-\alpha^{-1}\right)\frac{g_s^2}{g^2}+\alpha^{-1}\frac{g_{ss}}{g}\right]\\
&+\left[\frac{\gamma\bar{\lambda}\sigma S^{2\beta+1}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}-(\phi-\gamma)\sigma^2\alpha^{-1}S^{2\beta+2}\frac{g_s}{g}\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}}{\phi\frac{\gamma^2}{\left(x-Fe^{-r(T-t)}\right)^2}-\frac{\gamma^2(\gamma-1)}{\left(x-Fe^{-r(T-t)}\right)^2}}\right]\frac{\alpha^{-1}\frac{g_s}{g}\gamma}{\left(x-Fe^{-r(T-t)}\right)}=0\\
\Longrightarrow\quad&-\alpha^{-1}\frac{g_t}{g}+\gamma r+\frac{\gamma\bar{\lambda}^2S^{2\beta}-(\phi-\gamma)\bar{\lambda}\sigma\alpha^{-1}S^{2\beta+1}\frac{g_s}{g}}{\phi-\gamma+1}-\frac{\phi}{2\gamma}\left[\frac{\gamma\bar{\lambda}S^{\beta}-(\phi-\gamma)\sigma\alpha^{-1}S^{\beta+1}\frac{g_s}{g}}{\phi-\gamma+1}\right]^2\\
&-\frac{\phi}{\gamma}\left[\frac{\gamma\bar{\lambda}\sigma\alpha^{-1}S^{2\beta+1}\frac{g_s}{g}-(\phi-\gamma)\sigma^2\alpha^{-2}S^{2\beta+2}\frac{g_s^2}{g^2}}{\phi-\gamma+1}\right]+\frac{\gamma-1}{2\gamma}\left[\frac{\gamma\bar{\lambda}S^{\beta}-(\phi-\gamma)\sigma\alpha^{-1}S^{\beta+1}\frac{g_s}{g}}{\phi-\gamma+1}\right]^2\\
&+\left(rS\alpha^{-1}+\bar{\lambda}\sigma\alpha^{-1}S^{2\beta+1}\right)\frac{g_s}{g}-\frac{1}{2}\left(\frac{\phi}{\gamma}\sigma^2\alpha^{-2}S^{2\beta+2}-\sigma^2\alpha^{-2}S^{2\beta+2}+\sigma^2\alpha^{-1}S^{2\beta+2}\right)\frac{g_s^2}{g^2}\\
&+\frac{1}{2}\sigma^2\alpha^{-1}S^{2\beta+2}\frac{g_{ss}}{g}+\frac{\gamma\bar{\lambda}\sigma\alpha^{-1}S^{2\beta+1}\frac{g_s}{g}-(\phi-\gamma)\sigma^2\alpha^{-2}S^{2\beta+2}\frac{g_s^2}{g^2}}{\phi-\gamma+1}=0\\
\Longrightarrow\quad&-\alpha^{-1}\frac{g_t}{g}+\gamma r+\left(rS\alpha^{-1}+\bar{\lambda}\sigma\alpha^{-1}S^{2\beta+1}\right)\frac{g_s}{g}+\frac{1}{2}\sigma^2\alpha^{-1}S^{2\beta+2}\frac{g_{ss}}{g}\\
&+\left(\gamma^2\bar{\lambda}^2S^{2\beta}+(\gamma-\phi)^2\sigma^2\alpha^{-2}S^{2\beta+2}\frac{g_s^2}{g^2}+2\gamma\bar{\lambda}(\gamma-\phi)\sigma\alpha^{-1}S^{\beta+1}\frac{g_s}{g}\right)\frac{1}{2\gamma(\phi-\gamma+1)}\\
&-\frac{1}{2}\left(\frac{\phi}{\gamma}\sigma^2\alpha^{-2}S^{2\beta+2}-\sigma^2\alpha^{-2}S^{2\beta+2}+\sigma^2\alpha^{-1}S^{2\beta+2}\right)\frac{g_s^2}{g^2}=0\\
\Longrightarrow\quad&-\frac{g_t}{g}+\frac{1}{2}\alpha\sigma^2\alpha^{-1}S^{2\beta+2}\frac{g_{ss}}{g}+\alpha\left(rS\alpha^{-1}+\frac{\bar{\lambda}\sigma\alpha^{-1}S^{2\beta+1}}{\phi-\gamma+1}\right)\frac{g_s}{g}+\alpha\frac{\gamma\left(\bar{\lambda}S^{\beta}\right)^2}{2(\phi-\gamma+1)}+\alpha\gamma r\\
&+\alpha\left[\frac{(\gamma-\phi)\sigma^2\alpha^{-2}S^{2\beta+2}}{2\gamma(\phi-\gamma+1)}-\frac{1}{2}\sigma^2\alpha^{-1}S^{2\beta+2}\right]\frac{g_s^2}{g^2}=0
\end{aligned}
$$

$$\Longrightarrow -g_t + \frac{1}{2}\sigma^2 S^{2\beta+2} g_{ss} + \alpha S\left(r\frac{\gamma(\phi-\gamma+1)}{\gamma-\phi} + \frac{\bar{\lambda}\sigma\gamma S^{2\beta}}{\gamma-\phi}\right)g_s + \frac{\alpha\gamma\left(\bar{\lambda}S^{\beta}\right)^2}{2(\phi-\gamma+1)}g + \alpha\gamma r g = 0, \quad \text{(A11)}$$

where $\phi - \gamma + 1 \neq 0$, and the boundary condition is $g(S,0) = 1$.

Notably, when $\phi = 0$, Equation (A11) is simplified to the following form:

$$-g_t + \frac{1}{2}\sigma^2 S^{2\beta+2} g_{ss} + \alpha S\left(r - \gamma r + \bar{\lambda}\sigma S^{2\beta}\right)g_s + \frac{\alpha(\alpha-1)}{2}\left(\bar{\lambda}S^{\beta}\right)^2 g + r\gamma\alpha g = 0 \quad \text{(A12)}$$

where $1-\gamma \neq 0$. This corresponds to the corrected Cauchy problem (34) in Corollary 3 which was also documented in the reference [14]. The following notations are used from the source to our notation: $a = \sigma$, $\delta = \alpha$, $\alpha = r$, and $c = \frac{\bar{\lambda}}{\sigma}$. Thus, our robustness problem can be reduced to a non-robust scenario within the context of the M-CEV model by forcing $e_t = 0$ or equivalently $\phi = 0$.

In accordance with Theorem 2 from the reference [6], we can establish that function $g(S, T-t)$ satisfies the Cauchy problem defined by Equation (A11) when the terminal function is set as $g(S,0) = 1$. To facilitate a connection to the source, we can define $f(S,t) = g(S, T-t)$. The Cauchy problem (A11) can be rewritten as

$$\begin{cases} \mathcal{L}f(S,t) \equiv f_t + \frac{1}{2}\sigma^2 S^{2\beta+2} f_{ss} + \alpha S\left(r\frac{\gamma(\phi-\gamma+1)}{\gamma-\phi} + \frac{\bar{\lambda}\gamma\sigma S^{2\beta}}{\gamma-\phi}\right)f_s + \frac{\alpha\gamma\left(\bar{\lambda}S^{\beta}\right)^2}{2(\phi-\gamma+1)}f + \alpha\gamma r f = 0, \\ f(S,T) = 1. \end{cases} \quad \text{(A13)}$$

Appendix A.2. Solving Cauchy Problem (A13) for the M-CEV Case with Ambiguity

Appendix A.2.1. Scaling Transformation

In Appendix A.1, we derive the Cauchy problem (A13). Assume that $f(S,t)$ is the solution to Cauchy problem (A13). To simplify our analysis, we introduce scaled space and inverse variables, denoted as z and $\tilde{\tau}$, respectively. At first, let

$$\begin{aligned} & z = \omega S^x \Longrightarrow S = \left(\tfrac{z}{\omega}\right)^{\frac{1}{x}}, \quad \omega = ?, \quad \tilde{\tau} = B(T-t) \Longrightarrow t = T - \tfrac{\tilde{\tau}}{B}, \\ & \tfrac{\partial z}{\partial S} = x\omega S^{x-1}, \quad \tfrac{\partial \tilde{\tau}}{\partial t} = -B, \\ & \tfrac{\partial}{\partial t}f = \tfrac{\partial}{\partial \tilde{\tau}}f\tfrac{\partial \tilde{\tau}}{\partial t} = -B\tfrac{\partial}{\partial \tilde{\tau}}f, \\ & \tfrac{\partial}{\partial S}f = \tfrac{\partial}{\partial z}f\tfrac{\partial z}{\partial S} = x\omega S^{x-1}\tfrac{\partial}{\partial z}f = x\omega\left(\tfrac{z}{\omega}\right)^{1-\frac{1}{x}}\tfrac{\partial}{\partial z}f = x\omega^{\frac{1}{x}}z^{1-\frac{1}{x}}\tfrac{\partial}{\partial z}f, \\ & \tfrac{\partial^2}{\partial S^2}f = x\omega^{\frac{1}{x}}\left(z^{1-\frac{1}{x}}\tfrac{\partial}{\partial z}f\right)_S = x\omega^{\frac{1}{x}}\left[z^{1-\frac{1}{x}}\left(\tfrac{\partial}{\partial z}f\right)_S + \tfrac{\partial}{\partial z}f\left(z^{1-\frac{1}{x}}\right)_S\right] \\ & = x\omega^{\frac{1}{x}}\left[z^{1-\frac{1}{x}}\tfrac{\partial^2}{\partial z^2}f\tfrac{\partial z}{\partial S} + \tfrac{\partial}{\partial z}f\left(1-\tfrac{1}{x}\right)z^{1-\frac{1}{x}-1}\tfrac{\partial z}{\partial S}\right] \\ & = x^2\omega^{\frac{2}{x}}z^{2-\frac{2}{x}}\tfrac{\partial^2}{\partial z^2}f + \left(x^2 - x\right)\omega^{\frac{2}{x}}z^{1-\frac{2}{x}}\tfrac{\partial}{\partial z}f. \end{aligned}$$

According to (A13), the scaling transformed PDE of $f(S,t)$ is given by (only the key steps are shown):

$$\begin{aligned} & \tfrac{\partial}{\partial t}f(S,t) + \tfrac{1}{2}\sigma^2 S^{2\beta+2}\tfrac{\partial^2}{\partial S^2}f(S,t) + \alpha S\left(r\alpha^{-1} + \tfrac{\bar{\lambda}\gamma\sigma}{\gamma-\phi}S^{2\beta}\right)\tfrac{\partial}{\partial S}f(S,t) \\ & + \tfrac{\alpha\gamma\bar{\lambda}^2}{2(\phi-\gamma+1)}S^{2\beta}f(S,t) + \alpha\gamma r f(S,t) = 0 \\ \Longrightarrow \quad & (-B)\tfrac{\partial}{\partial\tilde{\tau}}f\left(\left(\tfrac{z}{\omega}\right)^{\frac{1}{x}}, T-\tfrac{\tilde{\tau}}{B}\right) + \tfrac{\sigma^2}{2}\left(\tfrac{z}{\omega}\right)^{\frac{1}{x}(2\beta+2)}x^2\omega^{\frac{2}{x}}z^{2-\frac{2}{x}}\tfrac{\partial^2}{\partial z^2}f\left(\left(\tfrac{z}{\omega}\right)^{\frac{1}{x}}, T-\tfrac{\tilde{\tau}}{B}\right) \\ & + \tfrac{\sigma^2}{2}\left(\tfrac{z}{\omega}\right)^{\frac{1}{x}(2\beta+2)}\left(x^2-x\right)\omega^{\frac{2}{x}}z^{1-\frac{2}{x}}\tfrac{\partial}{\partial z}f\left(\left(\tfrac{z}{\omega}\right)^{\frac{1}{x}}, T-\tfrac{\tilde{\tau}}{B}\right) \\ & + \left[\alpha\left(\tfrac{z}{\omega}\right)^{\frac{1}{x}}r\alpha^{-1} + \alpha\left(\tfrac{z}{\omega}\right)^{\frac{1}{x}}\tfrac{\bar{\lambda}\gamma\sigma}{\gamma-\phi}\left(\tfrac{z}{\omega}\right)^{\frac{2\beta}{x}}\right]x\omega^{\frac{1}{x}}z^{1-\frac{1}{x}}\tfrac{\partial}{\partial z}f\left(\left(\tfrac{z}{\omega}\right)^{\frac{1}{x}}, T-\tfrac{\tilde{\tau}}{B}\right) \\ & + \tfrac{\alpha\gamma\bar{\lambda}^2}{2(\phi-\gamma+1)}\left(\tfrac{z}{\omega}\right)^{\frac{2\beta}{x}}f\left(\left(\tfrac{z}{\omega}\right)^{\frac{1}{x}}, T-\tfrac{\tilde{\tau}}{B}\right) + \alpha\gamma r f\left(\left(\tfrac{z}{\omega}\right)^{\frac{1}{x}}, T-\tfrac{\tilde{\tau}}{B}\right) = 0 \end{aligned}$$

