mathematics

Article

Solutions of Umbral Dirac-Type Equations

Hongfen Yuan [1,*] and Valery Karachik [2]

[1] School of Mathematics and Physics, Hebei University of Engineering, Handan 056038, China
[2] Department of Mathematical Analysis, South Ural State University (NRU), 454080 Chelyabinsk, Russia; karachik@susu.ru
* Correspondence: yuanhongfen@hebeu.edu.cn

Abstract: The aim of this work is to study the method of the normalized systems of functions. The normalized systems of functions with respect to the Dirac operator in the umbral Clifford analysis are constructed. Furthermore, the solutions of umbral Dirac-type equations are investigated by the normalized systems.

Keywords: normalized systems of functions; umbral Dirac equation; umbral Clifford analysis

MSC: 30G35; 58C50; 35Q41

1. Introduction

Umbral calculus originated in the 17th Century and is an important method for studying polynomial sequences. It is a branch of combinatorial analysis [1]. In the mathematical study of umbral calculus, there are three approaches known in the literature. Firstly, umbral calculus can be seen as a type of "magic" for reducing and improving indicators in polynomials [2]. Secondly, the Appell polynomial is extended to the Sheffer polynomial [3]. However, the development of this polynomial is not sufficient because of the lack of computational tools. Thirdly, the abstract linear operators are used to study umbral calculus in functional analysis [4]. Later, these three routes were combined and binomial polynomial sequences studied using operator methods, revealing the mystery of umbral calculus [5]. Umbral calculus is based on modern concepts like linear functionals, linear operators, adjoints, and so on. It is used in fields such as combinatorics, homology algebra, statistics, Fourier analysis, physics, and invariant theory [6].

In [7], the authors introduced umbral calculus into the Clifford analysis. They defined the umbral Dirac operator using radial algebra and umbral calculus. In 2011, Faustino and Ren Guangbin used the operator composition method to study the decomposition theorems of umbral Dirac operators and Hamilton operators [8]. The Clifford analysis (see for instance [9–12]) is based on the study of the properties of monogenic functions, which are the higher dimensional analogue of holomorphic functions on the complex plane. While continuous Clifford analysis is a well-established theory with applications in many fields like electromagnetics and signal processing, discrete Clifford analysis is a theory used in discrete potential theory, numerical analysis, and combinatorics. Umbral Clifford analysis is seen as an abstract theory of Heisenberg exchange relations in quantum mechanics. This provides a framework unifying continuity and discreteness [13–15]. We verified this phenomena by studying normalized systems of functions with respect to the umbral Dirac operator in Clifford analysis and their applications.

The method of the f-normalized system of functions introduced by Karachik is used to construct polynomial solutions to linear partial differential equations with constant coefficients, such as the polyharmonic equation, the Helmholtz equation, the Poisson equation, and so on; see [16,17]. Generally speaking, the construction of polynomial solutions depends on the structure of the equation's operator. But, this method does not

check for updates

Citation: Yuan, H.; Karachik, V. Solutions of Umbral Dirac-Type Equations. *Mathematics* **2024**, *12*, 344. https://doi.org/10.3390/math12020344

Academic Editor: Yu Tian

Received: 20 December 2023
Revised: 11 January 2024
Accepted: 18 January 2024
Published: 20 January 2024

rely on the operator structure of the equation. Furthermore, the proposed method is also used in the study of polynomial solutions of boundary-value problems for polyharmonic equations and the Helmholtz equation, more specifically the Dirichlet problems, Neumann problems, and so on; see [18]. This paper is devoted to the applications of the method of the normalized systems of functions in constructing solutions to partial differential equations in umbral Clifford analysis.

The outline of this paper is as follows. In Section 2, we go over the basics of umbral Clifford analysis, including the umbral Dirac and Euler operators. For early research on umbral Clifford analysis, we refer the reader to [7,8]. In Section 4, applying the Sheffer operator, we obtain the intertwining relationship between umbral differential operators and classical differential operators. Furthermore, we construct 0-normalized systems of functions with respect to the umbral Dirac operator. In Section 5, by the system, we investigate Almansi-type expansions for umbral k-monogenic functions. Furthermore, we construct the solutions of inhomogeneous umbral poly-Dirac equations. In Section 6, we study the normalized system with the base $f(x)$. Moreover, we construct the solutions of umbral Dirac-type equations.

2. Preliminaries

In this section, we will review some basic notions with umbral Clifford analysis; see [7,8].

2.1. Umbral Dirac Operator

One of the interesting things about umbral Clifford analysis is the construction of a first-order operator, the so-called umbral Dirac operator. By taking O_{x_j} to be the partial derivative $\frac{\partial}{\partial x_j}$, we define the umbral Dirac operator by

$$D' := \sum_{j=1}^{n} e_j O_{x_j}, \tag{1}$$

where $e_j e_k + e_k e_j = -2\delta_{jk}, j, k = 1, 2, \cdots, n$. Here, δ_{jk} is the Kronecker symbol. The null solutions of this operator are umbral monogenic functions.

2.2. Umbral Euler Operator

Let $\underline{x} = (x_1, x_2, \ldots x_n) \in R^n$. Then, $\underline{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\alpha_1, \alpha_2, \ldots$ are nonnegative integers.

In umbral Clifford analysis, we take the delta operator O_{x_j} as the momentum operators and take

$$x_j' = \frac{1}{2}(x_j(O'_{x_j})^{-1} + (O'_{x_j})^{-1} x_j)$$

as the position operators, where $O'_{x_j} f(\underline{x}) = O_{x_j}(x_j f(\underline{x})) - x_j O_{x_j} f(\underline{x})$ is the Pincherle derivative. They satisfy the Heisenberg–Weyl relations:

$$[O_{x_j}, O_{x_k}] = 0 = [x_j', x_k'], \quad [O_{x_j}, x_k'] = \delta_{jk} id.$$

Let $x_k' := x_k(O'_{x_k})^{-1}$. Then, the basic polynomials are given by

$$V_\alpha(\underline{x}) = (\underline{x}')^\alpha, \tag{2}$$

where $(\underline{x}')^\alpha = \prod_{k=1}^{n} (x_k')^{\alpha_k}$. This is known as the Rodrigues formula.

The operator:

$$E' = \sum_{j=1}^{n} x_j' O_{x_j} \tag{3}$$

is called the umbral Euler operator. This operator allows us to have

$$E'V_\alpha(\underline{x}) = |\alpha| V_\alpha(\underline{x}).$$

2.3. Sheffer Operator

The Sheffer operator is defined by

$$\Psi_{\underline{x}} : \underline{x}^\alpha \to V_\alpha(\underline{x}), \tag{4}$$

where $\{V_\alpha(\underline{x}), \alpha \in N\}$ is a polynomial sequence. The inverse of this linear operator $\Psi_{\underline{x}}^{-1}$ is given by $\Psi_{\underline{x}}^{-1} : V_\alpha(\underline{x}) \to \underline{x}^\alpha$. Furthermore, we have the intertwining relations, i.e.,

$$O_{x_j}\Psi_{\underline{x}} = \Psi_{\underline{x}}\partial_{x_j}, \quad x_j'\Psi_{\underline{x}} = \Psi_{\underline{x}}x_j, \quad E'\Psi_{\underline{x}} = \Psi_{\underline{x}}E.$$

3. 0-Normalized System of Functions with Respect to the Umbral Dirac Operator

In this section, we set up the normalized systems of functions with respect to the Dirac operator in the setting of umbral Clifford analysis. In other words, we will establish a system of functions $\{F_k(x;f), k = 0,1,2,\cdots\}$ satisfying

$$D'F_k(x;f) = F_{k-1}(x;f), \quad k = 1,2,\cdots, \tag{5}$$

where $D'F_0(x;f) = D'f(x) = 0$. That is to say, the function $f(x)$ is an umbral monogenic function.

First of all, we give the following definitions:

Definition 1 ([19])**.** *Let the open connected set $\Omega \subset R^n$. If $x \in \Omega$ and $0 \le \alpha \le 1$ satisfy that $\alpha x \in \Omega$, then it is the so-called star domain with center 0. It is denoted by Ω_0.*

Definition 2. *Let $f(x) \in C(\Omega_0, Cl_{0,n})$. The operator J_l' is defined by*

$$J_l' = \Psi_{\underline{x}} J_l \Psi_{\underline{x}}^{-1} \tag{6}$$

where $J_l f(x) = \int_0^1 (1-\alpha)^{l-1}\alpha^{\frac{n}{2}-1} f(\alpha x)dx,\ l > 0$.

