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# A Method of Qualitative Analysis for Determining Monotonic Stability Regions of Particular Solutions of Differential Equations of Dynamic Systems

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**Abstract:** Developing stability analysis methods for modern dynamical system solutions has been a significant challenge in the field. This study aims to formulate a qualitative analysis approach for the monotone stability region of a specific solution to a single differential equation within a dynamical system. The system in question comprises two first-order nonlinear ordinary differential equations of a particular kind. The method proposed hinges on applying elements of combinatorics to the traditional mathematical investigation of a function with a single independent variable. This approach enables the exact determination of the different qualitative scenarios in which the desired solution changes, under the assumption that the function values monotonically diminish from a specified value down to a finite zero. This paper outlines the creation and decomposition of the monotone stability region associated with the solution under consideration.

**Keywords:** monotone stability; 2-function; stability region; particular solution; combinatorics

**MSC:** 34D20

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## 1. Introduction

Maintaining the stability of solutions in contemporary nonlinear dynamical systems has been a major challenge in their operation [1–4]. Conventionally [5–17], Lyapunov's second method is used to study the stability and optimization of solutions in systems composed of ordinary differential equations. For instance, in [5], Lyapunov's stability theory is employed to establish the global asymptotic stability of the periodic solution in a recognized ecosystem model. Paper [6] suggests a neural network controller for an adaptive radial basis, predicated on the use of Lyapunov's stability theory in uncertain fractional-order chaotic systems with varying time delays. Meanwhile, in [7], the Lyapunov approach is adopted to demonstrate the stability of control logic in a proposed unmanned vehicle.

In modeling and studying the operational patterns of damping devices, various methods are employed to analyze the stability of solutions to differential equations. It is important to mention that contemporary differential equation theory introduces the notions of strong and weak solution stability. These concepts allow for the categorization of various types of stable solution behaviors. Additionally, the concepts of strong and weak resonances are employed in solving certain types of mathematical physics equations, as explored in papers [18,19]. Notably, the external and internal stability analyses of resonances in perturbed motion of solid bodies theory are conducted independently. More specifically, the internal stability of resonances is examined in studies [20,21]. These referenced works provide an analysis of the stability of oscillations of coupled oscillators in the small vicinity of resonances.

In contemporary aviation and space technology, ensuring motion stability is often of paramount importance. Traditionally, Lyapunov's second method is frequently employed in this field. For instance, in [22], Lyapunov optimization is utilized in modeling the distribution of reception tasks among satellites in a constellation. In another example, [23], Lyapunov's second method is used to analyze the external stability of the primary resonance in the problem of a spacecraft's descent with slight asymmetry in Mars' atmosphere.

However, the practical implementation of Lyapunov's method often encounters complexities during the Lyapunov function selection procedure. It is worth noting that stability analysis in nonlinear differential equations can be built upon the properties of monotonically decreasing solutions of these equations [24–26]. Specifically, paper [24] presents a study of the external monotonic stability of average resonance in a perturbed dynamical system with one fast and one slow variable without employing Lyapunov's second method. In the paper [24], sufficient conditions for external stability of resonance were obtained while preserving the signs of the analyzed derivatives on the right-hand side of the differential equations throughout a nonresonant interval of independent variable change. The goal of [25] was to devise an asymptotic method to investigate the nonlinear monotonic stability of the amplitude of plane oscillations in the problem of a symmetric spacecraft's descent in Mars' atmosphere. This method was predicated on the use of the arbitrary constant variation method, the averaging method, and the classical method of mathematical investigation of a function with one independent variable. Paper [26] developed a method for scrutinizing the nonlinear monotonic simultaneous stability of solutions in a system of two ordinary autonomous differential equations. The approach employed the small parameter method, insights from combinatorics, and the classical method of mathematical investigation of functions with two variables. This method is applied in the paper to the problem of analyzing the nonlinear monotonic simultaneous stability of the unguided motion of an asymmetric spacecraft relative to its center of mass during descent in Mars' atmosphere by angles of attack and sideslip. In general, conducting a simultaneous examination of the signs of the first and second derivatives of the variables of interest, which retain their signs over the considered intervals of the independent variable's change, proves to be a fruitful approach in the stability investigation of these variables [24–26].

## 2. Calculation of the Number of Qualitatively Different Cases of Monotonic Stability

Consider a dynamical system written in the form of the following ordinary differential equations:

$$\frac{dy}{dt} = f(g(t)), \tag{1}$$

$$\frac{dg}{dt} = u(g(t)), \tag{2}$$

where $y = y(t) \geq 0$ is a non-negative and twice continuously differentiable function, which serves as a particular solution to Equation (1), defined within interval $t \in [t_0, t_1]$; $f(g(t)) \leq 0$ is a known nonpositive and continuously differentiable 2-function, defined within interval $t \in [t_0, t_1]$, where $g(t)$ is a continuously differentiable function, presenting a particular solution to Equation (2), defined within interval $t \in [t_0, t_1]$, $u(g(t))$ is a known continuous 2-function, defined, and of constant sign within the interval $t \in [t_0, t_1]$, t is an independent real variable, $t \in [0, +\infty)$.