$$\Longrightarrow B\frac{\partial}{\partial\tilde{\tau}}f\left(\left(\frac{z}{\omega}\right)^{\frac{1}{x}},T-\frac{\tilde{\tau}}{B}\right)-\frac{\sigma^2}{2}x^2\omega^{\frac{2}{x}-\frac{1}{x}(2\beta+2)}z^{\frac{1}{x}(2\beta+2)+2-\frac{2}{x}}\frac{\partial^2}{\partial z^2}f\left(\left(\frac{z}{\omega}\right)^{\frac{1}{x}},T-\frac{\tilde{\tau}}{B}\right)$$
$$-\frac{\sigma^2}{2}\left(x^2-x\right)\omega^{\frac{2}{x}-\frac{1}{x}(2\beta+2)}z^{\frac{1}{x}(2\beta+2)+1-\frac{2}{x}}\frac{\partial}{\partial z}f\left(\left(\frac{z}{\omega}\right)^{\frac{1}{x}},T-\frac{\tilde{\tau}}{B}\right)$$
$$-\left(rxz+x\omega^{-\frac{2\beta}{x}}z^{1+\frac{2\beta}{x}}\frac{\alpha\bar{\lambda}\gamma\sigma}{\gamma-\phi}\right)\frac{\partial}{\partial z}f\left(\left(\frac{z}{\omega}\right)^{\frac{1}{x}},T-\frac{\tilde{\tau}}{B}\right)$$
$$-\frac{\alpha\gamma\bar{\lambda}^2}{2(\phi-\gamma+1)}\omega^{-\frac{2\beta}{x}}z^{\frac{2\beta}{x}}f\left(\left(\frac{z}{\omega}\right)^{\frac{1}{x}},T-\frac{\tilde{\tau}}{B}\right)-\alpha\gamma rf\left(\left(\frac{z}{\omega}\right)^{\frac{1}{x}},T-\frac{\tilde{\tau}}{B}\right)=0. \quad \text{(A14)}$$

Denote

$$F(z,\tilde{\tau}):=f\left(\left(\frac{z}{\omega}\right)^{\frac{1}{x}},T-\frac{\tilde{\tau}}{B}\right),$$

and then Equation (A14) can be rewritten as

$$\frac{\partial}{\partial\tilde{\tau}}F(z,\tilde{\tau})-\underbrace{\frac{\sigma^2}{2B}x^2\omega^{-\frac{2\beta}{x}}z^{\frac{2\beta}{x}+2}}_{term\,2}\frac{\partial^2}{\partial z^2}F(z,\tilde{\tau})$$
$$-\underbrace{\left[\frac{\sigma^2}{2B}\left(x^2-x\right)\omega^{-\frac{2\beta}{x}}z^{\frac{2\beta}{x}+1}+\frac{1}{B}\left(rxz+x\omega^{-\frac{2\beta}{x}}z^{1+\frac{2\beta}{x}}\frac{\alpha\bar{\lambda}\gamma\sigma}{\gamma-\phi}\right)\right]}_{term\,3}\frac{\partial}{\partial z}F(z,\tilde{\tau})$$
$$-\underbrace{\left[\frac{\alpha\gamma\bar{\lambda}^2}{2B(\phi-\gamma+1)}\omega^{-\frac{2\beta}{x}}z^{\frac{2\beta}{x}}+\frac{\alpha\gamma r}{B}\right]}_{term\,4}F(z,\tilde{\tau})=0. \quad \text{(A15)}$$

To make Equation (A15) simplest, we assume

$$x=-2\beta$$

so that $z^{\frac{2\beta}{x}+1}=z^{-1+1}=z^0=1$ in *term* 3. Thus, the spatial variables can be scaled as

$$z=\omega S^{-2\beta}=\frac{\omega}{S^{2\beta}},$$

where ω is an unknown parameter. We will determine it later.

As a result, Equation (A15) can be rewritten as

$$\frac{\partial}{\partial\tilde{\tau}}F(z,\tilde{\tau})-\underbrace{\frac{2\sigma^2\beta^2\omega}{B}z}_{term\,2}\frac{\partial^2}{\partial z^2}F(z,\tilde{\tau})-\underbrace{\left[\frac{\sigma^2\left(4\beta^2+2\beta\right)}{2B}\omega-\frac{2\beta rz}{B}-\frac{2\beta\omega\alpha\bar{\lambda}\gamma\sigma}{B(\gamma-\phi)}\right]}_{term\,3}\frac{\partial}{\partial z}F(z,\tilde{\tau})$$
$$-\underbrace{\left[\frac{\alpha\gamma\bar{\lambda}^2}{2B(\phi-\gamma+1)}\omega z^{-1}+\frac{\alpha\gamma r}{B}\right]}_{term\,4}F(z,\tilde{\tau})=0. \quad \text{(A16)}$$

To simplify Equation (A16), we assume $B=\sigma^2\beta^2\omega$. That is, we can scale the inverse time variable as follows:

$$\tilde{\tau}=\sigma^2\beta^2\omega(T-t).$$

Therefore, the PDE for (A16) can be expressed as

$$\frac{\partial}{\partial\tilde{\tau}}F(z,\tilde{\tau}) - 2z\frac{\partial^2}{\partial z^2}F(z,\tilde{\tau}) - \left[\frac{2\beta+1}{\beta} - \frac{2r}{\sigma^2\beta\omega}z - \frac{2\alpha\bar{\lambda}\gamma}{\sigma\beta(\gamma-\phi)}\right]\frac{\partial}{\partial z}F(z,\tilde{\tau})$$
$$- \left[\frac{\alpha\gamma\bar{\lambda}^2}{2\sigma^2\beta^2(\phi-\gamma+1)}z^{-1} + \frac{\alpha\gamma r}{\sigma^2\beta^2\omega}\right]F(z,\tilde{\tau}) = 0. \tag{A17}$$

We determined two scaled variables, z and $\tilde{\tau}$. However, we have not yet specified parameter ω. We require this degree of freedom to simplify the PDEs.

Appendix A.2.2. Finding the Representation of $f(S,t)$ and the PDE of the Function h

In Appendix A.2.1, we discuss how to determine the two scaled variables z and $\tilde{\tau}$. For convenience, we use these two scaled space and inverse time variables, i.e.,

$$z = \frac{\omega}{S^{2\beta}}, \qquad \tilde{\tau} = \sigma^2\beta^2\omega(T-t),\ \omega = ?$$

in the process of solving the Cauchy problem (A13).

As per Equation (A17), if denoted as

$$F(z,\tilde{\tau}) := f(S,t) = f\left(\left(\frac{\omega}{z}\right)^{\frac{1}{2\beta}}, T - \frac{\tilde{\tau}}{a^2\beta^2\omega}\right).$$

we then re-express (A13) as

$$\begin{cases} \frac{\partial}{\partial\tilde{\tau}}F(z,\tilde{\tau}) - 2z\frac{\partial^2}{\partial z^2}F(z,\tilde{\tau}) - \left[\frac{2\beta+1}{\beta} - \frac{2r}{\sigma^2\beta\omega}z - \frac{2\alpha\bar{\lambda}\gamma}{\sigma\beta(\gamma-\phi)}\right]\frac{\partial}{\partial z}F(z,\tilde{\tau}) \\ -\left[\frac{\alpha\gamma\bar{\lambda}^2}{2\sigma^2\beta^2(\phi-\gamma+1)}z^{-1} + \frac{\alpha\gamma r}{\sigma^2\beta^2\omega}\right]F(z,\tilde{\tau}) = 0, \\ F(z,0) = 1. \text{ when } \tilde{\tau} = 0. \end{cases} \tag{A18}$$

To solve (A18) with the initial function $F(z,0) = 1$, we employ the Laplace transformation method to find its solution. The following steps outline this process:

Step I: Assume that the solution $f(S,t)$ for the Cauchy problem is represented as follows:

$$f(S,t) := F(z,\tilde{\tau}) = z^{\delta}\exp\{c + x\tilde{\tau} + yz\}h(z,\tilde{\tau}). \tag{A19}$$

At this point, the parameters δ, c, x and y are currently considered to be unknown, and their expression or values will be determined later.

From (A19) we have

$$\begin{aligned} \frac{\partial}{\partial\tilde{\tau}}F &= xF + z^{\delta}\exp\{c + x\tilde{\tilde{\tau}} + yz\}\frac{\partial}{\partial\tau}h = xF + z^{\delta}\exp\{c + x\tilde{\tau} + yz\}h\frac{1}{h}\frac{\partial}{\partial\tilde{\tau}}h \\ &= \left(x + \frac{h_{\tilde{\tau}}}{h}\right)F, \\ \frac{\partial}{\partial z}F &= z^{\delta}\exp\{c + x\tilde{\tau} + yz\}\frac{\partial}{\partial z}h + yF + \delta z^{\delta-1}\exp\{c + x\tilde{\tau} + yz\}h \\ &= \left(\frac{h_z}{h} + y + \frac{\delta}{z}\right)F, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial z^2}F &= \left(\frac{1}{h}\frac{\partial}{\partial z}h + y + \frac{\delta}{z}\right)^2 F + \left(\frac{\partial}{\partial z}\left(\frac{1}{h}\frac{\partial}{\partial z}h\right) - \frac{\delta}{z^2}\right)F \\ &= \left[y^2 + 2y\frac{\delta}{z} + \frac{(\delta^2-\delta)}{z^2} + \left(2y + 2\frac{\delta}{z}\right)\frac{h_z}{h} + \frac{h_{zz}}{h}\right]F. \end{aligned}$$

Substituting these into (A18) and dividing by F to yield

$$\begin{aligned}&\left(x+\tfrac{h_{\bar{\tau}}}{h}\right)-2z\left[y^2+2y\tfrac{\delta}{z}+\tfrac{(\delta^2-\delta)}{z^2}+\left(2y+2\tfrac{\delta}{z}\right)\tfrac{h_z}{h}+\tfrac{h_{zz}}{h}\right]\\&-\left[\tfrac{2\beta+1}{\beta}-\tfrac{2r}{\sigma^2\beta\omega}z-\tfrac{2\alpha\bar{\lambda}\gamma}{\sigma\beta(\gamma-\phi)}\right]\left(\tfrac{h_z}{h}+y+\tfrac{\delta}{z}\right)-\left[\tfrac{\alpha\gamma\bar{\lambda}^2}{2\sigma^2\beta^2(\phi-\gamma+1)}z^{-1}+\tfrac{\alpha\gamma r}{\sigma^2\beta^2\omega}\right]=0.\end{aligned}$$