Definition 3. *Let E' be as stated before. Let I be the identical operator. Then, the operator E_l' is defined by*

$$E_l' = E' + lI, \tag{7}$$

where $l > 0$. Note that $E'\Psi_{\underline{x}} = \Psi_{\underline{x}}E$. Thus, we have $E_l' = \Psi_{\underline{x}}E_l\Psi_{\underline{x}}^{-1}$.

Now, we consider the relations between the operators x', D' and E'.

Lemma 1 ([7])**.** *Let x', D', and E' be as stated before. Then, we have the following intertwining relations:*

$$\begin{aligned} x'D' + D'x' &= -2E' - n, \\ E'x' - x'E' &= x', \\ D'E' - E'D' &= D'. \end{aligned}$$

Lemma 2. *Let $f(x) \in C(\Omega_0, Cl_{0,n})$. Then,*

$$\begin{cases} D'[(x')^{2s}f(x)] = -2s(x')^{2s-1}f(x) + (x')^{2s}D'f(x), \\ D'[(x')^{2s-1}f(x)] = -2(x')^{2(s-1)}E'_{\frac{n}{2}+s-1}f(x) + (x')^{2(s-1)}D'f(x), \end{cases}$$

where $s \in N$.

For more details on the proof of Lemma 2, the reader can refer to [7].

Lemma 3. *Let $f(x) \in C(\Omega_0, Cl_{0,n})$. Then, for $s > 1$,*

$$E'_{\frac{n}{2}+s-1} J'_s f(x) = (s-1) J'_{s-1} f(x).$$

Proof of Lemma 3. Let $E'_s = \Psi_{\underline{x}} E_s \Psi_{\underline{x}}^{-1}$ and $J'_s = \Psi_{\underline{x}} J_s \Psi_{\underline{x}}^{-1}$. Then,

$$E'_{\frac{n}{2}+s-1} J'_s f(x) = \Psi_{\underline{x}} E_{\frac{n}{2}+s-1} \Psi_{\underline{x}}^{-1} \Psi_{\underline{x}} J_s \Psi_{\underline{x}}^{-1} f(x) = \Psi_{\underline{x}} (s-1) J_{s-1} \Psi_{\underline{x}}^{-1} f(x)$$

$$= (s-1) \Psi_{\underline{x}} J_{s-1} \Psi_{\underline{x}}^{-1} f(x) = (s-1) J'_{s-1} f(x).$$

The proof of this lemma is complete. □

Thus, we have the 0-normalized system of functions with respect to the umbral Dirac operator as follows.

Theorem 1. *Suppose that a function $f(x) \in C(\Omega_0, Cl_{0,n})$ satisfies the equation $D' f(x) = 0$. Then, the sequence of functions $F_k(x; f)$ in Ω_0 is the 0-normalized system of functions with respect to the operator D', where*

$$F_k(x; f) = \begin{cases} \dfrac{(x')^{2l}}{2^{2l} l!(l-1)!} J'_l f(x), \quad k = 2l, \\ -\dfrac{(x')^{2l-1}}{2^{2l-1}(l-1)!(l-1)!} J'_l f(x), \quad k = 2l-1, \\ f(x), \quad k = 0. \end{cases} \tag{8}$$

Proof of Theorem 1. It is easy to see that $D' F_0(x; f) = D' f(x) = 0$. We only need to prove that $D' F_k(x; f) = F_{k-1}(x; f)$ for any $k \in \mathbf{N}$. For $k = 2l-1$, applying Lemmas 2 and 3, we obtain

$$D' F_{2l-1}(x; f) = -\frac{1}{2^{2l-1}(l-1)!(l-1)!} D' \left[(x')^{2l-1} J'_l f(x) \right]$$

$$= -\frac{1}{2^{2l-1}(l-1)!(l-1)!} \left[-2(x')^{2(l-1)} E'_{\frac{n}{2}+l-1} J'_l f(x) - (x')^{2l-1} D' J'_l f(x) \right]$$

$$= \frac{2(x')^{2l-1}}{2^{2l-1}(l-1)!(l-1)!} E'_{\frac{n}{2}+l-1} J'_l f(x)$$

$$= \frac{(x')^{2(l-1)}}{2^{2(l-1)}(l-1)!(l-2)!} J'_{l-1} f(x) = F_{2l-2}(x; f).$$

For $k = 2l$, it is obvious to obtain the result by Lemma 2. □

4. Applications of 0-Normalized System of Functions with Respect to the Umbral Dirac Operator

4.1. The Almansi-Type Expansion for the Umbral Dirac Operator

In this section, we will derive the Almansi-type expansion for umbral k-monogenic functions by using the 0-normalized system of functions with respect to the umbral Dirac operator. We begin with the following lemma.

Lemma 4. *Let $f(x) \in C(\Omega_0, Cl_{0,n})$. Then*

$$E'_{l+1}\int_0^1 \frac{\alpha^l(1-\alpha)^p}{p!} f(\alpha x)d\alpha = \begin{cases} \displaystyle\int_0^1 \frac{\alpha^{l+1}(1-\alpha)^{p-1}}{(p-1)!} f(\alpha x)d\alpha, & p \in N, \\ f(x), & p = 0, \end{cases}$$

where $l > 0$.

Proof of Lemma 4. Let $T_{0,l}f = \int_0^1 \alpha^l f(\alpha x)d\alpha$. Then, $T'_{0,l} = \varphi_{\underline{x}} T_{0,l} \varphi_{\underline{x}}^{-1}$. By computation, we have

$$\begin{aligned} E'_{l+1}T'_{0,l}f &= \Psi_{\underline{x}} E_{l+1} \Psi_{\underline{x}}^{-1} \Psi_{\underline{x}} T_{0,l} \Psi_{\underline{x}}^{-1} f(x) \\ &= \Psi_{\underline{x}} E_{l+1} T_{0,l} \Psi_{\underline{x}}^{-1} f(x) = f(x). \end{aligned}$$

Let $T_{p,l}f = \int_0^1 \frac{(1-\alpha)^p}{p!}\alpha^l f(\alpha x)d\alpha$. Then, $T'_{p,l} = \Psi_{\underline{x}} T_{p,l} \Psi_{\underline{x}}^{-1}$, and

$$\begin{aligned} E'_{l+1}T'_{p,l}f &= \Psi_{\underline{x}} E_{l+1} \Psi_{\underline{x}}^{-1} \Psi_{\underline{x}} T_{p,l} \Psi_{\underline{x}}^{-1} f(x) \\ &= \Psi_{\underline{x}} E_{l+1} T_{p,l} \Psi_{\underline{x}}^{-1} f(x) = \varphi_{\underline{x}} T_{p-1,l+1} \Psi_{\underline{x}}^{-1} f(x) = T'_{p-1,l+1} f(x). \end{aligned}$$

Thus, we have the result. □

Theorem 2. *Let $F(x) \in C^k(\Omega_0, Cl_{0,n})$. If $(D')^k F(x) = 0$, then*

$$\begin{aligned} F(x) = f_0(x) &+ \sum_{i=1}^{[\frac{k}{2}]} \frac{(x')^{2i-1}}{2^{2i-1}(i-1)!(i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_{2i-1}(\alpha x)d\alpha \\ &+ \sum_{i=1}^{[\frac{k-1}{2}]} \frac{(x')^{2i}}{2^{2i} i!(i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_{2i}(\alpha x)d\alpha, \end{aligned} \tag{9}$$

where $f_j(x)(j = 0, \cdots, k-1)$ are umbral monogenic, and

$$\begin{aligned} f_j(x) = (D')^j F(x) &+ \sum_{s=1}^{[\frac{k-j}{2}]} \frac{(-1)^s (x')^{2s-1}}{2^{2s-1}} \int_0^1 \frac{(1-\beta)^{s-1}\beta^{s-1}}{(s-1)!(s-1)!} \beta^{\frac{n}{2}-1} (D')^{j+2s-1} F(\beta x)d\beta \\ &+ \sum_{s=1}^{[\frac{k-j-1}{2}]} \frac{(-1)^s (x')^{2s}}{2^{2s}} \int_0^1 \frac{(1-\beta)^{s-1}\beta^{s}}{s!(s-1)!} \beta^{\frac{n}{2}-1} (D')^{j+2s} F(\beta x)d\beta. \end{aligned} \tag{10}$$