Let us formulate the notion of monotone stability for a particular solution $y = y(t)$ of Equation (1) within the dynamical system (1)–(2).

Assume that for a non-negative solution $y = y(t)$ of system (1)–(2) the following conditions are satisfied on interval $t \in [t_0, t_1]$ (excluding the point g = 0):

(i). the function $y = y(t)$ are defined and twice continuously differentiable;

(ii). the derivative $\frac{d^2y}{dt^2}$ retains its strong signs consistently within intervals between inflection points $k = 0, 1, 2, \ldots, m,\ m < \infty$ or the derivative $\frac{d^2y}{dt^2} = 0$ ($\forall t \in [t_0, t_1]$).

**Definition 1.** *If a non-negative solution* $y = y(t)$ *of system (1)–(2) satisfies conditions (i)–(ii), and the solution decreases monotonically on interval*$t \in [t_0, t_1]$, *it is termed a monotone stable solution with respect to variable* $y(t)$ *on this interval.*

Let us examine a theorem outlining the sufficient condition for the monotonic stability of the solution $y = y(t)$ of Equation (1).

**Theorem 1.** *(Sufficient condition of the monotonic stability). If a non-negative solution* $y = y(t)$ *of system (1)–(2) satisfies conditions (i)–(ii), and the derivative* $\frac{dy(t)}{dt} < 0$ *on interval* $t \in [t_0, t_1]$, *then the solution is monotone stable within this interval.*

**Proof of Theorem 1.** As per Definition 1, the non-negative function $y = y(t)$ has a continuous first derivative on interval $t \in [t_0, t_1]$. If the derivative is negative, i.e., $\frac{dy(t)}{dt} < 0$, then the function $y = y(t)$ decreases on interval $t \in [t_0, t_1]$. This follows from the fundamental sufficient condition for a real variable function to decrease. Hence, the non-negative solution $y = y(t)$ of system (1)–(2) satisfies conditions (i)–(ii) and the solution decreases monotonically on interval $t \in [t_0, t_1]$. Consequently, according to Definition 1, the solution $y = y(t)$ is monotone stable on interval $t \in [t_0, t_1]$.

The theorem has been proven. □

Note. The phrase "qualitatively different cases of monotonic stability of solutions" pertains to monotone stable solutions of Equation (1) that exhibit a unique type of convexity compared to other solutions exhibiting monotonic stability.

Further analysis of monotone stability will utilize expressions of the first and second derivatives of the solution $y = y(t)$. For the count of qualitatively different cases of monotonic stability of solution $y = y(t)$, the following theorem holds.

**Theorem 2.** *If a particular solution* $y = y(t)$ *of Equation (1) from the system Equation (1)–(2) satisfies Definition 1, and the number of inflection points of the solution is 0, 1, 2, . . . , m, then the total number of qualitatively different cases of monotonic stability of the solution equals* $({}_{2m+3}C_1)$.

**Proof of Theorem 2.** We shall determine the count of all distinct cases of monotone stability for a particular solution $y = y(t)$ of Equation (1). Initially, we identifiy the number of distinct cases of monotonic stability in the absence of inflection points for the function $y = y(t)$ on the considered interval $[t_0, t_1]$. There are three such cases. Indeed, in the first linear case $\forall t \in [t_0, t_1]$, the derivatives of the particular solution have the following signs: $\frac{dy}{dt} < 0$ and $\frac{d^2y}{dt^2} = 0$. In the second case, the derivatives of the particular solution exhibit the signs: $\frac{dy}{dt} < 0$ and $\frac{d^2y}{dt^2} > 0$. In the third case, the derivatives of the particular solution have the signs: $\frac{dy}{dt} < 0$ and $\frac{d^2y}{dt^2} < 0$. Next, imagine that the function $y = y(t)$ has one inflection point within the interval $[t_0, t_1]$. Function $y = y(t)$ is continuously differentiable. Consequently, when passing through this point, a change in the nature of the convexity of the function's graph is observed. Adding one inflection point leads to the formation of a new interval with the constant convexity of two possible types of the function $y = y(t)$. As a result, we derive two new qualitatively distinct cases of monotonic stability, differentiated by the sign of the second derivative for all t belonging to the new interval. The formation of each subsequent inflection point also introduces the potential for two more qualitatively different cases of monotonic stability, and so on. In conclusion, for m inflection points, the number of qualitatively distinct cases of monotonic stability is ascertained by the following equality: ${}_{2m+3}C_1 = 2m + 3$.

The theorem has been proven. □

Let us consider an example of applying the obtained equality ${}_{2m+3}C_1 = 2m + 3$.

Sample. Determine the number of qualitatively different cases of monotonic stability given the presence of 1, 2, 3, and 4 inflection points on the solution curve within the specified interval.