Multiplying by h gives:

$$h_{zz}-\frac{1}{2z}\left[\left(\frac{2r}{\sigma^2\beta\omega}-4y\right)z+\frac{2\alpha\bar{\lambda}\gamma}{\sigma\beta(\gamma-\phi)}-\frac{2\beta+1}{\beta}-4\delta\right]h_z \tag{A20}$$

$$+\left[\begin{array}{c}\left(2y^2-\tfrac{2ry}{\sigma^2\beta\omega}\right)\tfrac{1}{2}+\left[2(\delta^2-\delta)+\tfrac{\alpha\gamma\bar{\lambda}^2}{2\sigma^2\beta^2(\phi-\gamma+1)}+\tfrac{(2\beta+1)\delta}{\beta}-\tfrac{2\alpha\bar{\lambda}\gamma\delta}{\sigma\beta(\gamma-\phi)}\right]\tfrac{1}{2z^2}\\+\left(4y\delta-x+\tfrac{\alpha r\gamma}{\sigma^2\beta^2\omega}+y\tfrac{2\beta+1}{\beta}-\tfrac{2r\delta}{\sigma^2\beta\omega}-y\tfrac{2\alpha\bar{\lambda}\gamma}{\sigma\beta(\gamma-\phi)}\right)\tfrac{1}{2z}\end{array}\right]h=\frac{1}{2z}h_{\bar{\tau}}.$$

Step II: The goal is to cancel the h_z term:
In order to eliminate the term of h_z, we set

$$\left(\frac{2r}{\sigma^2\beta\omega}-4y\right)z+\frac{2\alpha\bar{\lambda}\gamma}{\sigma\beta(\gamma-\phi)}-\frac{2\beta+1}{\beta}-4\delta=0.$$

This can be achieved by choosing appropriate values for δ and Q:

$$\begin{aligned}&\frac{2\alpha\bar{\lambda}\gamma}{\sigma\beta(\gamma-\phi)}-\frac{2\beta+1}{\beta}-4\delta=0\\ \Longrightarrow\ &\delta=\frac{2\alpha\bar{\lambda}\gamma}{4\sigma\beta(\gamma-\phi)}-\frac{2\beta+1}{4\beta}=\frac{\alpha\bar{\lambda}\gamma}{2\sigma\beta(\gamma-\phi)}-\frac{1}{2}-\frac{1}{4\beta}\\ &\quad=-\frac{1}{2}-\frac{1}{4\beta}+\frac{\alpha\bar{\lambda}\gamma}{2\sigma\beta(\gamma-\phi)}\\ \Longrightarrow\ &\delta=-\frac{1}{2}-\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\alpha\bar{\lambda}\gamma}{\sigma(\gamma-\phi)}\right),\end{aligned} \tag{A21}$$

and

$$\begin{aligned}&\frac{2r}{\sigma^2\beta\omega}-4y=0\\ \Longrightarrow\ &y=\frac{2r}{4\sigma^2\beta\omega}=\frac{1}{2}\frac{r}{\sigma^2\beta\omega}.\end{aligned}$$

We denote

$$Q:=\frac{r}{\sigma^2\beta\omega}. \tag{A22}$$

Then,

$$y=\frac{1}{2}Q \tag{A23}$$

Remark A1. *When $\phi=0$ (i.e., without ambiguity), the parameters δ and Q in (A21) and (A22) are identical to those presented in the reference [6]. Specifically, referring to [6], these two parameters were defined as follows:*

$$\begin{aligned}\lambda&=-\tfrac{1}{2}-\tfrac{1}{2\beta}\left(\tfrac{1}{2}-\delta c\right),\\ Q&=\tfrac{\delta(\alpha-\gamma r)}{\Lambda a^2\beta}.\end{aligned}$$

It should be noted that the expressions for δ and Q mentioned above correspond to the definitions provided in the reference [6]

We substitute (A21)–(A23), into Equation (A20) to obtain:

$$h_{zz} - \underbrace{\frac{1}{2z}\left[\left(\frac{2r}{\sigma^2\beta\omega} - 2Q\right)z + \frac{2\alpha\bar{\lambda}\gamma}{\sigma\beta(\gamma-\phi)} - \frac{2\beta+1}{\beta} - 4\delta\right]h_z}_{term\,2}$$
$$+\left[\begin{array}{c}\underbrace{\left(\frac{Q^2}{2} - \frac{rQ}{\sigma^2\beta\omega}\right)\frac{1}{2}}_{term\,3} + \underbrace{\left[2(\delta^2-\delta) + \frac{\alpha\gamma\bar{\lambda}^2}{2\sigma^2\beta^2(\phi-\gamma+1)} + \frac{(2\beta+1)\delta}{\beta} - \frac{2\alpha\bar{\lambda}\gamma\delta}{\sigma\beta(\gamma-\phi)}\right]\frac{1}{2z^2}}_{term\,4} \\ + \underbrace{\left(2Q\delta - x + \frac{\alpha r\gamma}{\sigma^2\beta^2\omega} + Q\frac{2\beta+1}{2\beta} - \frac{2r\delta}{\sigma^2\beta\omega} - Q\frac{2\alpha\bar{\lambda}\gamma}{2\sigma\beta(\gamma-\phi)}\right)\frac{1}{2z}}_{term\,5}\end{array}\right]h = \frac{1}{2z}h_{\tilde{\tau}}.$$

Based on the previous steps, it is clear that the coefficient of h_z is zero, allowing us to eliminate *term* 2. Therefore, the above equation can be expressed as

$$h_{zz} + \left[\begin{array}{c}\underbrace{\left(\frac{Q^2}{2} - \frac{rQ}{\sigma^2\beta\omega}\right)\frac{1}{2}}_{term\,3} + \underbrace{\left[2(\delta^2-\delta) + \frac{\alpha\gamma\bar{\lambda}^2}{2\sigma^2\beta^2(\phi-\gamma+1)} + \frac{(2\beta+1)\delta}{\beta} - \frac{2\alpha\bar{\lambda}\gamma\delta}{\sigma\beta(\gamma-\phi)}\right]\frac{1}{2z^2}}_{term\,4} \\ + \underbrace{\left(2Q\delta - x + \frac{\alpha r\gamma}{\sigma^2\beta^2\omega} + Q\frac{2\beta+1}{2\beta} - \frac{2r\delta}{\sigma^2\beta\omega} - Q\frac{2\alpha\bar{\lambda}\gamma}{2\sigma\beta(\gamma-\phi)}\right)\frac{1}{2z}}_{term\,5}\end{array}\right]h = \frac{1}{2z}h_{\tilde{\tau}}. \tag{A24}$$

Step III: Now, we have an unknown parameter ω in *term* 3. It has been observed that we can choose an appropriate parameter ω to transform Equation (A24), which governs the function h, into the well-known Whittaker equation. Hence, we assume that *term* 3 is equal to $-\frac{1}{4}$, i.e.,

$$\begin{aligned}&\left(\frac{Q^2}{2} - \frac{rQ}{\sigma^2\beta\omega}\right)\frac{1}{2} = -\frac{1}{4}\\ \Longrightarrow &\frac{Q^2}{2} - \frac{rQ}{\sigma^2\beta\omega} = -\frac{1}{2}\\ \Longrightarrow &\left(\frac{Q^2}{2} + \frac{1}{2}\right) - \frac{rQ}{\sigma^2\beta\omega} = 0.\end{aligned} \tag{A25}$$

It is known that $Q = \frac{r}{\sigma^2\beta\omega}$ in (A22). Substituting this into Equation (A25) yields

$$\begin{aligned}&\frac{r^2}{2\sigma^4\beta^2\omega^2} + \frac{1}{2} - \frac{r\frac{r}{\sigma^2\beta\omega}}{\sigma^2\beta\omega} = 0\\ \Longrightarrow &\omega = \frac{r}{\sigma^2 \mid \beta \mid},\end{aligned} \tag{A26}$$

where the parameter ω is determined.

Remark A2. *With the correspondence in notation from the source to our notation $\alpha = r$, $\delta = \alpha$ and $\Lambda = \omega$, the formula for the parameter ω in our M-CEV case is the same as that in the reference [6]. In that study, the Λ is given by*

$$\Lambda = \sqrt{\frac{\delta(\alpha^2 - \gamma r^2)}{a^4\beta^2}} = \frac{\sqrt{\delta}}{a^2 \mid \beta \mid}\sqrt{\alpha^2 - \gamma r^2}.$$

Step IV: To determine that the coefficient in *term* 4 is appropriate for Equation (A24), we begin by assuming that the coefficient of z^{-2} in *term* 4 is represented by an unknown parameter, denoted as k. That is

$$k = \frac{1}{2}\left(2(\delta^2 - \delta) + \frac{\alpha\gamma\bar{\lambda}^2}{2\sigma^2\beta^2(\phi - \gamma + 1)} + \frac{(2\beta+1)\delta}{\beta} - \frac{2\alpha\bar{\lambda}\gamma\delta}{\sigma\beta(\gamma - \phi)}\right) \tag{A27}$$

$$= \delta^2 - \delta + \frac{\alpha\gamma\bar{\lambda}^2}{4\sigma^2\beta^2(\phi - \gamma + 1)} + \frac{(2\beta+1)\delta}{2\beta} - \frac{2\alpha\bar{\lambda}\gamma\delta}{2\sigma\beta(\gamma - \phi)}. \tag{A28}$$

Due to $\delta = \frac{\alpha\bar{\lambda}\gamma}{2\sigma\beta(\gamma-\phi)} - \frac{2\beta+1}{4\beta}$, we have from Equation (A28)

$$\begin{aligned} k &= \delta^2 - \delta - \left[2\frac{\alpha\bar{\lambda}\gamma\delta}{2\sigma\beta(\gamma - \phi)} - 2\frac{(2\beta+1)}{4\beta}\delta\right] + \frac{\alpha\gamma\bar{\lambda}^2}{4\sigma^2\beta^2(\phi - \gamma + 1)} \\ &= -\delta^2 - \delta + \frac{\alpha\gamma\bar{\lambda}^2}{4\sigma^2\beta^2(\phi - \gamma + 1)} \\ \Longrightarrow \frac{1}{4} - k &= \left(\delta + \frac{1}{2}\right)^2 + \frac{\alpha\gamma\bar{\lambda}^2}{4\sigma^2\beta^2(\gamma - \phi - 1)}. \end{aligned} \tag{A29}$$

To obtain the form of the Whittaker equation, we introduce a new parameter η defined as

$$\begin{aligned} \eta^2 &= \left(\delta + \frac{1}{2}\right)^2 + \frac{\alpha\gamma\bar{\lambda}^2}{4\sigma^2\beta^2(\gamma - \phi - 1)} \\ \Longrightarrow \eta &= \sqrt{\left(\delta + \frac{1}{2}\right)^2 + \frac{\alpha\gamma\bar{\lambda}^2}{4\sigma^2\beta^2(\gamma - \phi - 1)}}. \end{aligned} \tag{A30}$$

Accordingly, we rewrite Equation (A29) as

$$\begin{aligned} & \tfrac{1}{4} - k = \eta^2 \\ \Longrightarrow \quad & k = \tfrac{1}{4} - \eta^2, \end{aligned}$$

which implies that Equation (A27) can be expressed as

$$\frac{1}{2}\left(2(\delta^2 - \delta) + \frac{\alpha\gamma\bar{\lambda}^2}{2\sigma^2\beta^2(\phi - \gamma + 1)} + \frac{(2\beta+1)\delta}{\beta} - \frac{2\alpha\bar{\lambda}\gamma\delta}{\sigma\beta(\gamma - \phi)}\right) = \frac{1}{4} - \eta^2.$$

Remark A3. *When $\phi = 0$, the parameter η (see Formula (A30)) is different from the reference [6]:*

$$\eta = \sqrt{\left(\lambda + \frac{1}{2}\right)^2 + \frac{\delta(1-\delta)c^2}{4a^4\beta^2}}.$$

This is because of the difference in the Cauchy problem.