Proof of Theorem 2. First of all, we first prove that $f_j(x)(j = 0, \cdots, k-1)$ are umbral monogenic. Using Lemmas 2 and 4, we have

$$\begin{aligned} f_j(x) &= (D')^{j+1} F(x) \\ &+ \sum_{s=1}^{[\frac{k-j}{2}]} \frac{(-1)^s (x')^{2(s-1)}}{2^{2(s-1)}} \left(E'_{\frac{n}{2}+s-1}\right) \int_0^1 \frac{(1-\beta)^{s-1}\beta^{\frac{n}{2}+s-2}}{(s-1)!(s-1)!} (D')^{j+2s-1} F(\beta x)d\beta \\ &- \sum_{s=1}^{[\frac{k-j-1}{2}]} \frac{(-1)^s (x')^{2s-1}}{2^{2s-1}} \int_0^1 \frac{(1-\beta)^{s-1}\beta^{s}}{(s-1)!(s-1)!} \beta^{\frac{n}{2}-1} (D')^{j+2s} F(\beta x)d\beta \end{aligned}$$

$$+\sum_{s=1}^{[\frac{k-j-1}{2}]}\frac{(-1)^s(x')^{2s-1}}{2^{2s-1}}\int_0^1\frac{(1-\beta)^{s-1}\beta^s}{(s-1)!(s-1)!}\beta^{\frac{n}{2}-1}(D')^{j+2s}F(\beta x)d\beta$$

$$+\sum_{s=1}^{[\frac{k-j}{2}-1]}\frac{(-1)^s(x')^{2s}}{2^{2s}}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+1}}{s!(s-1)!}\beta^{\frac{n}{2}-1}(D')^{j+2s+1}F(\beta x)d\beta.$$

By computing the first integral of the equality, we have

$$\sum_{s=1}^{[\frac{k-j}{2}]}\frac{(-1)^s(x')^{2(s-1)}}{2^{2(s-1)}}\left(E'_{\frac{n}{2}+s-1}\right)\int_0^1\frac{(1-\beta)^{s-1}\beta^{s-1}}{(s-1)!(s-1)!}\beta^{\frac{n}{2}-1}(D')^{j+2s-1}F(\beta x)d\beta$$

$$=-\left(E'_{\frac{n}{2}}\right)\int_0^1\beta^{\frac{n}{2}-1}(D')^{j+1}F(\beta x)d\beta$$

$$+\sum_{s=2}^{[\frac{k-j}{2}]}\frac{(-1)^s(x')^{2(s-1)}}{2^{2(s-1)}}\left(E'_{\frac{n}{2}+s-1}\right)\int_0^1\frac{(1-\beta)^{s-1}\beta^{s-1}}{(s-1)!(s-1)!}\beta^{\frac{n}{2}-1}(D')^{j+2s-1}F(\beta x)d\beta$$

$$=-(D')^{j+1}F(x)+\sum_{s=2}^{[\frac{k-j}{2}]}\frac{(-1)^s(x')^{2(s-1)}}{2^{2(s-1)}}\int_0^1\frac{(1-\beta)^{s-2}\beta^{s}}{(s-2)!(s-1)!}\beta^{\frac{n}{2}-1}(D')^{j+2s-1}F(\beta x)d\beta$$

$$=-(D')^{j+1}F(x)+\sum_{s=1}^{[\frac{k-j}{2}-1]}\frac{(-1)^{s+1}(x')^{2s}}{2^{2s}}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+1}}{(s-1)!s!}\beta^{\frac{n}{2}-1}(D')^{j+2s+1}F(\beta x)d\beta$$

$$=-(D')^{j+1}F(x)-\sum_{s=1}^{[\frac{k-j}{2}-1]}\frac{(-1)^{s}(x')^{2s}}{2^{2s}}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+1}}{(s-1)!s!}\beta^{\frac{n}{2}-1}(D')^{j+2s+1}F(\beta x)d\beta.$$

Thus, we have $D'f_j(x)=0$.

Substituting (10) into (9), we have

$$f_0(x)+\sum_{i=1}^{[\frac{k}{2}]}\frac{(x')^{2i-1}}{2^{2i-1}(i-1)!(i-1)!}\int_0^1(1-\alpha)^{i-1}\alpha^{\frac{n}{2}-1}f_{2i-1}(\alpha x)d\alpha$$

$$+\sum_{i=1}^{[\frac{k-1}{2}]}\frac{(x')^{2i}}{2^{2i}i!(i-1)!}\int_0^1(1-\alpha)^{i-1}\alpha^{\frac{n}{2}-1}f_{2i}(\alpha x)d\alpha$$

$$=F(x)+\sum_{i=1}^{[\frac{k}{2}]}\frac{(-1)^i(x')^{2i-1}}{2^{2i-1}}\int_0^1\frac{(1-\beta)^{i-1}\beta^{i-1}}{(i-1)!(i-1)!}\beta^{\frac{n}{2}-1}(D')^{2i-1}F(\beta x)d\beta$$

$$+\sum_{i=1}^{[\frac{k-1}{2}]}\frac{(-1)^i(x')^{2i}}{2^{2i}}\int_0^1\frac{(1-\beta)^{i-1}\beta^{i}}{i!(i-1)!}\beta^{\frac{n}{2}-1}(D')^{2i}F(\beta x)d\beta$$

$$+\sum_{i=1}^{[\frac{k}{2}]}\frac{(x')^{2i-1}}{2^{2i-1}}\int_0^1\frac{(1-\beta)^{i-1}}{(i-1)!(i-1)!}\beta^{\frac{n}{2}-1}(D')^{2i-1}F(\beta x)d\beta+F_1(x)+F_2(x)$$

$$+\sum_{i=1}^{[\frac{k-1}{2}]}\frac{(x')^{2i}}{2^{2i}i!(i-1)!}\int_0^1(1-\alpha)^{i-1}\alpha^{\frac{n}{2}-1}(D')^{2i}F(\alpha x)d\alpha+F_3(x)+F_4(x),$$

where

$$F_1(x) = \int_0^1 (1-\alpha)^{i-1}\alpha^{\frac{n}{2}-1}$$

$$\times \sum_{s=1}^{[\frac{k-2i+1}{2}]} \frac{(-1)^s[\alpha(x')]^{2s-1}}{2^{2s-1}} \int_0^1 \frac{(1-\beta)^{s-1}\beta^{\frac{n}{2}+s-2}}{(s-1)!(s-1)!}(D')^{2i+2s-2}F(\alpha\beta x)d\beta d\alpha.$$

We calculate

$$F_1(x) = \sum_{i=1}^{[\frac{k-1}{2}]}\sum_{s=1}^{[\frac{k-2i+1}{2}]} \frac{(-1)^s(x')^{2i+2s-2}}{4^{i+s-1}(i-1)!(i-1)!(s-1)!(s-1)!}$$

$$\times \int_0^1 (1-\alpha)^{i-1}\alpha^{\frac{n}{2}-1}\alpha^{2s-1}\int_0^1 (1-\beta)^{s-1}\beta^{s-1}\beta^{\frac{n}{2}-1}(D')^{2i+2s-2}F(\alpha\beta x)d\beta d\alpha$$

$$= \sum_{i=1}^{[\frac{k-1}{2}]}\sum_{s=1}^{[\frac{k-2i+1}{2}]} \frac{(-1)^s(x')^{2i+2s-2}}{4^{i+s-1}(i-1)!(i-1)!(s-1)!(s-1)!}$$

$$\times \int_0^1 \alpha(1-\alpha)^{i-1}\int_0^1 (\alpha-\alpha\beta)^{s-1}(\alpha\beta)^{s-1}(\alpha\beta)^{\frac{n}{2}-1}(D')^{2i+2s-2}F(\alpha\beta x)d\beta d\alpha.$$

Let $t = \alpha\beta$. Then,

$$B_1(x) = \int_0^1 \alpha(1-\alpha)^{i-1}\int_0^1 (\alpha-\alpha\beta)^{s-1}(\alpha\beta)^{s-1}(\alpha\beta)^{\frac{n}{2}-1}(D')^{2i+2s-2}F(\alpha\beta x)d\beta d\alpha$$

$$= \int_0^1 (1-\alpha)^{i-1}\int_0^\alpha (\alpha-t)^{s-1}t^{s-1}t^{\frac{n}{2}-1}(D')^{2i+2s-2}F(tx)dtd\alpha$$

$$= \int_0^1 t^{\frac{n}{2}+s-2}(D')^{2i+2s-2}F(tx)\int_t^1 (1-\alpha)^{i-1}(\alpha-t)^{s-1}d\alpha dt.$$

We consider the second integral of the above equality as follows.