Solution. According to Theorem 2, the number of qualitatively different cases of monotonic stability is obtained from the equation ${}_{2m+3}C_1 = 2m + 3$, where m is the number of the inflection points within the given interval. Applying this equation, we find the desired number of qualitatively different cases of monotonic stability: ${}_5C_1 = 2 + 3 = 5$ (for m = 1), ${}_7C_1 = 4 + 3 = 7$ (for m = 2), ${}_9C_1 = 6 + 3 = 9$ (for m = 3), and ${}_{11}C_1 = 8 + 3 = 11$ (for m = 4).

Note. In line with [26], for each characteristic case of monotonic stability of the particular solution $y = y(t)$, where the conditions $\frac{dy}{dt} < 0$; $\frac{d^2y}{dt^2} > 0$ and ($\frac{dy}{dt} < 0$; $\frac{d^2y}{dt^2} < 0$) are fulfilled within each interval of independent variable t change with a consistent convexity type, there exists another corresponding characteristic case of monotonic stability of the particular solution $\tilde{y} = \tilde{y}(t)$, for which the conditions $\frac{d\tilde{y}}{dt} < 0$; $\frac{d^2\tilde{y}}{dt^2} < 0$ and ($\frac{d\tilde{y}}{dt} < 0$; $\frac{d^2\tilde{y}}{dt^2} > 0$) are met within each similar interval with an unchanging convexity type. These two qualitatively different cases of monotone stability differ in the sign of the second derivatives, while preserving the sign of their first derivatives. Hence, they exhibit opposite convexity types within the same intervals of change in the independent variable. In this regard, we should mention a pairwise symmetry of characteristic cases of monotonic stability with the same number of inflection points, differing solely in the convexity type within the same intervals of independent variable variation.

Next, we will examine a method for qualitative analysis of the monotonic stability of the particular solution $y = y(t)$ of Equation (1).

**Definition 2.** *The "qualitative analysis of the monotonic stability of the solution* $y = y(t)$ *of Equation (1) within the interval* $t \in [t_0, t_1]$ *" refers to the study of convexity of a given strictly monotonically decreasing solution within this interval.*

The following theorem is established.

**Theorem 3.** *For a qualitative analysis of the monotonic stability of an unknown particular non-negative solution* $y = y(t)$ *of Equation (1), the following conditions must be satisfied:*

*(i)* *the particular solution* $y = y(t)$ *fulfills Definition 1 within the interval* $t \in [t_0, t_1]$ *and the solution is not linear;*

*(ii)* *the derivative* $\frac{dg(t)}{dt}$ *of the known continuously differentiable function* $g(t)$ *is defined within the interval* $t \in [t_0, t_1]$ *and maintains the same sign within this interval;*

*(iii)* *the initial condition* $y(0) > 0$*, the initial value* $g(0)$*, and the final value* $g(t_1)$ *are known.*

**Proof of Theorem 3.** Let us assume that the conditions of Definition 1 are met and that the solution $y = y(t)$ is not linear. Obviously, the function $y = y(t)$ decreases monotonically within the interval $t \in [t_0, t_1]$. The second derivative of this function $\frac{d^2y}{dt^2}$ either keeps its positive (or negative) sign or switches to its opposite only at a finite number of inflection points of the function $y = y(t)$ within the interval $t \in [t_0, t_1]$. Note that the function $y = y(t)$ maybe unknown. Let us formulate a method for analyzing the sign of the second derivative $\frac{d^2y}{dt^2}$ of the function $y = y(t)$ within the interval $t \in [t_0, t_1]$ for the above-mentioned case. We will express the second derivative $\frac{d^2y}{dt^2}$ of the function $y = y(t)$. As the function $f(g(t))$ is a twice differentiable function, we have:

$$\frac{d^2y}{dt^2} = \frac{df}{dg}\frac{dg}{dt}. \tag{3}$$

According to system (1)–(2), the sign of the derivative $\frac{dg}{dt}$ is known and remains unchanged $\forall t \in [t_0, t_1]$. Therefore, to determine the sign of the second derivative $\frac{d^2y}{dt^2}$ of the function $y = y(t)$ for all t in the interval $[t_0, t_1]$, we need to determine the sign of the derivative $\frac{df}{dg}$ for all t over the interval. It is important to note that the derivative $\frac{df}{dg}$ is a known smooth function $\frac{df}{dg} = F(g)$. Note, that the argument g of the function $F(g)$ changes strictly monotonically $\forall t \in [t_0, t_1]$. In this case, if the initial value $g(0)$ is known, then the sign of the derivative $\frac{df}{dg} = F(g)$ can be determined by directly calculating the values of the function $F(g)$ for all g within the given interval $[g(0), g_1]$. This provides a method for analyzing the sign of the second derivative $\frac{d^2y}{dt^2}$ of the function $y = y(t)$ over the interval $t \in [t_0, t_1]$. With the sign of the second derivative $\frac{d^2y}{dt^2}$ of the function $y = y(t)$ determined over the interval $t \in [t_0, t_1]$, we can also determine the type of convexity of the strictly decreasing solution $y = y(t)$ at each point $t \in [t_0, t_1]$. Thus, according to Definition 2, we have completed a qualitative analysis of the monotonic stability of the solution $y = y(t)$ of Equation (1), which is strictly decreasing within interval $t \in [t_0, t_1]$.