Step V: We aim to make the coefficient of z^{-1} zero in *term* 5 by selecting an appropriate parameter, x in Equation (A24). To determine the appropriate parameter x, we initially assume that the coefficient of *term* 5 is equal to zero, which can be expressed as

$$\begin{aligned} & \left[2Q\delta - x + \frac{\alpha r\gamma}{\sigma^2\beta^2\omega} + Q\frac{2\beta+1}{2\beta} - \frac{2r\delta}{\sigma^2\beta\omega} - Q\frac{2\alpha\bar{\lambda}\gamma}{2\sigma\beta(\gamma - \phi)}\right]\frac{1}{2} = 0 \\ \Longrightarrow & x = 2Q\delta + \frac{\alpha\gamma}{\sigma^2\beta^2\omega}r - 2Q\frac{\alpha\bar{\lambda}\gamma}{2\sigma\beta(\gamma - \phi)} + Q\frac{2\beta+1}{2\beta} - 2\frac{r\delta}{\sigma^2\beta\omega}. \end{aligned} \tag{A31}$$

From (A21) and (A22), we know that $\delta = \frac{\alpha\bar{\lambda}\gamma}{2\sigma\beta(\gamma-\phi)} - \frac{2\beta+1}{4\beta}$ and $Q = \frac{r}{\sigma^2\beta\omega}$, then Equation (A31) can be written as

$$\begin{aligned} x &= 2Q\delta + \tfrac{\alpha\gamma}{\sigma^2\beta^2\omega}r - 2Q\tfrac{\alpha\bar{\lambda}\gamma}{2\sigma\beta(\gamma-\phi)} + 2Q\tfrac{2\beta+1}{4\beta} - 2\tfrac{r}{\sigma^2\beta\omega}\delta \\ &= 2Q\delta + \tfrac{\alpha\gamma}{\sigma^2\beta^2\omega}r - 2Q\delta - 2Q\delta \\ &= \tfrac{r\alpha\gamma}{\sigma^2\beta^2\omega} - 2Q\delta. \end{aligned}$$

Denote

$$R^* := -2Q\delta. \tag{A32}$$

This means

$$x = \frac{r\alpha\gamma}{\sigma^2\beta^2\omega} + R^*.$$

Remark A4. *When $\alpha = r$, the R^* for (A32) is not the same as the reference [6]. We obtained*

$$R^* = -2Q\lambda - \frac{\delta(1-\delta)(\alpha-r)c}{\Lambda a^2\beta^2}.$$

As opposed to

$$R = -2Q\lambda - \frac{\delta(1-\delta)(\alpha-r)c}{\Lambda a^4\beta^2}.$$

Next, we determine the unknown parameter c:

Let $d := c + x\tilde{\tau}$. Because $\tilde{\tau} = \sigma^2\beta^2\omega(T-t)$ and the assumed representation of the solution $F(z,\tilde{\tau})$ is given by

$$\begin{aligned} F(z,\tilde{\tau}) &= z^\delta \exp\{c + x\tilde{\tau} + yz\}h(z,\tilde{\tau}) \\ &= z^\delta \exp\{d + yz\}h(z,\tilde{\tau}), \end{aligned}$$

then

$$\begin{aligned} d &= c + x\tilde{\tau} = c + \left(\tfrac{r\alpha\gamma}{\sigma^2\beta^2\omega} + R^*\right)\tilde{\tau} \\ &= c + r\alpha\gamma T - r\alpha\gamma t + R^*\tilde{\tau}. \end{aligned}$$

As a result, the educated guess of c can be given by

$$c = -r\alpha\gamma T,$$

which implies

$$d = -r\alpha\gamma t + R^*\tilde{\tau}.$$

Consequently, we can represent its solution $f(S,t)$ for the Cauchy problem as

$$f(S,t) = z^\delta \exp\left\{-r\alpha\gamma t + R^*\tilde{\tau} + \frac{Q}{2}z\right\}h(z,\tilde{\tau}), \tag{A33}$$

where $z = \omega S^{-2\beta}$ and $\tilde{\tau} = \sigma^2\beta^2\omega(T-t)$ with parameters

$$\begin{gathered} \omega = \tfrac{r}{\sigma^2|\beta|}, \quad \delta = -\tfrac{1}{2} - \tfrac{1}{2\beta}\left(\tfrac{1}{2} - \tfrac{\alpha\bar{\lambda}\gamma}{\sigma(\gamma-\phi)}\right), \\ \eta = \sqrt{\left(\delta + \tfrac{1}{2}\right)^2 + \tfrac{\alpha\gamma\bar{\lambda}^2}{4\sigma^2\beta^2(\gamma-\phi-1)}}, \\ Q = \tfrac{r}{\sigma^2\beta\omega}, \quad R^* = -2Q\delta. \end{gathered}$$

After ensuring the five terms in (A24), we express the PDE for (A24) as follows:

$$h_{zz} + \left[-\frac{1}{4} + \frac{\left(\frac{1}{4} - \eta^2\right)}{z^2} \right] h = \frac{1}{2z} h_{\tilde{\tau}} \tag{A34}$$

with initial condition

$$h(z,0) = \frac{F(z,0)}{z^{\delta} \exp\left\{\frac{Q}{2}(z)\right\}} = z^{-\delta} \exp\left\{ r\alpha\gamma T - \frac{Q}{2} z \right\}, \tag{A35}$$

which indicates that Equation (A34) is a Whittaker equation. Solving the Whittaker equation is provided below.

Let us denote the corresponding operator as

$$\mathcal{L}_h h(z,\tilde{\tau}) = h_{zz} + \left[-\frac{1}{4} + \left(\frac{1}{4} - \eta^2\right) \frac{1}{z^2} \right] h - \frac{1}{2z} h_{\tilde{\tau}}.$$

Appendix A.2.3. Applying the Laplace Transform $G(x;\zeta)$ to Find the Solution $h(z,\tilde{\tau})$

In Appendix A.2.2, we established the representation (A33) of the function $f(S,t)$ as follows:

$$f(S,t) = z^{\delta} \exp\left\{ -r\gamma\delta t + R^* \tilde{\tau} + \frac{Q}{2} z \right\} h(z,\tilde{\tau}),$$

where $h(z,\tilde{\tau})$ is the solution to the Cauchy problem in Equation (A34). In the following steps, our objective is to determine the solution for $h(z,\tilde{\tau})$:

Step 1: Find the Laplace transform of $h(z,\tilde{\tau})$, and produce its ODE.

According to the definition of the Laplace transform, we denote the Laplace transform of $h(z,\tilde{\tau})$ by $G(z;\zeta)$. That is,

$$G(z;\zeta) := \mathcal{L}_L h(z,\tilde{\tau}) = \int_0^{\infty} e^{-\zeta\tilde{\tau}} h(z,\tau) d\tilde{\tau}$$

with $Re(\zeta) > 0$.

Using the properties of Laplace transform methods, we know that the transform of a derivative w. r. t. z is a just differentiating the transformed function.

$$\begin{aligned} \mathcal{L}_L h_z(z,\tilde{\tau}) &= \int_0^{\infty} \exp\{-\zeta\tau\} h_z(z,\tilde{\tau}) d\tau \\ &= \frac{d}{dz} G(z;\zeta) = G_z(z;\zeta), \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_L h_{zz}(z,\tilde{\tau}) &= \int_0^{\infty} \exp\{-\zeta\tilde{\tau}\} h_{zz}(z,\tilde{\tau}) d\tilde{\tau} \\ &= \frac{d^2}{dz^2} G(z;\zeta) = G_{zz}(z;\zeta). \end{aligned}$$

To transform the derivative in $\tilde{\tau}$, we use the common rules (see the reference [17])

$$\mathcal{L}_L h_{\tilde{\tau}}(z,\tilde{\tau}) = \zeta G(z;\zeta) - h(z,0),$$

where

$$h(z,0) = \frac{F(z,0)}{z^{\delta} \exp\left\{\frac{Q}{2}(z)\right\}} = z^{-\delta} \exp\left\{ r\alpha\gamma T - \frac{Q}{2} z \right\}$$

Thus, we use the Laplace transform in Equation (A34) and produce the following ODE:

$$G_{zz}+\left[-\frac{1}{4}+\frac{\left(\frac{1}{4}-\eta^2\right)}{z^2}\right]G=\frac{1}{2z}\left(\zeta G-z^{-\delta}\exp\left\{r\alpha\gamma T-\frac{Q}{2}z\right\}\right)$$
$$\Longrightarrow G_{zz}+\left[-\frac{1}{4}-\frac{\frac{\zeta}{2}}{z}+\frac{\left(\frac{1}{4}-\eta^2\right)}{z^2}\right]G=-\chi(z) \quad \text{(A36)}$$

where denoting

$$\chi(z):=\frac{1}{2}z^{-1-\delta}\exp\left\{r\alpha\gamma T-\frac{Q}{2}z\right\}. \quad \text{(A37)}$$

Step 2: Solve the ODE for $G(z;\zeta)$.

In ODE (A36), the homogeneous equation for G is known as the Whittaker equation:

$$G_{zz}+\left[-\frac{1}{4}+\frac{-\zeta/2}{z}+\frac{\left(\frac{1}{4}-\eta^2\right)}{z^2}\right]G=0,$$

with two linearly independent solutions, namely the Whittaker functions $M_{-\frac{\zeta}{2},\eta}(z)$ and $W_{-\frac{\zeta}{2},\eta}(z)$ (see the reference [18]) .

Hence, the general solution of the homogeneous equation is given by

$$G_0(z;\zeta)=C_1M_{-\frac{\zeta}{2},\eta}(z)+C_2W_{-\frac{\zeta}{2},\eta}(z).$$

Let us return to the non-homogeneous Equation (A36). We seek its solution in the form:

$$G(z;\zeta)=C_1(z)M_{-\frac{\zeta}{2},\eta}(z)+C_2(z)W_{-\frac{\zeta}{2},\eta}(z). \quad \text{(A38)}$$

The functions $C_1(z)$ and $C_2(z)$ can be determined from the following system of equations:

$$\begin{cases}C_1'(z)M_{-\frac{\zeta}{2},\eta}(z)+C_2'(z)W_{-\frac{\zeta}{2},\eta}(z)=0,\\ C_1'(z)M'_{-\frac{\zeta}{2},\eta}(z)+C_2'(z)W'_{-\frac{\zeta}{2},\eta}(z)=-\chi(z).\end{cases} \quad \text{(A39)}$$

We express the derivative $C_1'(z)$ from the first equation

$$C_1'(z)=-C_2'(z)\frac{W_{-\frac{\zeta}{2},\eta}(z)}{M_{-\frac{\zeta}{2},\eta}(z)}.$$

Substituting back into the second equation, we find the derivative $C_2'(z)$:

$$-C_2'(z)\frac{W_{-\frac{\zeta}{2},\eta}(z)}{M_{-\frac{\zeta}{2},\eta}(z)}M'_{-\frac{\zeta}{2},\eta}(z)+C_2'(z)W'_{-\frac{\zeta}{2},\eta}(z)=-\chi(z)$$
$$\Longrightarrow C_2'(z)=-\frac{\chi(z)M_{-\frac{\zeta}{2},\eta}(z)}{M_{-\frac{\zeta}{2},\eta}(z)W'_{-\frac{\zeta}{2},\eta}(z)-W_{-\frac{\zeta}{2},\eta}(z)M'_{-\frac{\zeta}{2},\eta}(z)}. \quad \text{(A40)}$$

It can be seen that the denominators $M_{-\frac{\zeta}{2},\eta}(z)W'_{-\frac{\zeta}{2},\eta}(z) - W_{-\frac{\zeta}{2},\eta}(z)M'_{-\frac{\zeta}{2},\eta}(z)$ in (A40) are the Wronskians for these two Whittaker functions (see the reference [19]), which is given by

$$\begin{aligned} \mathcal{W}\left\{M_{-\frac{\zeta}{2},\eta}(z), W_{-\frac{\zeta}{2},\eta}(z)\right\} &= M_{-\frac{\zeta}{2},\eta}(z)W'_{-\frac{\zeta}{2},\eta}(z) - W_{-\frac{\zeta}{2},\eta}(z)M'_{-\frac{\zeta}{2},\eta}(z) \\ &= -\frac{\Gamma(1+2\eta)}{\Gamma(\frac{1}{2}+\eta+\frac{\zeta}{2})} \neq 0. \end{aligned}$$