$$C_1(t) = \int_t^1 (1-\alpha)^{i-1}(\alpha-t)^{s-1}d\alpha.$$

Let $\alpha = \beta + t$. Then,

$$C_1(t) = \int_0^{1-t} (1-\beta-t)^{i-1}\beta^{s-1}d\beta.$$

Let $\beta = \alpha(1-t)$. Then,

$$C_1(t) = \int_0^1 (1-\alpha(1-t)-t)^{i-1}[\alpha(1-t)]^{s-1}(1-t)d\alpha$$

$$= \int_0^1 (1-\alpha)^{i-1}(1-t)^{i-1}\alpha^{s-1}(1-t)^s d\alpha$$

$$= (1-t)^{i+s-1}\int_0^1 (1-\alpha)^{i-1}\alpha^{s-1}d\alpha.$$

It is well-known that the Euler beta function is given by

$$B(l,s) = \int_0^1 \alpha^{l-1}(1-\alpha)^{s-1}d\alpha.$$

After a simple calculation, we obtain

$$C_1(t) = (1-t)^{i+s-1} B(s,i).$$

Note that

$$B(l,s) = \frac{\Gamma(l)\Gamma(s)}{\Gamma(s+l)},$$

where

$$\Gamma(s) = (s-1)!.$$

We have

$$C_1(t) = (1-t)^{i+s-1} \frac{(i-1)!(s-1)!}{(s+i-1)!}.$$

Thus, we have

$$\begin{aligned}
F_1(x) = & \sum_{i=1}^{[\frac{k-1}{2}]} \sum_{s=1}^{[\frac{k-2i+1}{2}]} \frac{(-1)^s (x')^{2i+2s-2}}{4^{i+s-1}(i-1)!(i-1)!(s-1)!(s-1)!} \\
& \times \int_0^1 \frac{(i-1)!(s-1)!}{(s+i-1)!} t^{\frac{n}{2}+s-2} (D')^{2i+2s-2} F(tx) dt \\
= & \sum_{i=1}^{[\frac{k-1}{2}]} \sum_{s=1}^{[\frac{k-2i+1}{2}]} \frac{(-1)^s (x')^{2i+2s-2}}{4^{i+s-1}(i-1)!(s-1)!} \int_0^1 \frac{(1-t)^{i+s-1} t^{\frac{n}{2}+s-2}}{(s+i-1)!} (D')^{2i+2s-2} F(tx) dt \\
= & \sum_{j=1}^{[\frac{k-1}{2}]+1} \sum_{s=1}^{[j-1]} \frac{(-1)^s (x')^{2j-2}}{4^{j-1}(j-s-1)!(s-1)!} \int_0^1 \frac{(1-t)^{j-1} t^{s-1}}{(j-1)!} t^{\frac{n}{2}-1} (D')^{2j-2} F(tx) dt \\
= & \sum_{j=2}^{[\frac{k-1}{2}]+1} \frac{(x')^{2j-2}}{4^{j-1}(j-1)!} \int_0^1 \sum_{s=1}^{j-1} \frac{(-1)^s t^{s-1}}{(j-s-1)!(s-1)!} (1-t)^{j-1} t^{\frac{n}{2}-1} (D')^{2j-2} F(tx) dt \\
= & -\sum_{j=2}^{[\frac{k-1}{2}]+1} \frac{(x')^{2j-2}}{4^{j-1}(j-1)!} \int_0^1 \sum_{s=1}^{j-1} \frac{(-1)^{s-1} t^{s-1}}{(j-s-1)!(s-1)!} (1-t)^{j-1} t^{\frac{n}{2}-1} (D')^{2j-2} F(tx) dt \\
= & -\sum_{j=2}^{[\frac{k-1}{2}]+1} \frac{(x')^{2j-2}}{4^{j-1}(j-1)!} \int_0^1 \sum_{s=0}^{j-2} \frac{(-1)^s t^s}{(j-s-2)!s!} (1-t)^{j-1} t^{\frac{n}{2}-1} (D')^{2j-2} F(tx) dt \\
= & -\sum_{j=2}^{[\frac{k-1}{2}]+1} \frac{(x')^{2j-2}}{4^{j-1}(j-1)!} \int_0^1 \frac{(1-t)^{j-2}}{(j-2)!} (1-t)^{j-1} t^{\frac{n}{2}-1} (D')^{2j-2} F(tx) dt \\
= & -\sum_{j=1}^{[\frac{k-1}{2}]} \frac{(x')^{2j}}{4^j} \int_0^1 \frac{(1-t)^{j-1}}{(j-1)!} \frac{(1-t)^j}{j!} t^{\frac{n}{2}-1} (D')^{2j} F(tx) dt.
\end{aligned}$$

Similarly, we obtain $F_2(x), F_3(x), F_4(x)$ as follows. The theorem is proven. □

In the following part of this paper, we suppose that all infinite series converge absolutely and uniformly in Ω_0. For the discussion of the convergence of these series, the reader can refer to [17]. Now, we give the main theorem in this section as follows.

Theorem 3. *If $F(x) \in C^\infty(\Omega_0, R_{0,m})$, then*

$$\begin{aligned} F(x) = f_0(x) &- \sum_{i=1}^{\infty} \frac{(x')^{2i-1}}{2^{2i-1}(i-1)!(i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_{2i-1}(\alpha x) d\alpha \\ &+ \sum_{i=1}^{\infty} \frac{(x')^{2i}}{2^{2i} i!(i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_{2i}(\alpha x) d\alpha, \end{aligned} \tag{11}$$

where umbral monogenic functions $f_j(x), j = 0, 1, \cdots$, are given by

$$\begin{aligned} f_j(x) = (D')^j F(x) &- \sum_{s=1}^{\infty} \frac{(-1)^s (x')^{2s-1}}{2^{2s-1}} \int_0^1 \frac{(1-\beta)^{s-1} \beta^{s+\frac{n}{2}-2}}{(s-1)!(s-1)!} (D')^{j+2s-1} F(\beta x) d\beta \\ &+ \sum_{s=1}^{\infty} \frac{(-1)^s (x')^{2s}}{2^{2s} s!(s-1)!} \int_0^1 (1-\beta)^{s-1} \beta^{s+\frac{n}{2}-1} (D')^{j+2s} F(\beta x) d\beta. \end{aligned} \tag{12}$$

Remark 1. *From Theorem 3, we establish one representation of the functions by umbral monogenic functions, which is an Almansi formula of infinite order. As applications of the representation, we construct solutions of the equation $(D' + \lambda) f(x) = 0$ and the inhomogeneous umbral poly-Dirac equation.*

4.2. Solutions of the Equation $(D' + \lambda) f(x) = 0$

Let λ be a real number. Then, we consider the Dirac-type equation in umbral Clifford analysis:

$$(D' + \lambda) f(x) = 0. \tag{13}$$

Theorem 4. *If $F(x) \in C(\Omega_0, Cl_{0,n})$, then the solution of Equation (13) is given by*

$$\begin{aligned} F(x) = f_0(x) &+ \sum_{i=1}^{\infty} \frac{[\lambda(x')]^{2i}}{2^{2i} i!(i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_0(\alpha x) d\alpha \\ &+ \sum_{i=1}^{\infty} \frac{[\lambda(x')]^{2i-1}}{2^{2i-1}(i-1)!(i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_0(\alpha x) d\alpha, \end{aligned} \tag{14}$$

where

$$\begin{aligned} f_0(x) = F(x) &- \sum_{s=1}^{\infty} \frac{(-1)^s (x')^{2s-1}}{2^{2s-1}} \int_0^1 \frac{(1-\beta)^{s-1} \beta^{s+\frac{n}{2}-2}}{(s-1)!(s-1)!} (D')^{2s-1} F(\beta x) d\beta \\ &+ \sum_{s=1}^{\infty} \frac{(-1)^s (x')^{2s}}{2^{2s} s!(s-1)!} \int_0^1 (1-\beta)^{s-1} \beta^{s+\frac{n}{2}-1} (D')^{2s} F(\beta x) d\beta. \end{aligned} \tag{15}$$

Proof of Theorem 4. Let $F(x) \in C(\Omega_0, Cl_{0,n})$. Then, $D' f_0(x) = 0$ by Theorem 3. Furthermore, we have

$$D' \left(\int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_0(\alpha x) d\alpha \right) = 0.$$

From Lemma 2, we can see that

$$\begin{aligned} &D' \left[(x')^{2i} \int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_0(\alpha x) d\alpha \right] \\ &= -2i x^{2i-1} \int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_0(\alpha x) d\alpha, \end{aligned}$$

and

$$D'\left[(x')^{2i-1}\int_0^1(1-\alpha)^{i-1}\alpha^{\frac{n}{2}-1}f_0(\alpha x)d\alpha\right]$$

$$= -2x^{2(i-1)}E'_{\frac{n}{2}+i-1}\int_0^1(1-\alpha)^{i-1}\alpha^{\frac{n}{2}-1}f_0(\alpha x)d\alpha.$$

Differentiating both sides of Equation (14), we have

$$D'F(x) = -\sum_{i=1}^{\infty}\frac{\lambda^{2i}(x')^{2i-1}}{2^{2i-1}(i-1)!(i-1)!}\int_0^1(1-\alpha)^{i-1}\alpha^{\frac{m}{2}-1}f_0(\alpha x)d\alpha$$
$$-\sum_{i=1}^{\infty}\frac{\lambda^{2i-1}(x')^{2(i-1)}}{2^{2(i-1)}(i-1)!(i-1)!}E'_{\frac{n}{2}+i-1}\int_0^1(1-\alpha)^{i-1}\alpha^{\frac{m}{2}-1}f_0(\alpha x)d\alpha.$$