Therefore, the proof of Theorem 3 is completed. □

Note. Theorem 3 forms the mathematical basis for the method of analyzing the monotonic stability of solutions to Equation (1), a method that does not require direct calculation of these solutions.

Note. The proof of Theorem 3 is constructive, in that it describes all steps to be taken when using the method for analyzing monotonic stability of the solutions in question.

## 3. Construction and Decomposition of a Monotonic Stability Region

Assume that there are $({}_{2m+3}C_1)$ qualitatively distinct particular solutions of Equation (1) that satisfy the monotonic stability conditions outlined in Definition 1, Theorem 1, and Theorem 2. The goal is to identify the boundaries of the monotonic stability region for these solutions in the (t,y)-coordinate plane. In this context, the following theorem is applicable.

**Theorem 4.** *If all non-negative particular solutions of Equation (1) meet the requirements of Definition 1, then the boundaries of the monotonic stability region of these solutions* $y = y(t)$ *form a rectangle in the first quadrant of the coordinate system (t,y), with two sides lying along the coordinate axes.*

**Proof of Theorem 4.** The first derivative of any non-negative particular solution of Equation (1) that satisfies both Definition 1 and Theorem 1 equals $\frac{dy}{dt} = \frac{dy}{dg}\frac{dg}{dt} < 0$. Therefore, two distinct cases, in which the condition of the negativity of the first derivative $\frac{dy}{dt}$ is satisfied but the signs of the second derivative $\frac{d^2y}{dt^2}$ differ and need to be considered.

Case 1. Assume that for all $t \in [t_0, t_1]$, the following initial and final conditions are satisfied simultaneously: $\frac{dy}{dg} < 0$ and $\frac{dg}{dt} > 0$. Additionally, suppose the following initial and final conditions are $t_0 = 0$, $y_0 > 0$ and $t_1 > 0$, $y_1 = 0$, respectively. Now, let us transition from Cartesian coordinates (t,y) to polar coordinates (ρ,φ), where $\rho = \sqrt{t^2 + y^2}$, $tg\varphi = y/t$. Consequently, at the initial point t = 0, we get: $\rho_0 = y_0$. As we approach the limit (at $t \to 0 + 0$) and we find $\max\varphi = \varphi_0 = arctg(+\infty) = \pi/2$. At the endpoint, we get: $\rho_1 = t_1$, $\min\varphi = \varphi_1 = arctg(0) = 0$. Hence, as the angle $\phi$ decreases from $\pi/2$ to zero, it successively assumes all values in the first quadrant. The monotonic stability region of the solution $y = y(t)$ is constrained to the initial value $t_0 = 0$, the final value $t_1 > 0$, the initial value $y_0 > 0$, and the final value $y_1 = 0$. Note that the final value $t_1 > 0$ and the initial value $y_0 > 0$ limit the coordinate values (t,y) from above. Additionally, the initial value $t_0 = 0$ and the final value $y_1 = 0$ limit the coordinate values (t,y) from below. Therefore, the boundaries of the monotonic stability region of the particular solutions $y = y(t)$ of

Equation (1) form a rectangle on the plane (t,y), with vertices at the following points: (0,0), ($t_1$,0), (0,$y_0$), and ($t_1$,$y_0$).

Case 2. Assume that for all $t \in [t_0, t_1]$, the following initial and final conditions are simultaneously met: $\frac{dy}{dg} < 0$ and $\frac{dg}{dt} < 0$. The proof of Case 2 follows the same procedure as Case 1.

Therefore, under both possible scenarios, the boundaries of the monotonic stability region of the solution $y = y(t)$ form a rectangle contained in the first quadrant of the coordinate system (t,y), with two sides along the coordinate axes.

This concludes the proof of the theorem. □

Let us say that the particular solution $y = y(t)$ of Equation (1) satisfies Definition 1 and Theorem 1, and also has n ($n \neq 0$) inflection points in the interval $[t_0, t_1]$. With this premise, the following theorem about the decomposition of the monotonic stability interval $[t_0, t_1]$ of the function $y = y(t)$ stands:

**Theorem 5.** *Suppose that the twice smooth function* $y = y(t)$ *strictly decreases in the interval* $[t_0, t_1]$ *from point* $(t_0, y_0)$*, where* $t_0 = 0, y_0 > 0$*, all the way to point* $(t_1, y_1)$*, where* $t_1 > 0, y_1 = 0$*, and this function has n (n ≠ 0) inflection points within this interval. In this case, the inflection points divide the interval of decreasing* $[t_0, t_1]$ *into n + 1 segments, each with a constant type of convexity for the function* $y = y(t)$.

