mathematics

Article

An Algorithm for the Numbers of Homomorphisms from Paths to Rectangular Grid Graphs

Hatairat Yingtaweesittikul , Sayan Panma and Penying Rochanakul *

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand; hatairat.y@cmu.ac.th (H.Y.); sayan.panma@cmu.ac.th (S.P.)
* Correspondence: penying.rochanakul@cmu.ac.th

Abstract: Let G and H be graphs. A mapping f from the vertices of G to the vertices of H is known as a *homomorphism* from G to H if, for every pair of adjacent vertices x and y in G, the vertices $f(x)$ and $f(y)$ are adjacent in H. A *rectangular grid graph* is the Cartesian product of two path graphs. In this paper, we provide a formula to determine the number of homomorphisms from paths to rectangular grid graphs. This formula gives the solution to the problem concerning the number of walks in the rectangular grid graphs.

Keywords: homomorphisms; graph homomorphisms; path graphs; rectangular grid graphs; grid graphs

MSC: 20M10; 05C25; 05C76; 05C85

check for updates

Citation: Yingtaweesittikul, H.; Panma, S.; Rochanakul, P. An Algorithm for the Numbers of Homomorphisms from Paths to Rectangular Grid Graphs. *Mathematics* **2023**, *11*, 2587. https://doi.org/10.3390/math11112587

Academic Editors: Ruo-Wei Hung and Bo Zhou

Received: 1 April 2023
Revised: 1 June 2023
Accepted: 2 June 2023
Published: 5 June 2023

1. Introduction

In mathematics, the image is the set of the values of a mapping at all elements in the domain. In such an image, some structures of the domain are preserved. A mapping that preserves a structure, the one that we need to study, is usually known as a homomorphism. For graphs, a homomorphism is defined as follows.

Throughout this paper, all graphs are finite and simple, and we denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. Let G and H be two graphs. A mapping f from $V(G)$ to $V(H)$ is known as a *homomorphism* from G to H if $\{f(x), f(y)\} \in E(H)$ for all $\{x,y\} \in E(G)$. When $G = H$, f is an *endomorphism* on G. The composition of homomorphisms is also known as a homomorphism. This leads to a preorder on graphs and a category [1]. We use the symbol Hom(G,H) to denote the set of all homomorphisms from G to H and End(G) to denote the set of all endomorphisms on G.

In a simple graph, a *walk* is a sequence of consecutive adjacent vertices. A *path* is a walk in which no vertex is repeated. We shall also use the word 'path' to denote a graph where the first and the last vertices have a degree one, and the other vertices have a degree two. Here, P_n stands for a path of order n with $V(P_n) = \{0,1,...,n-1\}$ and $E(P_n) = \{\{i,i+1\} \mid i = 0,1,...,n-2\}$. Let us denote the path P_n with an edge-labeling ϕ by P_n^{ϕ}. Furthermore, refer to [1,2] for more basic definitions and results regarding graphs and algebraic graphs.

The formula for the number of endomomorphisms on P_n, $|\mathrm{End}(P_n)|$, was introduced by Arworn [3] in 2009. This number is calculated by the summation of the numbers of shortest paths from point $(0,0)$ to any point (i,j) in a square lattice and an r-ladder square lattice. Moreover, in the same year, Arworn and Wojtylak [4] proposed a formula for the number of homomorphisms from P_m to P_n, $|\mathrm{Hom}(P_m,P_n)|$, in terms of $|\mathrm{Hom}_j^i(P_m,P_n)|$, where $\mathrm{Hom}_j^i(P_m,P_n) = \{f \in \mathrm{Hom}(P_m,P_n) \mid f(0) = i, f(m-1) = j\}$ for all $i,j \in \{0,1,\ldots,n-1\}$. In 2012, Lina and Zeng [5] constructed another formula for $|\mathrm{Hom}(P_m,P_n)|$, which was obtained by proving the conjecture in [6]. In 2014, Eggleton and Morayne [7] also gave another formula for $|\mathrm{Hom}(P_m,P_n)|$. Moreover, they considered finite Laurent series to be generating functions that can move homomorphisms of a finite path into any path, finite or infinite.