We calculate the second sum in the above expression as follows.

$$-\sum_{i=1}^{\infty}\frac{\lambda^{2i-1}(x')^{2(i-1)}}{2^{2(i-1)}(i-1)!(i-1)!}E'_{\frac{n}{2}+i-1}\int_0^1(1-\alpha)^{i-1}\alpha^{\frac{m}{2}-1}f_0(\alpha x)d\alpha$$
$$= -\lambda E'_{\frac{n}{2}}\int_0^1\alpha^{\frac{n}{2}-1}f_0(\alpha x)d\alpha - \sum_{i=2}^{\infty}\frac{\lambda^{2i-1}(x')^{2(i-1)}}{2^{2(i-1)}(i-1)!}E'_{\frac{n}{2}+i-1}\int_0^1\frac{(1-\alpha)^{i-1}}{(i-1)!}\alpha^{\frac{n}{2}-1}f_0(\alpha x)d\alpha$$
$$= -\lambda f_0(x) - \sum_{i=2}^{\infty}\frac{\lambda^{2i-1}(x')^{2(i-1)}}{2^{2(i-1)}(i-1)!(i-2)!}\int_0^1(1-\alpha)^{i-2}\alpha^{\frac{n}{2}-1}f_0(\alpha x)d\alpha$$
$$= -\lambda f_0(x) - \sum_{i=1}^{\infty}\frac{\lambda^{2i+1}(x')^{2i}}{2^{2i}i!(i-1)!}\int_0^1(1-\alpha)^{i-1}\alpha^{\frac{n}{2}-1}f_0(\alpha x)d\alpha.$$

To sum up, we have that the function $F(x)$ is a solution of Equation (14). □

4.3. Solutions of Inhomogeneous Umbral Poly-Dirac Equations

In this section, we investigate the inhomogeneous umbral poly-Dirac equation:

$$(D')^k g = f(x)u \tag{16}$$

where $f(x) \in C^{\infty}(\Omega_0, R_{0,n})$.

Theorem 5. *Assume that $f(x) \in C^{\infty}(\Omega_0, R_{0,n})$ is a real analytic function. Then, the function $F(x)$ is given by*

$$\begin{aligned}F(x) = &\sum_{i=0}^{\infty}\frac{(-1)^{i+k}(x')^{2i+k}}{2^{2i+k}i!(i+k-1)!}\int_0^1(1-\alpha)^{i+k-1}\alpha^{\frac{n}{2}+i-1}(D')^{2i}f(\alpha x)d\alpha\\ &+\sum_{i=0}^{\infty}\frac{(-1)^{i+k-1}(x')^{2i+k+1}}{2^{2i+k+1}(i+1)!(i+k-1)!}\int_0^1(1-\alpha)^{i+k-1}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha\end{aligned} \tag{17}$$

Proof of Theorem 5. We argue by induction. For $k = 1$, we prove that the solution of Equation $D' g(x) = f(x)$ is given by

$$\begin{aligned} g(x) = &\sum_{i=0}^{\infty} \frac{(-1)^{i+1}\big(x'\big)^{2i+1}}{2^{2i+1} i! i!} \int_0^1 (1-\alpha)^i \alpha^{\frac{n}{2}+i-1} (D')^{2i} f(\alpha x) d\alpha \\ &+ \sum_{i=0}^{\infty} \frac{(-1)^i \big(x'\big)^{2(i+1)}}{2^{2(i+1)}(i+1)! i!} \int_0^1 (1-\alpha)^i \alpha^{\frac{n}{2}+i} (D')^{2i+1} f(\alpha x) d\alpha. \end{aligned} \tag{18}$$

Then, it follows by Theorem 3 that

$$\begin{aligned} G(x) = f_0(x) &- \sum_{i=1}^{\infty} \frac{(x')^{2i-1}}{2^{2i-1}(i-1)!(i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_{2i-1}(\alpha x) d\alpha \\ &+ \sum_{i=1}^{\infty} \frac{(x')^{2i}}{2^{2i} i!(i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{\frac{n}{2}-1} f_{2i}(\alpha x) d\alpha, \end{aligned} \tag{19}$$

where $f_j(x)$ are umbral monogenic functions in Ω_0 given by the relation:

$$\begin{aligned} f_j(x) = (D')^j G(x) &- \sum_{s=1}^{\infty} \frac{(-1)^s (x')^{2s-1}}{2^{2s-1}} \int_0^1 \frac{(1-\beta)^{s-1}\beta^{s-1}}{(s-1)!(s-1)!} \beta^{\frac{n}{2}-1} (D')^{j+2s-1} G(\beta x) d\beta \\ &+ \sum_{s=1}^{\infty} \frac{(-1)^s (x')^{2s}}{2^{2s}} \int_0^1 \frac{(1-\beta)^{s-1}\beta^{s}}{s!(s-1)!} \beta^{\frac{n}{2}-1} (D')^{j+2s} G(\beta x) d\beta. \end{aligned} \tag{20}$$

Using (19) and (20), we obtain

$$\begin{aligned} G(x) - f_0(x) &= \sum_{s=1}^{\infty} \frac{(-1)^s (x')^{2s-1}}{2^{2s-1}} \int_0^1 \frac{(1-\beta)^{s-1}\beta^{s-1}}{(s-1)!(s-1)!} \beta^{\frac{n}{2}-1} (D')^{2s-1} G(\beta x) d\beta \\ &\quad - \sum_{s=1}^{\infty} \frac{(-1)^s (x')^{2s}}{2^{2s}} \int_0^1 \frac{(1-\beta)^{s-1}\beta^{s}}{s!(s-1)!} \beta^{\frac{n}{2}-1} (D')^{2s} G(\beta x) d\beta \\ &= \sum_{s=0}^{\infty} \frac{(-1)^{s+1}\big(x'\big)^{2s+1}}{2^{2s+1} s! s!} \int_0^1 (1-\alpha)^s \alpha^{\frac{n}{2}+s-1} (D')^{2s} f(\alpha x) d\alpha \\ &\quad + \sum_{s=0}^{\infty} \frac{(-1)^s \big(x'\big)^{2(s+1)}}{2^{2(s+1)}(s+1)! s!} \int_0^1 (1-\alpha)^s \alpha^{\frac{n}{2}+s} (D')^{2s+1} f(\alpha x) d\alpha. \end{aligned}$$

The left-hand side of the above expression is a solution of Equation (16) for $k = 1$; therefore, its right-hand side is a solution as well.

Assume that Formula (17) holds for $k = p$. Then, we prove that Formula (17) also holds for $k = p + 1$.

By setting $D^p u = g(x)$ for $g(x)$, we obtain the equation $Dg = f(x)$; therefore, by Theorem 1, the function $g(x)$ is given by

$$\begin{aligned} g(x) = &\sum_{i=0}^{\infty} \frac{(-1)^{i+1}\big(x'\big)^{2i+1}}{2 \cdot 4^i \cdot i! \cdot i!} \int_0^1 (1-\alpha)^i \alpha^{\frac{n}{2}+i-1} (D')^{2i} f(\alpha x) d\alpha \\ &+ \sum_{i=0}^{\infty} \frac{(-1)^i \big(x'\big)^{2(i+1)}}{4^{i+1} \cdot (i+1)! \cdot i!} \int_0^1 (1-\alpha)^i \alpha^{\frac{n}{2}+i} (D')^{2i+1} f(\alpha x) d\alpha \end{aligned} \tag{21}$$

Moreover, by the inductive assumption, we have

$$
\begin{aligned}
u(x) &= \sum_{i=0}^{\infty} \frac{(-1)^{i+p}(x')^{2i+p}}{2^{2i+p} i!(i+p-1)!} \int_0^1 (1-\alpha)^{i+p-1} \alpha^{\frac{n}{2}+i-1} (D')^{2i} g(\alpha x) d\alpha \\
&\quad + \sum_{i=0}^{\infty} \frac{(-1)^{i+p-1}(x')^{2i+p+1}}{2^{2i+p+1}(i+p-1)!(i+1)!} \int_0^1 (1-\alpha)^{i+p-1} \alpha^{\frac{n}{2}+i} (D')^{2i+1} g(\alpha x) d\alpha
\end{aligned} \tag{22}
$$