**Proof of Theorem 5.** According to the necessary condition for the existence of an inflection point, if the function $y = y(t)$ has a continuous second derivative $\frac{d^2y}{dt^2}$ at inflection points, then this second derivative is zero, $\frac{d^2y}{dt^2} = 0$ at these points. As the function $\frac{d^2y}{dt^2}$ is continuous in the segment $[t_0, t_1]$, it retains its sign in segments between every pair of inflection points, or in segments between the boundary point ($t_0$ or $t_1$) and the nearest inflection point. Consider one of these intervals $[t_a, t_b]$, which is part of the segment $[t_0, t_1]$. Based on the sufficient condition for convexity, if the function $y = y(t)$ is twice continuously differentiable at all points in the interval $[t_a, t_b]$ and has a constant second derivative $\frac{d^2y}{dt^2}$ with an unchanging sign throughout the segment, and also if the second derivative $\frac{d^2y}{dt^2}$ is positive (negative) $\forall t \in [t_a, t_b]$, then the function $y = y(t)$ is convex downwards (upwards) at all points of the segment. If there are n inflection points in the interval $[t_0, t_1]$, then n + 1 segments are created, each with an unchanging type of convexity for the function $y = y(t)$.

The theorem is thus proven. □

Next, let us examine the theorem regarding the decomposition of a plane region under the curve $y = y(t)$, which meets the requirements of Definition 1 and Theorem 1 and 2.

**Theorem 6.** *If a non-negative and twice continuously differentiable function* $y = y(t)$ *strictly decreases in the interval* $[t_0, t_1]$ *from point* $(t_0, y_0)$*, where* $t_0 = 0, y_0 > 0$*, to point* $(t_1, y_1)$*, where* $t_1 > 0, y_1 = 0$*, and the function has n (n ≠ 0) inflection points within this interval, then these inflection points divide the planar region between the curve* $y = y(t)$ *and the x-axis into n curvilinear trapezoids (over the first n segments) and one curvilinear triangle (over the last n + 1 segment).*

**Proof of Theorem 6.** Consider the twice continuously differentiable function that decreases monotonically in the interval $[t_0, t_1]$, which we discussed in Theorem 2, that has n inflection points within this interval. Therefore, the function has n + 1 segments with an unchanging type of convexity in the interval $[t_0, t_1]$. It is evident that the first n intervals divide the planar region between the curve $y = y(t)$ and the $x$-axis into n curvilinear trapezoids. In fact, each of these intervals forms a curvilinear trapezoid that has an upper curvilinear side, a lower side formed by a segment of the $x$-axis, and two vertical parallel sides formed by linear intervals of nonzero lengths, equivalent to the values of the function $y = y(t)$ at points t that define the boundaries of this segment of the *x*-axis. At the last n + 1 interval, at the endpoint $(t_1, y_1)$, the equation $y_1 = 0$ is fulfilled, and the penultimate point of the interval $[t_0, t_1]$ has $y \neq 0$. Hence, in the last n + 1 interval between the curve $y = y(t)$ and the *x*-axis, we get a curvilinear triangle. Therefore, if the function $y = y(t)$ strictly decreases in the interval $[t_0, t_1]$ from point $(t_0, y_0)$, where $t_0 = 0, y_0 > 0$, to point $(t_1, y_1)$, where $t_1 > 0, y_1 = 0$, and has n ($n \neq 0$) inflection points within this interval, then these inflection points divide the planar region between the curve $y = y(t)$ and the *x*-axis into n curvilinear trapezoids (over the first n intervals) and one curvilinear triangle-shaped region (over the last n + 1 interval).

This concludes the proof of the theorem. □

Note. Let us consider the twice smooth monotonically decreasing function $y = y(t)$, defined in the interval $[t_0, t_1]$ and having $[t_0, t_1]$ n ($n \neq 0$) inflection points within this interval, similar to our consideration in Theorem 6, as the majorant for several twice smooth monotonically decreasing non-negative functions $y_i = y_i(t)$ also defined within this interval (i.e., the conditions $y(t) > y_i(t)$, i = 1, m, $m < \infty$ are met). Then, the general boundary of the stability region of the decreasing functions $y(t), y_i(t)$ contains n curvilinear trapezoids (over the first n intervals) and one curvilinear triangle-shaped region (over the last n + 1 interval).