Denote

$$\begin{aligned} \Xi\left(-\tfrac{\zeta}{2},\eta\right) &:= -\frac{1}{\mathcal{W}\left\{M_{-\frac{\zeta}{2},\eta}(z), W_{-\frac{\zeta}{2},\eta}(z)\right\}} \\ &= \frac{\Gamma(\frac{1}{2}+\eta+\frac{\zeta}{2})}{\Gamma(1+2\eta)}, \end{aligned}$$

and then it follows that

$$M_{-\frac{\zeta}{2},\eta}(z)W'_{-\frac{\zeta}{2},\eta}(z) - W_{-\frac{\zeta}{2},\eta}(z)M'_{-\frac{\zeta}{2},\eta}(z) = -\frac{1}{\Xi\left(-\frac{\zeta}{2},\eta\right)}.$$

Thus,

$$\begin{aligned} C_1(z) &= \int_z^\infty \frac{\chi(\psi)W_{-\frac{\zeta}{2},\eta}(\psi)}{\mathcal{W}\left\{M_{-\frac{\zeta}{2},\eta}(\psi), W_{-\frac{\zeta}{2},\eta}(\psi)\right\}}\mathrm{d}\psi = -\Xi\left(-\tfrac{\zeta}{2},\eta\right)\int_z^\infty \chi(\psi)W_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi, \\ &= \Xi\left(-\tfrac{\zeta}{2},\eta\right)\int_0^z \chi(\psi)W_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi \\ C_2(z) &= -\int_z^\infty \frac{\chi(\psi)M_{-\frac{\zeta}{2},\eta}(\psi)}{\mathcal{W}\left\{M_{-\frac{\zeta}{2},\eta}(\psi), W_{-\frac{\zeta}{2},\eta}(\psi)\right\}}\mathrm{d}\psi = \Xi\left(-\tfrac{\zeta}{2},\eta\right)\int_z^\infty \chi(\psi)M_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi. \end{aligned}$$

From (A38) the solution $G(z;\zeta)$ can be obtained as

$$\begin{aligned} G(z;\zeta) &= C_1(z)M_{-\frac{\zeta}{2},\eta}(z) + C_2(z)W_{-\frac{\zeta}{2},\eta}(z) \\ &= M_{-\frac{\zeta}{2},\eta}(z)\Xi\left(-\frac{\zeta}{2},\eta\right)\int_0^z \chi(\psi)W_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi + W_{-\frac{\zeta}{2},\eta}(z)\Xi\left(-\frac{\zeta}{2},\eta\right)\int_z^\infty \chi(\psi)M_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi \\ &= \Xi\left(-\frac{\zeta}{2},\eta\right)\left(M_{-\frac{\zeta}{2},\eta}(z)\int_0^z \chi(\psi)W_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi + W_{-\frac{\zeta}{2},\eta}(z)\int_z^\infty \chi(\psi)M_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi\right). \end{aligned} \tag{A41}$$

See the references [20,21] (such as [20]: 6.669.4], the following relationship exists between Whittaker functions and modified Bessel functions:

$$\begin{aligned} \int_0^\infty e^{-\frac{1}{2}(a_1+a_2)t\cosh x}\left[\coth\left(\tfrac{x}{2}\right)\right]^{2\nu} I_{2\mu}\left(t\sqrt{a_1a_2}\sinh x\right)\mathrm{d}x &= \frac{\Gamma(\frac{1}{2}+\mu-\nu)}{t\sqrt{a_1a_2}\Gamma(1+2\mu)}W_{\nu,\mu}(a_1t)M_{\nu,\mu}(a_2t), \\ &\left[Re\left(\tfrac{1}{2}+\mu-\nu\right) > 0,\ Re(\mu) > 0, a_1 > a_2\right]. \end{aligned}$$

Hence, when $\nu = -\frac{\zeta}{2}$, $a_1t = \psi$ and $a_2t = z$, we have

$$\begin{aligned} &\frac{\Gamma(\frac{1}{2}+\eta+\frac{\zeta}{2})}{\sqrt{\psi z}\Gamma(1+2\eta)}W_{-\frac{\zeta}{2},\eta}(\psi)M_{-\frac{\zeta}{2},\eta}(z) = \int_0^\infty e^{-\frac{1}{2}(\psi+z)\cosh\Psi}\left[\coth\left(\frac{\Psi}{2}\right)\right]^{2(-\frac{\zeta}{2})} I_{2\eta}\left(\sqrt{\psi z}\sinh\Psi\right)\mathrm{d}\Psi \\ \Longrightarrow\ &\Xi\left(-\tfrac{\zeta}{2},\eta\right)W_{-\frac{\zeta}{2},\eta}(\psi)M_{-\frac{\zeta}{2},\eta}(z) = \sqrt{\psi z}\int_0^\infty e^{-\frac{\psi+z}{2}\cosh\Psi}\left[\tanh^\zeta\left(\frac{\Psi}{2}\right)\right] I_{2\eta}\left(\sqrt{\psi z}\sinh\Psi\right)\mathrm{d}\Psi, \\ &\left[Re\left(\tfrac{1}{2}+\eta+\tfrac{\zeta}{2}\right)\right) > 0,\ Re(\eta) > 0, \psi > z\right]. \end{aligned}$$

It follows that

$$
\begin{aligned}
G(z;\zeta) &= \Xi\left(-\frac{\zeta}{2},\eta\right)\left(M_{-\frac{\zeta}{2},\eta}(z)\int_0^z \chi(\psi)W_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi + W_{-\frac{\zeta}{2},\eta}(z)\int_z^\infty \chi(\psi)M_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi\right)\\
&= \Xi\left(-\frac{\zeta}{2},\eta\right)\left(M_{-\frac{\zeta}{2},\eta}(z)\int_0^z \frac{1}{2}\psi^{-1-\delta}e^{r\alpha\gamma T-\frac{Q}{2}\psi}W_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi + W_{-\frac{\zeta}{2},\eta}(z)\int_z^\infty \frac{1}{2}\psi^{-1-\delta}e^{r\alpha\gamma T-\frac{Q}{2}z}M_{-\frac{\zeta}{2},\eta}(\psi)\mathrm{d}\psi\right)\\
&= \frac{\sqrt{z}}{2}\left(\int_0^\infty\int_0^\infty \sqrt{\psi}\psi^{-1-\delta}e^{r\alpha\gamma T-\frac{Q}{2}\psi}e^{-\frac{\psi+z}{2}\cosh\Psi}\left[\tanh^\zeta\left(\frac{\Psi}{2}\right)\right]I_{2\eta}(\sqrt{\psi z}\sinh\Psi)\mathrm{d}\psi\mathrm{d}\Psi\right.\\
&= \frac{\sqrt{z}}{2}e^{r\alpha\gamma T}\left(\int_0^\infty\int_0^\infty e^{-\frac{z\cosh\Psi}{2}}\psi^{-\frac{1}{2}-\delta}e^{-\frac{Q+\cosh\Psi}{2}\psi}I_{2\eta}(\sqrt{\psi z}\sinh\Psi)\tanh^\zeta\left(\frac{\Psi}{2}\right)\mathrm{d}\psi\mathrm{d}\Psi.\right.
\end{aligned}
\tag{A42}
$$

Again, we know the relationship formula (6.643.2) in the reference [20]

$$
\begin{aligned}
\int_0^\infty x^{\mu-\frac{1}{2}}e^{-\alpha x}I_{2\nu}(2b\sqrt{x})dx \quad &= \frac{\Gamma(\mu+\nu+\frac{1}{2})}{\Gamma(2\nu+1)}b^{-1}e^{\frac{b^2}{2\alpha}}\alpha^{-\mu}M_{-\mu,\nu}\left(\frac{b^2}{\alpha}\right),\\
&\left[Re\left(\mu+\nu+\tfrac{1}{2}\right)>0\right].
\end{aligned}
$$

Based on (A42), we have

$$
\begin{aligned}
\int_0^\infty \psi^{\frac{1}{2}-\delta}e^{-\alpha\psi}I_{2\eta}(2\lambda\sqrt{\psi})d\psi = \Xi(\delta,\eta)\lambda^{-1}e^{\frac{\lambda^2}{2\alpha}}\alpha^\delta M_{\delta,\eta}\left(\frac{\lambda^2}{\alpha}\right),\\
\left[Re\left(\frac{1}{2}-\delta+\eta\right)>0\right].
\end{aligned}
\tag{A43}
$$

with $\alpha = \frac{\cosh\Psi+Q}{2}$ and $\lambda = \frac{\sqrt{z}\sinh\Psi}{2}$.

From (A42) and (A43), the representation for $G(z;\zeta)$ is given by

$$
\begin{aligned}
G(z;\zeta) &= e^{r\alpha\gamma T}\Xi(\delta,\eta)\frac{\sqrt{z}}{2}\left(\int_0^\infty e^{-\frac{z\cosh\Psi}{2}}\lambda^{-1}e^{\frac{\lambda^2}{2\alpha}}\alpha^\delta M_{\delta,\eta}\left(\frac{\lambda^2}{\alpha}\right)\tanh^\zeta\left(\frac{\Psi}{2}\right)\mathrm{d}\Psi\right.\\
&= e^{r\alpha\gamma T}\Xi(\delta,\eta)\left(\int_0^\infty e^{-\frac{z\cosh\Psi}{2}+\frac{z\sinh^2(\Psi)}{4(\cosh\Psi+Q)}}\tanh^\zeta\left(\frac{\Psi}{2}\right)\left(\frac{\cosh\Psi+Q}{2}\right)^\delta M_{\delta,\eta}\left(\frac{z\sinh^2\Psi}{2(\cosh\Psi+Q)}\right)\frac{\mathrm{d}\Psi}{\sinh\Psi}.\right.
\end{aligned}
\tag{A44}
$$

Appendix A.2.4. Finding the Solution $f(S,t)$ for the Cauchy Problem

We achieved the function $G(z;\zeta)$ as shown in (A44) in Appendix A.2.3. For convenience, it can also be written as

$$
\begin{aligned}
&e^{\frac{Qz}{2}}G(z;\zeta)\\
&= e^{r\alpha\gamma T}\Xi(\delta,\eta)\left(\int_0^\infty e^{\frac{Qz}{2}}e^{-\frac{z\cosh\Psi}{2}+2\frac{z\sinh^2(\Psi)}{4(\cosh\Psi+Q)}}e^{-\frac{z\sinh^2(\Psi)}{4(\cosh\Psi+Q)}}\tanh^\zeta\left(\frac{\Psi}{2}\right)\left(\frac{\cosh\Psi+Q}{2}\right)^\delta M_{\delta,\eta}\left(\frac{z\sinh^2\Psi}{2(\cosh\Psi+Q)}\right)\frac{\mathrm{d}\Psi}{\sinh\Psi}.\right.
\end{aligned}
\tag{A45}
$$

Let $Y(\Psi) := \frac{Q}{2} - \frac{\cosh\Psi}{2} + 2\frac{\sinh^2(\Psi)}{4(\cosh\Psi+Q)}$, and we simplify $Y(\Psi)$ to obtain

$$
\begin{aligned}
Y(\Psi) \quad &= \frac{Q}{2} + \frac{\sinh^2(\Psi)}{2(\cosh\Psi+Q)} - \frac{\cosh\Psi}{2}\\
&= \frac{Q^2-1}{2(\cosh\Psi+Q)}.
\end{aligned}
$$