Note that $(D')^p g(x) = (D')^{p-1} f(x)$. From (22), we have

$$
\begin{aligned}
u(x) &= \frac{(-1)^p (x')^p}{2^p (p-1)!} \int_0^1 (1-\alpha)^{p-1} \alpha^{\frac{n}{2}-1} g(\alpha x) d\alpha \\
&\quad + \sum_{i=1}^{\infty} \frac{(-1)^{i+p}(x')^{2i+p}}{2^{2i+p} i!(i+p-1)!} \int_0^1 (1-\alpha)^{i+p-1} \alpha^{\frac{n}{2}+i-1} (D')^{2i-1} f(\alpha x) d\alpha \\
&\quad + \sum_{i=0}^{\infty} \frac{(-1)^{i+p-1}(x')^{2i+p+1}}{2^{2i+p+1}(i+p-1)!(i+1)!} \int_0^1 (1-\alpha)^{i+p-1} \alpha^{\frac{n}{2}+i} (D')^{2i} f(\alpha x) d\alpha
\end{aligned} \tag{23}
$$

Using (21), we transform the first integral in (23) as follows.

$$
\begin{aligned}
&\frac{(-1)^p (x')^p}{2^p (p-1)!} \int_0^1 (1-\alpha)^{p-1} \alpha^{\frac{n}{2}-1} g(\alpha x) d\alpha \\
&= \frac{(-1)^p (x')^p}{2^p (p-1)!} \int_0^1 (1-\alpha)^{p-1} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}(x')^{2i+1}}{2 \cdot 4^i i! i!} \int_0^1 \alpha(\alpha-\alpha\beta)^i (\alpha\beta)^{\frac{n}{2}+i-1} (D')^{2i} f(\alpha\beta x) d\beta d\alpha \\
&+ \frac{(-1)^p (x')^p}{2^p (p-1)!} \int_0^1 (1-\alpha)^{p-1} \sum_{i=0}^{\infty} \frac{(-1)^{i}(x')^{2i+2}}{4^{i+1} (i+1)! i!} \int_0^1 \alpha(\alpha-\alpha\beta)^i (\alpha\beta)^{\frac{n}{2}+i} (D')^{2i+1} f(\alpha\beta x) d\beta d\alpha \\
&= J_1 + J_2.
\end{aligned} \tag{24}
$$

We calculate the sum of the two terms in the above equation separately.

$$
\begin{aligned}
&\frac{(-1)^p (x')^p}{2^p (p-1)!} \sum_{i=0}^{\infty} \frac{(-1)^{i+1} x^{2i+1}}{2 \cdot 4^i i! i!} \int_0^1 \int_0^1 (1-\alpha)^{p-1} (\alpha-\alpha\beta)^i (\alpha\beta)^{\frac{n}{2}+i-1} (D')^{2i} f(\alpha\beta x) d(\alpha\beta) d\alpha \\
&= \frac{(-1)^p (x')^p}{2^p (p-1)!} \sum_{i=0}^{\infty} \frac{(-1)^{i+1} x^{2i+1}}{2 \cdot 4^i i! i!} \int_0^1 \int_0^{\alpha} (1-\alpha)^{p-1} (\alpha-\beta)^i (\beta)^{\frac{n}{2}+i-1} (D')^{2i} f(\beta x) d\beta d\alpha \\
&= \frac{(-1)^p (x')^p}{2^p (p-1)!} \sum_{i=0}^{\infty} \frac{(-1)^{i+1} x^{2i+1}}{2 \cdot 4^i i! i!} \int_0^1 \beta^{\frac{n}{2}+i-1} (D')^{2i} f(\beta x) \int_{\beta}^1 (1-\alpha)^{p-1} (\alpha-\beta)^i d\alpha d\beta \\
&= \frac{(-1)^p (x')^p}{2^p} \sum_{i=0}^{\infty} \frac{(-1)^{i+1} (x')^{2i+1}}{2 \cdot 4^i i! (i+p)!} \int_0^1 (1-\alpha)^{p+i} \alpha^{\frac{n}{2}+i-1} (D')^{2i} f(\alpha x) d\alpha,
\end{aligned} \tag{25}
$$

and

$$\frac{(-1)^p(x')^p}{2^p(p-1)!}\int_0^1(1-\alpha)^{p-1}\sum_{i=0}^{\infty}\frac{(-1)^i(x')^{2i+2}}{4^{i+1}(i+1)!i!}\int_0^1(\alpha-\alpha\beta)^i(\alpha\beta)^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha\beta x)d(\alpha\beta)d\alpha$$

$$=\frac{(-1)^px^p}{2^p(p-1)!}\sum_{i=0}^{\infty}\frac{(-1)^ix^{2i+2}}{4^{i+1}(i+1)!i!}\int_0^1\int_0^{\alpha}(1-\alpha)^{p-1}(\alpha-\beta)^i\beta^{\frac{n}{2}+i}(D')^{2i+1}f(\beta x)d\beta d\alpha$$

$$=\frac{(-1)^p(x')^p}{2^p(p-1)!}\sum_{i=0}^{\infty}\frac{(-1)^ix^{2i+2}}{4^{i+1}(i+1)!i!}\int_0^1\beta^{\frac{n}{2}+i}(D')^{2i+1}f(\beta x)d\beta\int_{\beta}^1(1-\alpha)^{p-1}(\alpha-\beta)^id\alpha$$

$$=\frac{(-1)^p(x')^p}{2^p}\sum_{i=0}^{\infty}\frac{(-1)^i(x')^{2i+2}}{4^{i+1}(i+1)!(i+p)!}\int_0^1(1-\beta)^{p+i}\beta^{\frac{n}{2}+i}(D')^{2i+1}f(\beta x)d\beta.$$

By replacing $i\to i+1$, we compute the second integral in Formula (23):

$$\begin{aligned}
u(x)=&\frac{(-1)^p(x')^p}{2^p}\sum_{i=0}^{\infty}\frac{(-1)^{i+1}(x')^{2i+1}}{2^{2i+1}i!(i+p)!}\int_0^1(1-\alpha)^{p+i}\alpha^{\frac{n}{2}+i-1}(D')^{2i}f(\alpha x)d\alpha\\
&+\frac{(-1)^p(x')^p}{2^p}\sum_{i=0}^{\infty}\frac{(-1)^i(x')^{2i+2}}{2^{2(i+1)}(i+1)!(i+p)!}\int_0^1(1-\alpha)^{p+i}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\beta x)d\beta\\
&+\sum_{i=0}^{\infty}\frac{(-1)^{i+p}(x')^{2i+2+p}}{2^{2i+p+2}(i+1)!(i+p)!}\int_0^1(1-\alpha)^{i+p}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha\\
&+\sum_{i=0}^{\infty}\frac{(-1)^{i+p}(x')^{2i+p+1}}{2^{2i+p+1}(i+p-1)!(i+1)!}\int_0^1(1-\alpha)^{i+p-1}\alpha^{\frac{n}{2}+i}(D')^{2i}f(\alpha x)d\alpha\\
=&\sum_{i=0}^{\infty}\frac{(-1)^{i+p+1}(x')^{2i+p+1}}{2^{2i+p+1}i!(i+p)!}\int_0^1(1-\alpha)^{i+p}\alpha^{\frac{n}{2}+i-1}(D')^{2i}f(\alpha x)d\alpha\\
&+\sum_{i=0}^{\infty}\frac{(-1)^{i+p}(x')^{2i+p+2}}{2^{2i+p+2}(i+1)!(i+p)!}\int_0^1(1-\alpha)^{i+p}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha
\end{aligned}$$

Thus, we have the result. □

5. Normalized System with the Base $f(x)$ and Its Applications

5.1. Normalized System with the Base $f(x)$

In this section, we construct normalized systems with the base $f(x)$ in the setting of umbral Clifford analysis. That is to say, we establish a system of functions $\{G_m(x;f), m=1,2,\cdots\}$ satisfying

$$D'G_m(x;f)=G_{m-1}(x;f),\quad m=1,2,\cdots, \tag{26}$$

where

$$D'G_1(x;f)=G_0(x;f)=f(x).$$

For $m = 2s - 1$,

$$G_{2s-1}(x;f) = \sum_{i=0}^{\infty} \frac{(-1)^{i+1}(x')^{2i+2s-1}}{2^{2i+s}i!(i+s-1)!} \int_0^1 (1-\alpha)^{i+s-1}\alpha^{\frac{n}{2}+i-1}(D')^{2i}f(\alpha x)d\alpha$$
$$+ \sum_{i=0}^{\infty} \frac{(-1)^{i}(x')^{2(i+s)}}{2^{2i+s+1}(i+1)!(i+s-1)!} \int_0^1 (1-\alpha)^{i+s-1}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha. \quad (27)$$