## 4. Comparison of the Monotonic Stability Analysis Method and the Second Lyapunov Method

Let us analyze the stability of a particular solution $y = y(t)$ of system (1)–(2) using the second Lyapunov method. For this purpose, we introduce the function $W = f^2(g(t))/2$, which serves as the Lyapunov function. Indeed, the function $f(g(t))$ is definitely negative on the interval $t \in [t_0, t_1]$. Therefore, the function $W = f^2(g(t))/2$ is a definitely positive function and can be considered as the Lyapunov function on this interval $t \in [t_0, t_1]$. According to the second Lyapunov method, the stability condition for the solution $y = y(t)$ of system (1)–(2) on the interval $t \in [t_0, t_1]$ is written as: $\frac{dW}{dt} = f(t)\frac{df(t)}{dt} \leq 0$. In the system (1)–(2) under consideration, the following condition is satisfied: $\frac{dy}{dt} = f(t) \leq 0$. Therefore, in order to fulfill the stability condition $\frac{dW}{dt} = f(t)\frac{df(t)}{dt} \leq 0$, it is required that the condition $\frac{d^2y}{dt^2} = \frac{df(t)}{dt} > 0$ be satisfied across the entire interval $t \in [t_0, t_1]$. The only exception is the point g = 0, where $\frac{d^2y}{dt^2} = 0$. Indeed, a negative definite function $\frac{dy}{dt} = f(g(t))$ has a maximum at the point g = 0.

Thus, the stability condition of a particular solution $y = y(t)$ of system (1)–(2) on the interval $t \in [t_0, t_1]$ according to the Lyapunov method can be written in the following form: $\frac{dy}{dt} = f(g(t)) < 0$, $\frac{d^2y}{dt^2} = \frac{df(g(t))}{dt} > 0$ (for $g(t) \neq 0$), or $\frac{dy}{dt} = f(g(t)) = 0, \frac{d^2y}{dt^2} = \frac{df(g(t))}{dt} = 0$ (for $g = 0$).

Considering these conditions in comparison with the conditions in Definition 1 and Theorems 1–6, we can draw the following conclusions: The method of studying monotonic stability described in this paper, such as the second Lyapunov method, allows us to analyze the stability of solutions to nonlinear equations. Unlike the second Lyapunov method, this method does allow for the analysis of the stability of solutions with various convexity types. The primary condition for the existence of solutions in this method is that the first derivative of the desired solution is nonpositive. Unlike the second Lyapunov method, this method facilitates a qualitative analysis of the nonlinear behavior of solutions. An additional distinctive feature of this method is the construction of a region of monotonic stability. Moreover, the monotonic stability analysis method offers the possibility of decomposition and analysis of the stability region. However, the method described in the work is developed solely for dynamical systems comprising two first-order ordinary differential equations. In this case, the right-hand sides of these equations must retain their signs over the considered intervals of the independent variable change. It should be highlighted that the second Lyapunov method is applicable to a dynamical system comprising numerous differential equations. In this sense, the Lyapunov method represents a more general method of stability analysis.

## 5. Example of Monotonic Stability in One Variable in Some Dynamic System

Let us consider an example of the application of the method of analysis of monotone stability in one variable for an integrable dynamical system. In this case, we will construct regions of monotonic stability with respect to the variable under study. In this example, we consider a system of two differential equations of the following form:

$$\frac{dy}{dt} = -\arctan(x), \tag{4}$$

$$\frac{dx}{dt} = 2\sqrt{x}. \tag{5}$$

Here, $y(t) \geq 0$ is the non-negative particular solution, defined in the interval $t \in [t_0, t_1]$; $x(t) > 0$ is the strictly positive particular solution, defined in the interval $t \in [t_0, t_1]$. Note that the solution $y(t)$ satisfies Definition 1 and Theorem 1.

Equation (5) is a differential equation with separable variables. It has the following particular solution:

$$x(t) = (t-3)^2. \tag{6}$$

Substituting solution (6) into Equation (4), we obtain a differential equation with separated variables:

$$dy = -\arctan((t-3)^2)dt. \tag{7}$$

As a result of finding indefinite intervals from the right and left sides of Equation (7), we obtain the general solution of this equation:

$$\begin{aligned} y(t) = {} & \frac{\ln\left((t-3)\left(t-\sqrt{2}-3\right)+1\right)-\ln\left((t-3)\left(t+\sqrt{2}-3\right)+1\right)}{2^{\frac{3}{2}}} \\ & -\arctan\left((t-3)^2\right)(t-3) + \frac{\arctan\left(\sqrt{2}(t-3)+1\right)+\arctan\left(\sqrt{2}(t-3)-1\right)}{\sqrt{2}} + C. \end{aligned} \tag{8}$$

Let us analyze differential Equation (7) and its particular solution (8) together. The right-hand side of Equation (7) is negative over the entire interval $t \in [t_0, t_1]$ and takes on a zero value at $t = 3$. Therefore, the function $y(t)$ does not increase on the interval $t \in [t_0, t_1]$. This regularity of change in function (8) is shown in Figure 1. This dependence describes a particular solution obtained from function (8) at C = 3. The initial value of y(0) is equal to 5.82. The final zero value y(t) = 0 is obtained at t = 6.12. Therefore, we have $t_0 = 0$ and $t_1 = 6.12$.

Let us calculate the first derivative of the particular solution (8) with respect to the variable t. We obtain the derivative $\frac{dy}{dt}$ in the following form:

$$\frac{dy}{dt} = -\arctan((t-3)^2). \tag{9}$$

From the right-hand side of Equation (9), we can deduce that the derivative of the particular solution is $\frac{dy}{dt} < 0$ for all $t \in [t_0, t_1]$ except the point $t = 3$, where $\frac{dy}{dt} = 0$.