Thus, Equation (A45) can be expressed as

$$
z^\delta e^{\frac{Qz}{2}}G(z;\zeta) = e^{r\alpha\gamma T}\Xi(\delta,\eta)\left(\int_0^\infty e^{zY(\Psi)}Z^\delta(\Psi)\tanh^\zeta\left(\frac{\Psi}{2}\right)(zI(\Psi))^\lambda e^{-\frac{zI(\Psi)}{2}}M_{\lambda,\eta}(zI(\Psi))\frac{\mathrm{d}\Psi}{\sinh\Psi},\right.
\tag{A46}
$$

where

$$I(\Psi) = \frac{\sinh^2 \Psi}{2(\cosh \Psi + Q)}, \quad Y(\Psi) = \frac{Q^2 - 1}{2(\cosh \Psi + Q)}, \quad Z(\Psi) = \frac{(\cosh \Psi + Q)^2}{\sinh^2 \Psi}.$$

Let the integration variable $v = \ln \tanh\left(\frac{\Psi}{2}\right)$. Since

$$\begin{aligned}
\mathrm{d}v &= \frac{1}{\tanh\left(\frac{\Psi}{2}\right)} \frac{\mathrm{d}}{\mathrm{d}\left(\frac{\Psi}{2}\right)} \tanh\left(\frac{\Psi}{2}\right) \frac{\mathrm{d}}{\mathrm{d}\Psi}\left(\frac{\Psi}{2}\right) \\
&= \frac{\cosh \Psi + 1}{\sinh \Psi} \frac{1}{2\cosh^2\left(\frac{\Psi}{2}\right)} \mathrm{d}\Psi, \\
e^{v} &= \tanh\left(\frac{\Psi}{2}\right) = \frac{\sinh \Psi}{\cosh \Psi + 1} = \frac{\cosh \Psi - 1}{\sinh \Psi}, \\
e^{-v} &= \left(\tanh\left(\frac{\Psi}{2}\right)\right)^{-1} = \frac{\cosh \Psi + 1}{\sinh \Psi} = \frac{\sinh \Psi}{\cosh \Psi - 1}, \\
e^{-v} - e^{v} &= \frac{\sinh \Psi}{\cosh \Psi - 1} - \frac{\cosh \Psi - 1}{\sinh \Psi} = \frac{2(\cosh \Psi - 1)}{(\cosh \Psi - 1)\sinh \Psi} = \frac{2}{\sinh \Psi}, \\
e^{v} + e^{-v} &= \frac{\cosh \Psi - 1}{\sinh \Psi} + \frac{\sinh \Psi}{\cosh \Psi - 1} = \frac{2\cosh^2 \Psi - 2\cosh \Psi}{(\cosh \Psi - 1)\sinh \Psi} = \frac{2\cosh \Psi}{\sinh \Psi},
\end{aligned}$$

it follows that

$$\begin{aligned}
\mathrm{d}v &= \frac{\cosh \Psi + 1}{\sinh \Psi} \frac{1}{2\cosh^2\left(\frac{\Psi}{2}\right)} \mathrm{d}\Psi = \frac{\cosh \Psi + 1}{\sinh \Psi} \frac{1}{2\frac{\cosh \Psi + 1}{2}} \mathrm{d}\Psi \\
&= \frac{\cosh \Psi + 1}{\sinh \Psi} \frac{1}{\cosh \Psi + 1} \mathrm{d}\Psi = \frac{\mathrm{d}\Psi}{\sinh \Psi}, \\
\frac{1}{\sinh(-v)} &= -\frac{1}{\sinh(v)} = \frac{2}{e^{-v} - e^{v}} = \frac{2}{\frac{2}{\sinh \Psi}} = \sinh \Psi \\
\Longrightarrow \sinh \Psi &= \frac{1}{\sinh(-v)}, \\
\coth(-v) = -\coth v &= \frac{e^{v} + e^{-v}}{e^{-v} - e^{v}} = \frac{\frac{2\cosh \Psi}{\sinh \Psi}}{\frac{2}{\sinh \Psi}} = \cosh \Psi \\
\Longrightarrow \cosh \Psi &= \coth(-v).
\end{aligned}$$

And yet,

$$\begin{aligned}
I(\Psi) &= \frac{\sinh^2 \Psi}{2(\cosh \Psi + Q)} = \frac{\frac{1}{\sinh^2(-v)}}{2(\coth(-v) + Q)} = \frac{1}{2\sinh^2(-v)(\coth(-v) + Q)}, \\
Y(\Psi) &= \frac{Q^2 - 1}{2(\cosh \Psi + Q)} = \frac{Q^2 - 1}{2(\coth(-v) + Q)}, \\
Z(\Psi) &= \frac{(\cosh \Psi + Q)^2}{\sinh^2 \Psi} = \frac{4(\coth(-v) + Q)(\coth(-v) + Q)}{4\frac{1}{\sinh^2(-v)}} \\
&= \frac{1}{4}\left(\frac{Q^2 - 1}{\frac{1}{2\sinh^2(-v)(\cosh(-v) + Q)} \frac{Q^2 - 1}{2(\cosh(-v) + Q)}}\right) \\
&= \frac{Q^2 - 1}{4A(-v)B(-v)}.
\end{aligned}$$

Then, denote

$$\begin{aligned}
A(-v) &:= \frac{1}{2\sinh^2(-v)(\coth(-v) + Q)}, \\
B(-v) &= \frac{Q^2 - 1}{2(\coth(-v) + Q)}, \\
D(-v) &:= \frac{Q^2 - 1}{4A(-v)B(-v)}.
\end{aligned}$$

Subsequently, introducing the integration variable $v = \ln \tanh\left(\frac{\Psi}{2}\right)$ results in

$$\begin{aligned}
&\frac{\mathrm{d}\Psi}{\sinh \Psi} = \mathrm{d}v, \quad \sinh \Psi = \frac{1}{\sinh(-v)}, \quad \cosh \Psi = \coth(-v), \\
&I(\Psi) = A(-v), \quad Y(\Psi) = B(-v), \quad Z(\Psi) = D(-v),
\end{aligned} \tag{A47}$$

and

$$\tanh^{\zeta}\left(\frac{\Psi}{2}\right) = \frac{(\cosh\Psi - 1)^{\zeta}}{\sinh^{\zeta}(\Psi)} = \frac{(\coth(-v) - 1)^{\zeta}}{\frac{1}{\sinh^{\zeta}(-v)}}$$
$$= [\sinh(-v)(\coth(-v) - 1)]^{\zeta}$$
$$= \left(\frac{e^{-v} + e^{v} - e^{-v} + e^{v}}{2}\right)^{\zeta} = e^{\zeta v}. \tag{A48}$$

By (A47) and (A48) we rewrite Equation (A46) as

$$z^{\delta} e^{\frac{Qz}{2}} G(z;\zeta) = e^{r\alpha\gamma T}\Xi(\delta,\eta)\int_0^{\infty} e^{zY(\Psi)} Z^{\delta}(\Psi)\tanh^{\zeta}\left(\tfrac{\Psi}{2}\right)(zI(\Psi))^{\delta} e^{-\frac{zI(\Psi)}{2}} M_{\delta,\eta}(zI(\Psi))\tfrac{\mathrm{d}\Psi}{\sinh\Psi}$$
$$\Longrightarrow \quad z^{\delta} e^{\frac{Qz}{2}} G(z;\zeta) = e^{r\alpha\gamma T}\Xi(\delta,\eta)\int_{-\infty}^{0} e^{zB(-v) - \frac{zA(-v)}{2} + \zeta v} D^{\delta}(-v)(zA(-v))^{\delta} M_{\delta,\eta}(zA(-v))\mathrm{d}v.$$

Applying the following inverse Laplace transform

$$\begin{aligned} h(z,\tilde{\tau}) &= \mathcal{L}^{-1}\{G(z;\zeta)\}(\tilde{\tau}) = \tfrac{1}{2\pi i}\lim_{T\longrightarrow\infty}\int_{N-iT}^{N+iT} e^{\zeta\tilde{\tau}} G(z;\zeta)\mathrm{d}\zeta \\ &= \tfrac{1}{2\pi i}\int_{N-i\infty}^{N+i\infty} e^{\zeta\tilde{\tau}} G(z;\zeta)\mathrm{d}\zeta \\ &= z^{-\delta} e^{r\alpha\gamma T - \frac{Qz}{2}}\tfrac{\Xi(\delta,\eta)}{2\pi i}\int_{N-i\infty}^{N+i\infty}\int_{-\infty}^{0} e^{zB(-v) - \frac{zA(-v)}{2} + \zeta v + \zeta\tilde{\tau}} D^{\delta}(-v)(zA(-v))^{\delta} M_{\delta,\eta}(zA(-v))\mathrm{d}v\mathrm{d}\zeta \end{aligned}$$

to the function (A33), we obtain

$$\begin{aligned} f(S,t) &= z^{\delta} e^{-r\alpha\gamma t + R^{*}\tilde{\tau} + \frac{Q}{2}z} h(z,\tilde{\tau}) \\ &= \frac{\Xi(\delta,\eta)}{2\pi i} e^{r\alpha\gamma(T-t) + R^{*}\tilde{\tau}}\int_{N-i\infty}^{N+i\infty}\int_{-\infty}^{0} e^{zB(-v) - \frac{zA(-v)}{2} + \zeta(v+\tilde{\tau})}(zD(-v)A(-v))^{\delta} M_{\delta,\eta}(zA(-v))\mathrm{d}v\mathrm{d}\zeta, \end{aligned} \tag{A49}$$

where N is chosen such that all singularities of the integrand expression are to the left of the straight line $(N - i\infty, N + i\infty)$ in the complex plane.

Additionally, as we know, by analytic continuation of the Fourier transform, the Laplace transform of the delta-function satisfies

$$\int_0^{\infty}\delta(t-a)e^{-st}\mathrm{d}t = e^{-sa}.$$

By using the inverse Laplace transform, the Dirac delta function for the M-CEV case with ambiguity is given by

$$\frac{1}{2\pi i}\int_{N-i\infty}^{N+i\infty} e^{z\zeta}\mathrm{d}\zeta = \delta(z),$$

and changing the order of integration in (A49) yields

$$\begin{aligned} f(S,t) &= \frac{\Xi(\delta,\eta)}{2\pi i} e^{r\alpha\gamma(T-t) + R^{*}\tilde{\tau}}\int_{N-i\infty}^{N+i\infty}\int_{-\infty}^{0} e^{zB(-v) - \frac{zA(-v)}{2}} e^{\zeta(v+\tilde{\tau})}(zD(-v)A(-v))^{\delta} M_{\delta,\eta}(zA(-v))\mathrm{d}v\mathrm{d}\zeta \\ &= \Xi(\delta,\eta)e^{r\alpha\gamma(T-t) + R^{*}\tilde{\tau}}\int_{-\infty}^{0}\delta((-v) - \tilde{\tau})e^{zB(-v) - \frac{zA(-v)}{2}}(zD(-v)A(-v))^{\delta} M_{\delta,\eta}(zA(-v))\mathrm{d}(-v). \end{aligned} \tag{A50}$$

Note that $\tilde{\tau} \geq 0$, the integration interval can be extended to the entire line. Using the properties of the delta-function

$$\int_{-\infty}^{\infty}\delta(\zeta - z)g(\zeta)\mathrm{d}\zeta = g(z), \tag{A51}$$

we express (A50) as

$$f(S,t) = \Xi(\delta,\eta)e^{r\alpha\gamma(T-t)+R\tilde{\tau}}e^{zB(\tilde{\tau})-\frac{zA(\tilde{\tau})}{2}}(zD(\tilde{\tau})A(\tilde{\tau}))^{\delta}M_{\delta,\eta}(zA(\tilde{\tau}))$$
$$= e^{r\alpha\gamma(T-t)+R^*\tilde{\tau}+zB(\tilde{\tau})}D^{\delta}(\tilde{\tau})\frac{\Gamma(\eta-\delta+\frac{1}{2})}{\Gamma(1+2\eta)}e^{-\frac{zA(\tilde{\tau})}{2}}(zA(\tilde{\tau}))^{\delta}M_{\delta,\eta}(zA(\tilde{\tau})). \quad \text{(A52)}$$