For $m = 2s$,

$$G_{2s}(x;f) = -\sum_{i=0}^{\infty} \frac{(-1)^{i+1}(x')^{2i+2s}}{2^{2i+s+1}i!(i+s)!} \int_0^1 (1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i-1}(D')^{2i}f(\alpha x)d\alpha$$
$$- \sum_{i=0}^{\infty} \frac{(-1)^{i}(x')^{2(i+s)+1}}{2^{2i+s+2}(i+1)!(i+s)!} \int_0^1 (1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha. \quad (28)$$

Theorem 6. *The function system $G_m(x;f)$ is the normalized system with respect to the umbral Dirac operator with the base $f(x)$.*

Proof of Theorem 6. For $m = 1$,

$$\begin{aligned}
D'G_1(x;f) &= \sum_{i=0}^{\infty} \frac{(-1)^{i}(x')^{2i}}{2^{2i}i!i!} E'_{\frac{n}{2}+i} \int_0^1 (1-\alpha)^{i}\alpha^{\frac{n}{2}+i-1}(D')^{2i}f(\alpha x)d\alpha \\
&\quad + \sum_{i=0}^{\infty} \frac{(-1)^{i}(x')^{2i+1}}{2^{2i+1}i!i!} \int_0^1 (1-\alpha)^{i}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha \\
&\quad + \sum_{i=0}^{\infty} \frac{(-1)^{i+1}2(i+1)(x')^{2i+1}}{2^{2(i+1)}(i+1)!i!} \int_0^1 (1-\alpha)^{i}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha \\
&\quad + \sum_{i=0}^{\infty} \frac{(-1)^{i}(x')^{2(i+1)}}{2^{2(i+1)}(i+1)!i!} \int_0^1 (1-\alpha)^{i}\alpha^{\frac{n}{2}+i+1}(D')^{2i+2}f(\alpha x)d\alpha \\
&= f(x) + \sum_{i=1}^{\infty} \frac{(-1)^{i}(x')^{2i}}{2^{2i}i!(i-1)!} \int_0^1 (1-\alpha)^{i-1}\alpha^{\frac{n}{2}+i}(D')^{2i}f(\alpha x)d\alpha \\
&\quad + \sum_{i=0}^{\infty} \frac{(-1)^{i}(x')^{2i+1}}{2^{2i+1}i!i!} \int_0^1 (1-\alpha)^{i}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha \\
&\quad + \sum_{i=0}^{\infty} \frac{(-1)^{i+1}(x')^{2i+1}}{2^{2i+1}i!i!} \int_0^1 (1-\alpha)^{i}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha \\
&\quad + \sum_{i=1}^{\infty} \frac{(-1)^{i+1}(x')^{2i}}{2^{2i}i!(i-1)!} \int_0^1 (1-\alpha)^{i-1}\alpha^{\frac{n}{2}+i}(D')^{2i}f(\alpha x)d\alpha \\
&= f(x).
\end{aligned}$$

For $m = 2s$,

$$
\begin{aligned}
D'G_{2s}(x;f) = &-\sum_{i=0}^{\infty}\frac{(-1)^{i+1}}{2^{2i+s+1}i!(i+s)!}D'[(x')^{2i+2s}\int_0^1(1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i-1}(D')^{2i}f(\alpha x)d\alpha] \\
&-\sum_{i=0}^{\infty}\frac{(-1)^{i}}{2^{2i+s+2}(i+1)!(i+s)!}D'[(x')^{2(i+s)+1}\int_0^1(1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha] \\
&=\sum_{i=0}^{\infty}\frac{(-1)^{i+1}2(i+s)(x')^{2i+2s-1}}{2^{2i+s+1}i!(i+s)!}\int_0^1(1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i-1}(D')^{2i}f(\alpha x)d\alpha \\
&-\sum_{i=1}^{\infty}\frac{(-1)^{i+1}(x')^{2(i+s)}}{2^{2i+s+1}i!(i+s)!}\int_0^1(1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha \\
&+\sum_{i=0}^{\infty}\frac{(-1)^{i}(x')^{2(i+s)}}{2^{2i+s+2}(i+1)!(i+s)!}E'_{\frac{n}{2}+i+s}\int_0^1(1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha \\
&-\sum_{i=0}^{\infty}\frac{(-1)^{i}(x')^{2(i+s)+1}}{2^{2i+s+2}(i+1)!(i+s)!}\int_0^1(1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i+1}(D')^{2i+2}f(\alpha x)d\alpha.
\end{aligned}
\tag{29}
$$

We calculate the first sum and the forth sum in (29).

$$
\begin{aligned}
I_1+I_4 &= \sum_{i=0}^{\infty}\frac{(-1)^{i+1}(x')^{2i+2s-1}}{2^{2i+s}i!(i+s-1)!}\int_0^1(1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i-1}(D')^{2i}f(\alpha x)d\alpha \\
&+\sum_{i=1}^{\infty}\frac{(-1)^{i+1}(x')^{2i+2s-1}}{2^{2i+s}(i+s-1)!i!}\int_0^1(1-\alpha)^{i+s-1}\alpha^{\frac{n}{2}+i}(D')^{2i}f(\alpha x)d\alpha \\
&=\sum_{i=0}^{\infty}\frac{(-1)^{i+1}(x')^{2i+2s-1}}{2^{2i+s}i!(i+s-1)!}\int_0^1(1-\alpha)^{i+s-1}\alpha^{\frac{n}{2}+i-1}(D')^{2i}f(\alpha x)d\alpha.
\end{aligned}
$$

We calculate the second sum and the third sum in (29).

$$
\begin{aligned}
I_2+I_3 = &-\sum_{i=0}^{\infty}\frac{(-1)^{i+1}(x')^{2(i+s)}}{2^{2i+s+1}i!(i+s)!}\int_0^1(1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha \\
&+\sum_{i=0}^{\infty}\frac{(-1)^{i+1}(x')^{2(i+s)}}{2^{2i+s+1}(i+1)!(i+s-1)!}\int_0^1(1-\alpha)^{i+s-1}\alpha^{\frac{n}{2}+i+1}(D')^{2i+1}f(\alpha x)d\alpha \\
&+\sum_{i=0}^{\infty}\frac{(-1)^{i+1}(x')^{2(i+s)}}{2^{2i+s+1}(i+1)!(i+s)!}\int_0^1(1-\alpha)^{i+s}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha \\
&=\sum_{i=0}^{\infty}\frac{(-1)^{i+1}(x')^{2(i+s)}}{2^{2i+s+1}(i+1)!(i+s-1)!}\int_0^1(1-\alpha)^{i+s-1}\alpha^{\frac{n}{2}+i}(D')^{2i+1}f(\alpha x)d\alpha.
\end{aligned}
$$

To sum up, we have the result. □

5.2. Applications of the Normalized System with the Base $f(x)$

Consider the inhomogeneous umbral Dirac equation:

$$(D^{'} + \lambda)g(x) = f(x), \tag{30}$$

where $f(x) \in C(\Omega_0, Cl_{0,n})$ and $\lambda \in R$.

Theorem 7. *Let $f(x) \in C(\Omega_0, Cl_{0,n})$. Then, the solution of Equation (30) is given by*

$$\begin{aligned}
g(x) = &\sum_{l=0}^{\infty}\sum_{i=0}^{l} \frac{(-1)^{i+1}\lambda^{l-i}(x^{'})^{2l+1}}{2^{l+i+1}i!l!} \int_0^1 (1-\alpha)^l \alpha^{\frac{n}{2}+i-1}(D^{'})^{2i} f(\alpha x)d\alpha \\
&+ \sum_{l=0}^{\infty}\sum_{i=0}^{l} \frac{(-1)^{i}\lambda^{l-i}(x^{'})^{2(l+1)}}{2^{l+i+2}(i+1)!l!} \int_0^1 (1-\alpha)^l \alpha^{\frac{n}{2}+i}(D^{'})^{2i+1} f(\alpha x)d\alpha \\
&- \sum_{l=0}^{\infty}\sum_{i=0}^{l} \frac{(-1)^{i+1}\lambda^{l-i+1}(x^{'})^{2l+2}}{2^{l+i+2}i!(l+1)!} \int_0^1 (1-\alpha)^{l+1} \alpha^{\frac{n}{2}+i-1}(D^{'})^{2i} f(\alpha x)d\alpha \\
&- \sum_{l=0}^{\infty}\sum_{i=0}^{l} \frac{(-1)^{i}\lambda^{l-i+1}(x^{'})^{2(l+1)+1}}{2^{l+i+2}(i+1)!l!} \int_0^1 (1-\alpha)^{l+1} \alpha^{\frac{n}{2}+i}(D^{'})^{2i+1} f(\alpha x)d\alpha.
\end{aligned} \tag{31}$$