Suppose that a non-negative and twice continuously differentiable solution (8) of Equation (7) exists and is unique (for given initial conditions). Moreover, let solution (6) be a continuously differentiable function, defined in the interval $t \in [t_0, t_1]$, which also exists and is unique (for given initial conditions).

Now, let us show that a particular solution (8) of Equation (7) can satisfy Definition 1 of monotonic stability for this solution and Theorem 1. Let the particular solution (8) be defined $\forall t \in [t_0, t_1]$. The right-hand side of Equation (4) is the 2-function. Moreover, the first derivative of this particular solution $\frac{dy}{dt}$ takes negative values at $\forall t \in [t_0, t_1]$ (except the point $t = 3$). Additionally, the second continuous derivative of this function $\frac{d^2y}{dt^2}$ keeps its positive (or negative) sign or changes it to the opposite one only at the inflection point t = 3 of function (8) in the interval. Thus, Definition 1 and Theorem 1 are satisfied.

Figure 1 shows one of the characteristic cases of changing the function (8) in the implementation of monotonic stability. It can be seen from Figure 1 that in this case there is only one inflection point on the monotonic decrease curve y(t).

For particular solutions y(t) of Equation (7), Theorem 2 on the number of qualitatively different cases of monotonic stability of particular solution holds. Indeed, if all particular solutions y(t) of Equation (7) satisfy Definition 1 and the number of inflection points of functions y = y(t) is 0 or 1, then the number of all qualitatively different cases of monotonic stability of solutions y = y(t) equals $({}_5C_1) = 5$. Note that the case of monotone linear stability is not realized in this solution. Therefore, the number of qualitatively different cases of monotonic stability is 4.

Next, let us find the boundaries of the monotonic stability of particular solutions y(t) on the coordinate plane (t,y). In this case, Theorem 3 is applicable. Indeed, if all particular solutions y(t) of Equation (7) satisfy the conditions of Definition 1 and Theorem 1, then the boundaries of the monotonic stability region of the solutions y = y(t) form a rectangle contained in the first quadrant of the coordinate system (t,y) and having two sides lying on the coordinate axes. Figure 1 plots this rectangle, which must include all 5 qualitatively different cases of monotonic stability of solutions y = y(t). In this case, the number of inflection points of functions y = y(t) equals 0 or 1.

Now let us consider the application of Theorem 5 to the decomposition of the monotonic stability interval $[t_0, t_1]$ of the considered function y = y(t). Theorem 4 is also valid for solutions y = y(t). Figure 1 shows the case of one inflection point on the curve y(t) and two intervals with constant convexity. On the interval $[0, 3)$, the curve y(t) has a downward convexity, and in the interval $(3, 6.12]$, the curve y(t) has an upward convexity.

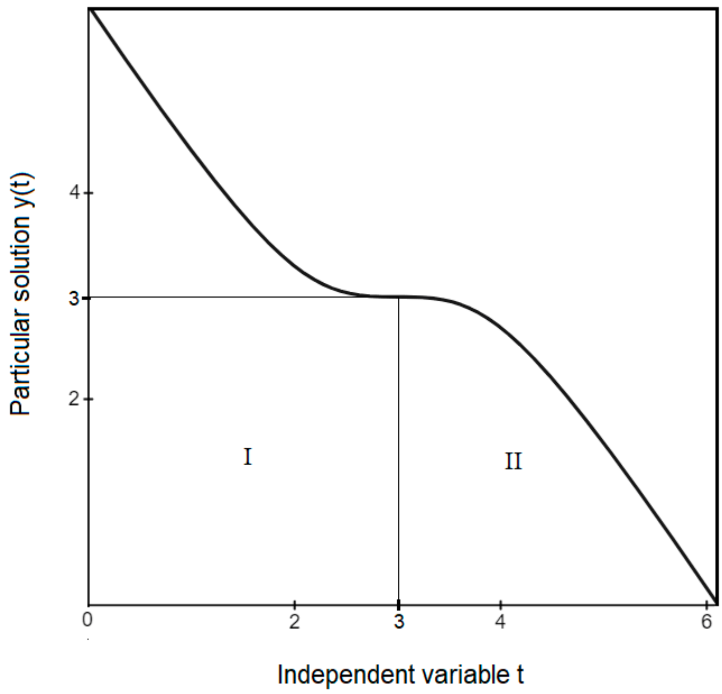


**Figure 1.** Dependence of the variable y on the independent variable t in the study of the monotonic stability of the solution (8) at C = 3.

Note that the conditions of Theorem 6 are also satisfied for the solution (8). In Figure 1, the curvilinear trapezoid and curvilinear triangle-shaped regions are denoted by Roman numerals I and II, respectively.