Due to $T-t=\frac{\tilde{\tau}}{\sigma^2\beta^2\omega}$, the expression (A52) can be written as

$$\begin{aligned} f(S,t) &= e^{\frac{r\alpha\gamma}{\sigma^2\beta^2\omega}\tilde{\tau}+R^*\tilde{\tau}+zB(\tilde{\tau})}D^{\delta}(\tilde{\tau})\frac{\Gamma(\eta-\delta+\frac{1}{2})}{\Gamma(1+2\eta)}e^{-\frac{zA(\tilde{\tau})}{2}}(zA(\tilde{\tau}))^{\delta}M_{\delta,\eta}(zA(\tilde{\tau})) \\ &= e^{\left(\frac{r\alpha\gamma}{\sigma^2\beta^2\omega}+R^*\right)\tilde{\tau}+zB(\tilde{\tau})}D^{\delta}(\tilde{\tau})\frac{\Gamma(\eta-\delta+\frac{1}{2})}{\Gamma(1+2\eta)}e^{-\frac{zA(\tilde{\tau})}{2}}(zA(\tilde{\tau}))^{\delta}M_{\delta,\eta}(zA(\tilde{\tau})). \end{aligned}$$

Denote

$$R := \frac{r\alpha\gamma}{\sigma^2\beta^2\omega} + R^*.$$

Again since (see (A32))

$$R^* = -2Q\delta,$$

it follows that

$$R = \frac{r\alpha\gamma}{\sigma^2\beta^2\omega} - 2Q\delta. \quad \text{(A53)}$$

Therefore, we obtain the solution $f(S,t)$ for Cauchy problem (A13) as follows:

$$f(S,t) = e^{R\tilde{\tau}+zB(\tilde{\tau})}D^{\delta}(\tilde{\tau})\frac{\Gamma(\eta-\delta+\frac{1}{2})}{\Gamma(1+2\eta)}e^{-\frac{zA(\tilde{\tau})}{2}}(zA(\tilde{\tau}))^{\delta}M_{\delta,\eta}(zA(\tilde{\tau})), \quad \text{(A54)}$$

where $z=\omega S^{-2\beta}$, $\tilde{\tau}=\sigma^2\beta^2\omega(T-t)$, $\Gamma(z)$ is the gamma function, $M_{\delta,\eta}(z)$ is the Whittaker function with parameters

$$\delta = -\frac{1}{2} - \frac{1}{2\beta}\left(\frac{1}{2} - \frac{\alpha\bar{\lambda}\gamma}{\sigma(\gamma-\phi)}\right), \qquad \eta = \sqrt{\left(\delta+\frac{1}{2}\right)^2 + \frac{\alpha\gamma\bar{\lambda}^2}{4\sigma^2\beta^2(\gamma-\phi-1)}}.$$

The remaining constants and functions are given by

$$\begin{aligned} \omega &= \frac{r}{\sigma^2|\beta|}, \quad Q = \frac{r}{\sigma^2\beta\omega}, \quad R = \frac{r\alpha\gamma}{\sigma^2\beta^2\omega} - 2Q\delta, \\ A(\tilde{\tau}) &= \frac{1}{2\sinh^2\tilde{\tau}(\coth\tilde{\tau}+Q)}, \quad B(\tilde{\tau}) = \frac{Q^2-1}{2(\coth\tilde{\tau}+Q)}, \quad D(\tilde{\tau}) = \frac{Q^2-1}{4A(\tilde{\tau})B(\tilde{\tau})}. \end{aligned}$$

To compute the optimal exposure, we must determine the ratio $\frac{f_S}{f}$ in the next section.

Appendix A.2.5. Calculating the Ratio $\frac{f_S(S,t)}{f(S,t)}$

As documented in the reference [22], the ratio $\frac{f_s}{f}$ can be determined using the differential rules for Whittaker functions, i.e.,

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}z\right)^n\left(e^{-\frac{z}{2}}z^{k-1}M_{k,\mu}(z)\right) = \frac{\Gamma(\mu+k+n+\frac{1}{2})}{\Gamma(\mu+k+\frac{1}{2})}e^{-\frac{z}{2}}z^{k+n-1}M_{k+n,\mu}(z).$$

When $n=1$, it follows that

$$\begin{aligned} &\left(z\frac{\mathrm{d}}{\mathrm{d}z}z\right)\left(e^{-\frac{z}{2}}z^{k-1}M_{k,\mu}(z)\right) = \frac{\Gamma(\mu+k+\frac{1}{2}+1)}{\Gamma(\mu+k+\frac{1}{2})}e^{-\frac{z}{2}}z^{k+1-1}M_{k+1,\mu}(z) \\ \Longrightarrow\ &\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(e^{-\frac{z}{2}}z^{k-1+1}M_{k,\mu}(z)\right) = \frac{(\mu+k+\frac{1}{2})\Gamma(\mu+k+\frac{1}{2})}{\Gamma(\mu+k+\frac{1}{2})}e^{-\frac{z}{2}}z^{k}M_{k+1,\mu}(z) \\ \Longrightarrow\ &\frac{\mathrm{d}}{\mathrm{d}z}\left(e^{-\frac{z}{2}}z^{k}M_{k,\mu}(z)\right) = (\mu+k+\tfrac{1}{2})e^{-\frac{z}{2}}z^{k-1}M_{k+1,\mu}(z). \end{aligned}$$

In the previous section, we derived the solution $f(S,t)$ that satisfies the following expression (that is, function (A54)):

$$f(S,t) = e^{R\tilde{\tau}+zB(\tilde{\tau})} D^{\delta}(\tilde{\tau}) \frac{\Gamma(\eta-\delta+\frac{1}{2})}{\Gamma(1+2\eta)} e^{-\frac{z}{2}A(\tilde{\tau})} (zA(\tilde{\tau}))^{\delta} M_{\delta,\eta}(zA(\tilde{\tau})),$$

where $\Gamma(z)$ is the Euler gamma function, $M_{\lambda,\eta}(z)$ is the Whittaker M-function, and functions $A(\tilde{\tau})$, $B(\tilde{\tau})$, and $D(\tilde{\tau})$ are obtained in Appendix A.2.4; namely,

$$A(\tilde{\tau}) = \frac{1}{2\sin h^2(\tilde{\tau})\left(\coth(\tilde{\tau})+Q\right)},$$
$$B(\tilde{\tau}) = \frac{Q^2-1}{2\left(\coth(\tilde{\tau})+Q\right)}, \quad D(\tilde{\tau}) = \frac{Q^2-1}{4A(\tilde{\tau})B(\tilde{\tau})}.$$

For convenience in calculations, denote

$$G := D^{\delta}(\tilde{\tau}) \frac{\Gamma(\eta-\lambda+\frac{1}{2})}{\Gamma(1+2\eta)}$$

and thus, we rewrite the function (A54) as

$$f(S,t) = Ge^{R\tilde{\tau}+zB(\tilde{\tau})} e^{-\frac{z}{2}A(\tilde{\tau})} (zA(\tilde{\tau}))^{\delta} M_{\delta,\eta}(zA(\tilde{\tau})). \tag{A55}$$

The derivative of the function $f(S,t)$ with respect to S is given by

$$\begin{aligned} f_s(S,t) &= G\left[\frac{\partial e^{R\tilde{\tau}+zB(\tilde{\tau})}}{\partial S}\left(e^{-\frac{z}{2}A(\tilde{\tau})}(zA(\tilde{\tau}))^{\delta}M_{\delta,\eta}(zA(\tilde{\tau}))\right) + e^{R\tilde{\tau}+zB(\tilde{\tau})}\frac{\partial}{\partial S}\left(e^{-\frac{z}{2}A(\tilde{\tau})}(zA(\tilde{\tau}))^{\delta}M_{\delta,\eta}(zA(\tilde{\tau}))\right)\right] \\ &= Ge^{R\tilde{\tau}+zB(\tilde{\tau})}e^{-\frac{z}{2}A(\tilde{\tau})}(zA(\tilde{\tau}))^{\delta}M_{\delta,\eta}(zA(\tilde{\tau}))\left[B(\tilde{\tau}) + \frac{\left(\eta+\delta+\frac{1}{2}\right)}{z}\frac{M_{\delta+1,\eta}(zA(\tilde{\tau}))}{M_{\delta,\eta}(zA(\tilde{\tau}))}\right]\frac{\mathrm{d}z}{\mathrm{d}S}. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{g_s}{g} &:= \frac{g_s(S,T-t)}{g(S,T-t)} = \frac{f_s(S,t)}{f(S,t)} \\ &= \frac{Ge^{R\tilde{\tau}+zB(\tilde{\tau})}e^{-\frac{z}{2}A(\tilde{\tau})}(zA(\tilde{\tau}))^{\delta}M_{\delta,\eta}(zA(\tilde{\tau}))\left[B(\tilde{\tau}) + \frac{\left(\eta+\delta+\frac{1}{2}\right)}{z}\frac{M_{\delta+1,\eta}(zA(\tilde{\tau}))}{M_{\delta,\eta}(zA(\tilde{\tau}))}\right]\frac{\mathrm{d}z}{\mathrm{d}S}}{Ge^{R\tilde{\tau}+zB(\tau)}e^{-\frac{z}{2}A(\tilde{\tau})}(zA(\tilde{\tau}))^{\delta}M_{\delta,\eta}(zA(\tilde{\tau}))} \\ &= \left[B(\tilde{\tau}) + \frac{\delta+\eta+\frac{1}{2}}{z}\frac{M_{\delta+1,\eta}\left(zA(\tilde{\tau})\right)}{M_{\delta,\eta}\left(zA(\tilde{\tau})\right)}\right]\frac{\mathrm{d}z}{\mathrm{d}S}. \end{aligned} \tag{A56}$$

Appendix A.3. The Optimal Exposure and Worse-Case Measure

Building upon the preceding analysis, we derive the following optimal exposure from (A10):

$$\theta^* = \frac{\bar{\lambda}S^{\beta}}{(\phi-\gamma+1)}\frac{\left(x-Fe^{-r(T-t)}\right)}{x} + \sigma S^{\beta+1}\frac{\left(x-Fe^{-r(T-t)}\right)}{x}\frac{g_s}{g}. \tag{A57}$$

Substituting $\frac{\mathrm{d}z}{\mathrm{d}S} = -2\beta\omega S^{-2\beta-1}$ and (A56) into (A57), we can determine the portfolio's optimal exposure to the fundamental risk factor Z_t as follows:

$$\begin{aligned}\theta^* &= \frac{\bar{\lambda}S^\beta}{(\phi-\gamma+1)}\frac{\left(x-Fe^{-r(T-t)}\right)}{x}+\sigma S^{\beta+1}\frac{\left(x-Fe^{-r(T-t)}\right)}{x}\left[B(\tilde{\tau})+\frac{\delta+\eta+\frac{1}{2}}{z}\frac{M_{\delta+1,\eta}\left(zA(\tilde{\tau})\right)}{M_{\delta,\eta}\left(zA(\tilde{\tau})\right)}\right]\frac{\mathrm{d}z}{\mathrm{d}S}\\ &= \frac{\bar{\lambda}S^\beta}{(\phi-\gamma+1)}\frac{\left(x-Fe^{-r(T-t)}\right)}{x}-2\sigma\omega\beta S^{-\beta}\left[B(\tilde{\tau})+\frac{\delta+\eta+\frac{1}{2}}{z}\frac{M_{\delta+1,\eta}\left(zA(\tilde{\tau})\right)}{M_{\delta,\eta}\left(zA(\tilde{\tau})\right)}\right]\frac{\left(x-Fe^{-r(T-t)}\right)}{x}\\ &= \frac{x-Fe^{-r(T-t)}}{x}\left[\frac{\bar{\lambda}S^\beta}{\phi-\gamma+1}-2\sigma\omega\beta S^{-\beta}B(\tilde{\tau})-2\sigma\beta S^\beta(\delta+\eta+\frac{1}{2})\frac{M_{\delta+1,\eta}\left(\omega A(\tilde{\tau})S^{-2\beta}\right)}{M_{\delta,\eta}\left(\omega A(\tilde{\tau})S^{-2\beta}\right)}\right].\end{aligned} \tag{A58}$$