Proof of Theorem 7. From Theorem 6, we have

$$\begin{aligned}
g(x) = &\sum_{s=1}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i+1}\lambda^{s-1}(x^{'})^{2i+2s-1}}{2^{2i+s}i!(i+s-1)!} \int_0^1 (1-\alpha)^{i+s-1} \alpha^{\frac{n}{2}+i-1}(D^{'})^{2i} f(\alpha x)d\alpha \\
&+ \sum_{s=1}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i}\lambda^{s-1}(x^{'})^{2(i+s)}}{2^{2i+s+1}(i+1)!(i+s-1)!} \int_0^1 (1-\alpha)^{i+s-1} \alpha^{\frac{n}{2}+i}(D^{'})^{2i+1} f(\alpha x)d\alpha \\
&- \sum_{s=1}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i+1}\lambda^{s}(x^{'})^{2i+2s}}{2^{2i+s+1}i!(i+s)!} \int_0^1 (1-\alpha)^{i+s} \alpha^{\frac{n}{2}+i-1}(D^{'})^{2i} f(\alpha x)d\alpha \\
&- \sum_{s=1}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i}\lambda^{s}(x^{'})^{2(i+s)+1}}{2^{2i+s+2}(i+1)!(i+s)!} \int_0^1 (1-\alpha)^{i+s} \alpha^{\frac{n}{2}+i}(D^{'})^{2i+1} f(\alpha x)d\alpha.
\end{aligned}$$

By changing the summation index as $s \to s+1$, we have

$$\begin{aligned}
g(x) = &\sum_{s=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i+1}\lambda^{s}(x^{'})^{2i+2s+1}}{2^{2i+s+1}i!(i+s)!} \int_0^1 (1-\alpha)^{i+s} \alpha^{\frac{n}{2}+i-1}(D^{'})^{2i} f(\alpha x)d\alpha \\
&+ \sum_{s=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i}\lambda^{s}(x^{'})^{2(i+s+1)}}{2^{2i+s+2}(i+1)!\cdot(i+s)!} \int_0^1 (1-\alpha)^{i+s} \alpha^{\frac{n}{2}+i}(D^{'})^{2i+1} f(\alpha x)d\alpha \\
&- \sum_{s=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i+1}\lambda^{s+1}(x^{'})^{2i+2s+2}}{2^{2i+s+2}i!(i+s+1)!} \int_0^1 (1-\alpha)^{i+s+1} \alpha^{\frac{n}{2}+i-1}(D^{'})^{2i} f(\alpha x)d\alpha \\
&- \sum_{s=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i}\lambda^{s+1}(x^{'})^{2(i+s+1)+1}}{2^{2i+s+2}(i+1)!(i+s)!} \int_0^1 (1-\alpha)^{i+s+1} \alpha^{\frac{n}{2}+i}(D^{'})^{2i+1} f(\alpha x)d\alpha
\end{aligned}$$

Note that $\sum_{s,i=0}^{\infty} = \sum_{l=0}^{\infty} \sum_{s+i=l} = \sum_{l=0}^{\infty} \sum_{i=0}^{l}$. Then, we have

$$g(x) = \sum_{l=0}^{\infty} \sum_{i=0}^{l} \frac{(-1)^{i+1} \lambda^{l-i} (x')^{2l+1}}{2^{l+i+1} i! l!} \int_0^1 (1-\alpha)^l \alpha^{\frac{n}{2}+i-1} (D')^{2i} f(\alpha x) d\alpha$$

$$+ \sum_{l=0}^{\infty} \sum_{i=0}^{l} \frac{(-1)^{i} \lambda^{l-i} (x')^{2(l+1)}}{2^{l+i+2} (i+1)! l!} \int_0^1 (1-\alpha)^l \alpha^{\frac{n}{2}+i} (D')^{2i+1} f(\alpha x) d\alpha$$

$$- \sum_{l=0}^{\infty} \sum_{i=0}^{l} \frac{(-1)^{i+1} \lambda^{l-i+1} (x')^{2l+2}}{2^{l+i+2} i! (l+1)!} \int_0^1 (1-\alpha)^{l+1} \alpha^{\frac{n}{2}+i-1} (D')^{2i} f(\alpha x) d\alpha$$

$$- \sum_{l=0}^{\infty} \sum_{i=0}^{l} \frac{(-1)^{i} \lambda^{l-i+1} (x')^{2(l+1)+1}}{2^{l+i+2} (i+1)! l!} \int_0^1 (1-\alpha)^{l+1} \alpha^{\frac{n}{2}+i} (D')^{2i+1} f(\alpha x) d\alpha.$$

This completes the proof. □

6. Conclusions

The method of the normalized systems of functions is devoted to the construction of the solutions of initial- and boundary-value problems for real-valued partial differential equations. In this paper, applying the Sheffer operator, we constructed the normalized systems of functions to study Clifford-valued partial differential equations in the frame of umbral calculus. Umbral Clifford analysis based on umbral calculus is a bridge between continuous and discrete Clifford analysis. One may further bring this method to the field of discrete Clifford analysis.

Author Contributions: Writing—original draft, H.Y.; writing—review and editing, V.K. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by the Natural Science Foundation of Hebei Province (No. A2022402007), the National Natural Science Foundation of China (No. 11426082), and the Project of Handan Municipal Science and Technology Bureau (No. 19422031008-13).

Data Availability Statement: All the data is present within the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Riordan, J. *An Introduction to Combinatorial Analysis*; Wiley: New York, NY, USA, 1958; pp. 123–176.
2. Bell, E.T. The history of Blissards symbolic calculus, with a sketch of the inventors life. *Amer. Math. Mon.* **1938**, *45*, 414–421. [CrossRef]
3. Sheffer I.M. Some properties of polynomial sets of type zero. *Duke Math. J.* **1939**, *5*, 590–622. [CrossRef]
4. Pincherle S.; Amalde U. *Le Operazioni Distributive e le loro Applicazioni al Analisi*; Zanichelli: Bologana, Italy, 1901.
5. Mullin, R.; Rota G.C. *On the Foundations of Combinatorial Theory. III. Theory of Binomial Enumeration, Graph Theory and Its Applications*; Academic Press: New York, NY, USA, 1970; pp. 167–213.
6. Di Bucchianico A.; Loeb, D.E.; Rota, G.C. Umbral Calculus in Hilbert Space . In *Mathematical Essays in Honor of Gian- Carlo Rota*; Sagan, B., Stanley, R.P., Eds.; Birkhuser: Boston, MA, USA, 1998; pp. 213–238.
7. Ren G.; Faustino N. Almansi theorems in umbral Clifford analysis and the quantum harmonic oscillator. *Mathematic* **2009**, *1*, 1–22.
8. Faustino N.; Ren G. Discrete Almansi type decompositions: An umbral calculus framework based on osp(1|2) symmetries. *Mathematics* **2011**, 2, 1–26. [CrossRef]
9. Brackx, F.; Delanghe, R.; Sommen F. *Clifford Analysis*; Pitman Advanced Publishing Program: London, UK, 1982.
10. Delanghe, R.; Sommen, F.; Soucek, V. *Clifford Algebra and Spinor-Valued Functions*; Kluwer Acad. Publ.: Amsterdam, The Netherlands, 1992.
11. Gilbert, J.; Murray M. *Clifford Algebra and Dirac Operators in Harmonic Analysis*; Cambridge University Press: Cambridge, UK, 1991.
12. Gurlebeck,K.; Sprossig, W. *Quaternionic Analysis and Elliptic Boundary Value Problems*; Akademie-Verlag: Berlin, Germany, 1989.
13. Roman, S. *The Umbral Calculus*; Academic Press: San Diego, CA, USA, 1984.
14. Roman, S.; Rota, G.C. The umbral calculus. *Adv. Math.* **1978**, *27*, 95–188. [CrossRef]
15. Rota, G.-C.; Kahane, D.; Odlyzko, A. Finite operator calculus. *J. Math. Anal. Appl.* **1973**, *42*, 685–760. [CrossRef]

16. Karachik, V.V. Normalized system of functions with respect to the Laplace operator and its applications. *J. Math. Anal. Appl.* **2003**, *287*, 577–592. [CrossRef]
17. Karachik, V.V. *Method of Normalized Systems of Functions*; Publishing Center of SUSU: Chelaybinsk, Russia, 2014. (In Russian)
18. Karachik, V.V. Dirichlet and Neumann boundary value problems for the polyharmonic equation in the unit ball. *Mathematics* **2021**, *9*, 1907. [CrossRef]
19. Aronszajn, N.; Creese, T.M.; Lipkin, L.J. *Polyharmonic Functions*; Oxfordmathematics Monographs; The Clarendon Press: New York, NY, USA; Oxford University Press: Oxford, UK, 1983.