Finally, let us consider the stability analysis of the solution y(t) using the second Lyapunov method. In this example, we consider the system (4)–(5). However, we choose a particular solution of Equation (5) in the form $x(t) = t^2$. Substituting the solution into Equation (4), we obtain a differential equation $\frac{dy}{dt} = -\arctan(t^2)$. The right-hand side of this differential equation is a negative definite function. After separation of variables and integration, we obtain the following particular solution (for constant C = 3):

$$\begin{aligned} y(t) = &-(\arctan(t^2))(t) + \left(\sqrt{2}\right)^{-3}\ln\left(t^2 - \sqrt{2}\,t + 1\right) - \left(\sqrt{2}\right)^{-3}\ln\left(t^2 + \sqrt{2}\,t + 1\right) \\ &+(\sqrt{2})^{-1}\arctan\left(\sqrt{2}t + 1\right) + \left(\sqrt{2}\right)^{-1}\arctan\left(\sqrt{2}t - 1\right) + 3. \end{aligned} \tag{10}$$

The regularity of change in function (10) is shown in Figure 2. The initial value of y(0) is equal to 3. The final zero value y(t) = 0 is obtained at t = 3.12. In Figure 2, the curvilinear triangle region is denoted by Roman numeral II.

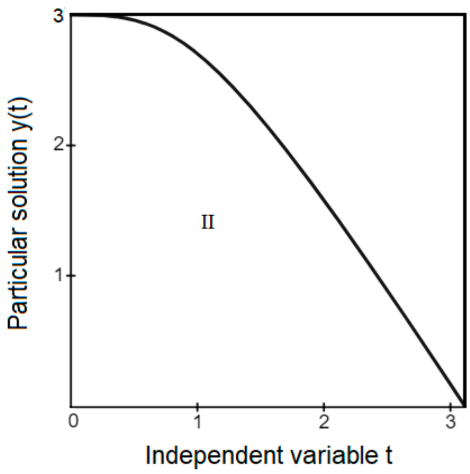


**Figure 2.** Dependence of the variable y on the independent variable t in the study of the monotonic stability of the solution (10).

Let us denote the right side of the Equation (4) as follows: $F(t) = -\arctan(t^2)$. Since the function $F(t)$ is negative definite, the positive definite function $W = F^2(t)/2$ can be considered as a Lyapunov function. When we differentiate the function, keeping in mind the equations of system (4)–(5), we derive the Lyapunov method's stability condition: $\frac{dW}{dt} = \frac{dy}{dt}\frac{d^2y}{dt^2} \le 0$. Considering the solution (8), it can be seen that this stability condition is satisfied in the interval $[t_0, t_1]$.

Indeed, this stability condition can be represented in the following form: $\frac{dy}{dt} = F(t) < 0$, $\frac{d^2y}{dt^2} = F(t)\frac{dF(t)}{dt} > 0$ (for $t \neq 0$) or $\frac{dy}{dt} = F(t) = 0, \frac{d^2y}{dt^2} = F(t)\frac{dF(t)}{dt} = 0$ (for $t = 0$).

Therefore, employing the second Lyapunov method helps us ascertain the stability condition for the solution y(t) of the Equation (4) of the dynamic system (4)–(5). Conversely, utilizing the monotone stability analysis method not only allows us to determine the stability conditions for the investigated solution y(t), but also facilitates a comprehensive analysis of all kinds of monotonically stable solutions. Additionally, it aids in the construction and decomposition of their stability region.

## 6. Discussion

This study presents a method for the qualitative investigation of the monotonic nonlinear stability region of a dynamical system. The system in question comprises two first-order ordinary differential equations. The right side of the first equation is defined as a known nonpositive 2-function that is both defined and continuously differentiable across the entire interval of the independent variable. For a finite number of stationary points, the right side of the first equation equals to zero. The right side of the second equation is a known continuous 2-function that maintains its sign consistently throughout the entire interval of the independent variable change. This study explores the conditions for monotonic stability of specific solutions of the first differential equation within the dynamic system. The classical method of mathematical investigation of a function of a single independent variable is employed, along with elements of combinatorics. As a result, a formula is determined that accurately defines the total number of qualitatively different solutions, assuming their monotonic decrease from an initial positive value to a final zero value. The study discerns a rectangular region of monotonic stability for qualitatively different and monotonically decreasing particular solutions of the first differential equation. Within this rectangular stability region, there's a pairwise symmetry of characteristic cases of monotonic stability with the same number of inflection points. However, these cases differ only in their type of convexity within the same independent variable variation intervals. The area beneath the curve of the particular solution of the first equation is segmented into various areas shaped as curvilinear trapezoids and one area shaped as a curvilinear triangle. In this case, the number of areas shaped as curvilinear trapezoids matches the number of inflection points of the given particular solution's curve. It is noteworthy that if a particular solution that monotonically decreases to zero serves as the upper bound for several monotonically decreasing particular solutions of the first differential equation, then the area beneath the upper bound represents the common region of monotonic stability. This paper utilizes the theoretical results under examination for a qualitative analysis of the monotonic stability region of a particular solution of a differential equation within one nonlinear dynamical system.

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