By Formulas (A4) and (A57), we have

$$\begin{aligned}\epsilon^* &= \frac{\phi}{\gamma J}\left[xS^{-\beta}\left(\frac{\bar{\lambda}S^\beta}{(\phi-\gamma+1)}\frac{\left(x-Fe^{-r(T-t)}\right)}{x}+\sigma S^{\beta+1}\frac{\left(x-Fe^{-r(T-t)}\right)}{x}\frac{g_s}{g}\right)\frac{\gamma}{\left(x-Fe^{-r(T-t)}\right)}J+\sigma S\alpha^{-1}\frac{g_s}{g}J\right]\\ &= \frac{\phi\bar{\lambda}}{\phi-\gamma+1}+\frac{\phi\sigma S}{\gamma-\phi}\frac{g_s}{g}.\end{aligned} \tag{A59}$$

We substitute (A56) into (A59) to obtain

$$\begin{aligned}\epsilon^* &= \frac{\phi\bar{\lambda}}{\phi-\gamma+1}+\frac{\phi\sigma S}{\gamma-\phi}\left[B(\tilde{\tau})+\frac{\delta+\eta+\frac{1}{2}}{z}\frac{M_{\delta+1,\eta}\left(zA(\tilde{\tau})\right)}{M_{\delta,\eta}\left(zA(\tilde{\tau})\right)}\right]\frac{\mathrm{d}z}{\mathrm{d}S}\\ &= \frac{\phi\bar{\lambda}}{\phi-\gamma+1}+\frac{\phi\sigma S}{\gamma-\phi}\left[-2\omega\beta S^{-2\beta-1}B(\tilde{\tau})-\frac{2\beta(\delta+\eta+\frac{1}{2})}{S}\frac{M_{\delta+1,\eta}\left(\omega A(\tilde{\tau})S^{-2\beta}\right)}{M_{\delta,\eta}\left(\omega A(\tilde{\tau})S^{-2\beta}\right)}\right].\end{aligned}$$

As a result, the worse-case measure is given by

$$e^{s_*} = \epsilon^* S^\beta$$

$$= \frac{\phi\bar{\lambda}S^\beta}{\phi-\gamma+1}+\frac{\phi\sigma S^{\beta+1}}{\gamma-\phi}\left[B(\tilde{\tau})+\frac{\delta+\eta+\frac{1}{2}}{z}\frac{M_{\delta+1,\eta}\left(zA(\tilde{\tau})\right)}{M_{\delta,\eta}\left(zA(\tilde{\tau})\right)}\right]\frac{\mathrm{d}z}{\mathrm{d}S} \tag{A60}$$

$$= \frac{\phi\bar{\lambda}S^\beta}{\phi-\gamma+1}+\frac{\phi\sigma S^\beta}{\phi-\gamma}\left[-2\omega\beta S^{-2\beta}B(\tilde{\tau})-2\beta(\delta+\eta+\frac{1}{2})\frac{M_{\delta+1,\eta}\left(\omega A(\tilde{\tau})S^{-2\beta}\right)}{M_{\delta,\eta}\left(\omega A(\tilde{\tau})S^{-2\beta}\right)}\right] \tag{A61}$$

Appendix A.4. Special Case: Zero Interest-Free Rate

By assuming an interest-free rate, $r=0$, we can simplify the solution $f(S,t)$ and the optimal exposure θ^*. In this special case, we have $\omega=\tilde{\tau}=z=0$, and the constants Q and R are both zero. The limit values are

(a)

$$\begin{aligned}\lim_{\omega\longrightarrow 0}D(\tilde{\tau}) &= \lim_{\omega\longrightarrow 0}\frac{Q^2-1}{4A(\tau)B(\tilde{\tau})} = \lim_{\omega\longrightarrow 0}\frac{-1}{4\frac{1}{2\sinh^2\tilde{\tau}(\coth\tilde{\tau})}\frac{-1}{2\left(\coth\tilde{\tilde{\tau}}\right)}}\\ &= \lim_{\omega\longrightarrow 0}\cosh^2\tilde{\tau} = \left(\frac{e^0+1}{2e^0}\right)^2\\ &= \left(\frac{1+1}{2}\right)^2 = 1.\end{aligned} \tag{A62}$$

(b)

$$\begin{aligned}\varphi(S,t) &= \lim_{\omega\longrightarrow 0} zA(\tilde{\tau}) = \lim_{\omega\longrightarrow 0}\frac{z}{2\sinh^2\tilde{\tau}(\coth\tilde{\tau}+Q)} = \lim_{\omega\longrightarrow 0}\frac{\omega S^{-2\beta}}{2\sinh^2\tilde{\tau}\coth\tilde{\tau}}\\ &= \lim_{\omega\longrightarrow 0}\frac{S^{-2\beta}}{2\sigma^2\beta^2(T-t)\cosh^2\tilde{\tau}+2\sigma^2\beta^2(T-t)\sinh^2\tilde{\tau}}\\ &= \tfrac{1}{2}\sigma^{-2}\beta^{-2}S^{-2\beta}(T-t)^{-1},\end{aligned}$$

because of $\sinh\tilde{\tau} = \frac{e^{\tilde{\tau}} - e^{-\tilde{\tau}}}{2}$.

(c)

$$\begin{aligned}
\Omega(S,t) &= \lim_{\omega\longrightarrow 0} zB(\tilde{\tau}) = \lim_{\omega\longrightarrow 0} \tfrac{z(Q^2-1)}{2(\coth\tilde{\tau}+Q)} = \lim_{\omega\longrightarrow 0} \tfrac{\omega S^{-2\beta}(\frac{r^2}{\sigma^4\beta^2\omega^2}-1)}{2(\coth\tilde{\tau}+\frac{r}{\sigma^2\beta\omega})} \\
&= \lim_{\omega\longrightarrow 0} \tfrac{(-2\omega\sigma^4\beta^2 S^{-2\beta})\sinh\tilde{\tau}+(S^{-2\beta}r^2-\omega^2\sigma^4\beta^2 S^{-2\beta})\sigma^2\beta^2(T-t)\cosh\tilde{\tau}}{2\sigma^4\beta^2\cosh\tilde{\tau}+2\sigma^4\beta^2\sigma^2\beta^2(T-t)\omega\sinh\tilde{\tau}+2r\sigma^2\beta\sigma^2\beta^2(T-t)\cosh\tilde{\tau}} \\
&= \lim_{\omega\longrightarrow 0} -\left(\tfrac{2r\sigma^2|\beta|S^{-2\beta}\sinh\tilde{\tau}}{2\sigma^2\cosh\tilde{\tau}+2\sigma^4\beta^2(T-t)\omega\sinh\tilde{\tau}+2r\sigma^2\beta(T-t)\cosh\tilde{\tau}}\right) \\
&= \tfrac{0}{2\sigma^2} = 0.
\end{aligned}$$

(d)

$$\begin{aligned}
\Psi(t) = \lim_{\omega\longrightarrow 0} R\tilde{\tau} &= \lim_{\omega\longrightarrow 0}\left(\frac{r\gamma\alpha}{\sigma^2\beta^2\omega} - 2Q\delta\right)\sigma^2\beta^2\omega(T-t) \\
&= \lim_{\omega\longrightarrow 0}\left[\frac{r\gamma\alpha}{\sigma^2\beta^2\omega}\sigma^2\beta^2\omega(T-t) - 2\frac{r\delta}{\omega\beta\sigma^2}\sigma^2\beta^2\omega(T-t)\right] \\
&= \lim_{\omega\longrightarrow 0}\left[r\gamma\alpha(T-t) - 2r\delta\beta(T-t)\right] \\
&= 0.
\end{aligned} \tag{A63}$$

Thus, the solution $f(S,t)$ is given by

$$\begin{aligned}
f(S,t) &= e^{R\tilde{\tau}+zB(\tilde{\tau})} D^{\delta}(\tilde{\tau}) \tfrac{\Gamma(\eta-\delta+\frac{1}{2})}{\Gamma(1+2\eta)} e^{-\frac{zA(\tilde{\tau})}{2}} \big(zA(\tilde{\tau})\big)^{\delta} M_{\delta,\eta}\big(zA(\tilde{\tau})\big) \\
&= \tfrac{\Gamma\left(\eta-\delta+\frac{1}{2}\right)}{\Gamma(1+2\eta)} e^{\Psi(t)+\Omega(S,t)-\frac{\varphi(S,t)}{2}} \varphi^{\delta}(S,t) M_{\delta,\eta}\big(\varphi(S,t)\big) \\
&= \Xi(\delta,\eta) e^{-\frac{\varphi(S,t)}{2}} \varphi^{\delta}(S,t) M_{\delta,\eta}\big(\varphi(S,t)\big).
\end{aligned}$$

Then,

$$\begin{aligned}
\tfrac{g_s}{g} &= \tfrac{f_s(S,t)}{f(S,t)} \\
&= \frac{\Xi(\delta,\eta)e^{-\frac{\varphi(S,t)}{2}}\varphi^{\delta}(S,t)M_{\delta,\eta}\left(\varphi(S,t)\right)\left[B(\tilde{\tau})+\frac{\left(\eta+\delta+\frac{1}{2}\right)}{z}\frac{M_{\delta+1,\eta}(\varphi(S,t))}{M_{\delta,\eta}(\varphi(S,t))}\right]\frac{\mathrm{d}z}{\mathrm{d}S}}{\Xi(\delta,\eta)e^{-\frac{\varphi(S,t)}{2}}\varphi^{\delta}(S,t)M_{\delta,\eta}(\varphi(S,t))} \\
&= -\tfrac{2\beta\left(\delta+\eta+\frac{1}{2}\right)}{S}\tfrac{M_{\delta+1,\eta}\left(\varphi(S,t)\right)}{M_{\delta,\eta}\left(\varphi(S,t)\right)}
\end{aligned}$$

and thereby, if $r=0$, we have

$$\begin{aligned}
\theta^* &= \frac{\bar{\lambda}S^{\beta}}{(\phi-\gamma+1)}\frac{(x-F)}{x} + \sigma S^{\beta+1}\frac{(x-F)}{x}\frac{g_s}{g} \\
&= \frac{(x-F)S^{\beta}}{x}\left[\frac{\bar{\lambda}}{\phi-\gamma+1} - 2\sigma\beta\big(\delta+\eta+\frac{1}{2}\big)\frac{M_{\delta+1,\eta}\big(\varphi(S,t)\big)}{M_{\delta,\eta}\big(\varphi(S,t)\big)}\right].
\end{aligned} \tag{A64}$$

Meanwhile,

$$\begin{aligned}
\epsilon^* &= \tfrac{\phi\bar{\lambda}}{\phi-\gamma+1} + \tfrac{\phi\sigma S}{\gamma-\phi}\tfrac{g_s}{g} \\
&= \tfrac{\phi\bar{\lambda}}{\phi-\gamma+1} - \tfrac{2\phi\sigma\beta\left(\delta+\eta+\frac{1}{2}\right)}{\gamma-\phi}\tfrac{M_{\delta+1,\eta}\left(\varphi(S,t)\right)}{M_{\delta,\eta}\left(\varphi(S,t)\right)},
\end{aligned}$$

which implies that the worse case is given by

$$\begin{aligned} e^{s_*} &= \epsilon^* S^\beta \\ &= \frac{\phi\bar{\lambda}S^\beta}{\phi-\gamma+1} - \frac{2\phi\sigma\beta S^\beta\left(\delta+\eta+\frac{1}{2}\right)}{\gamma-\phi}\frac{M_{\delta+1,\eta}(\varphi(S,t))}{M_{\delta,\eta}(\varphi(S,t))}. \end{aligned} \tag{A65}$$

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