In 2018, Knauer and Pipattanajinda [8] studied a generalization of path endomorphisms, namely weak path endomorphisms. The number of weak path endomorphisms is calculated by the summation of the numbers of shortest paths from point $(0,0,0)$ to any point (i,j,k) in a cubic lattice and in an r-ladder cubic lattice. Recently, in 2022, Pomsri et al. [9] proposed a formula for the number of weak homomorphisms from P_m to P_n in recursive form.

The *Cartesian product* $G \times H$ of the graphs G and H is a graph with $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{\{(a,b),(a,c)\} \mid a \in V(G), \{b,c\} \in E(H)\} \cup \{\{(a,b),(d,b)\} \mid \{a,d\} \in E(G), b \in V(H)\}$. A *rectangular grid graph* or an $m \times n$ *grid graph* is the Cartesian product of two path graphs on m and n vertices. There is one-to-one correspondence between the set of homomorphisms $f : P_n \to G_1 \Box G_2$ and the set of walks of n vertices in $G_1 \Box G_2$. Thus, the number of homomorphisms from a path P_n to a grid graph gives the number of walks of n vertices in the rectangular grid graph.

In 2023, Keshavarz-Kohjerdi and Bagheri [10] studied a rectangular grid graph in which some rectangles are removed from its corners, namely a truncated rectangular grid graph. They provided a linear-time algorithm for finding a Hamiltonian cycle problem in a truncated rectangular grid graph. These could be extended to the lower bound for the number of homomorphisms from a cycle to a rectangular grid graph.

Our purpose is to find a formula for the number of homomorphisms from a path P_m to another path P_n and to a rectangular grid graph $P_n \Box P_k$.

2. The Number of Homomorphisms from Paths to Paths with $f(0) = j$

In this section, we provide the formula for finding the number of homomorphisms from paths P_m to P_n, which maps 0 to j. We denote the set of homomorphisms from P_m to P_n, which maps 0 to j, by $\mathrm{Hom}^j(P_m, P_n)$.

For $0 \le j \le n-1$, let

$$\mathrm{Hom}^j(P_m, P_n) = \{f \in \mathrm{Hom}(P_m, P_n) \mid f(0) = j\}. \tag{1}$$

By the symmetry of P_n, we obtain the following lemma:

Lemma 1. *Let j and n be integers such that $0 \le j < n$.*

$$|\mathrm{Hom}^j(P_m, P_n)| = |\mathrm{Hom}^{(n-j-1)}(P_m, P_n)|. \tag{2}$$

Here, we transform the cardinal number of $|\mathrm{Hom}^j(P_m, P_n)|$ to count the shortest paths on square lattices. Figure 1a–c show the possible homomorphisms from P_4 to P_5, which map 0 to 0, 1, and 2, respectively. The numbers on the top are elements of the domain set $V(P_4)$, and the tuples on the left are elements of the image set $V(P_5)$. These become square lattices, as shown in Figure 2a–c after rotating 45° counterclockwise.

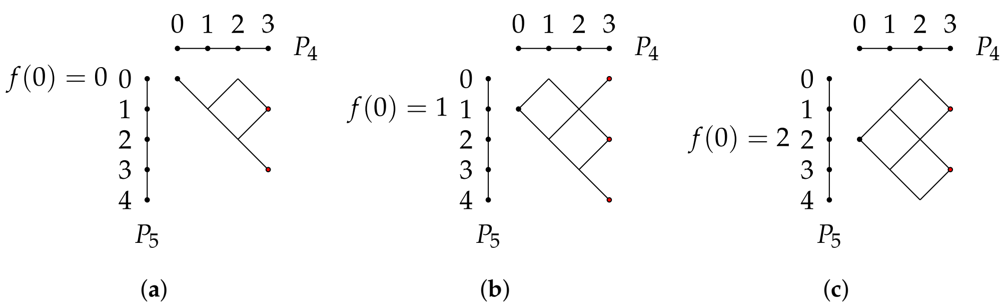


Figure 1. Graphical presentation of the domain and image of all possible homomorphisms $f \in \mathrm{Hom}(P_4, P_5)$. (**a**) $f(0) = 0$. (**b**) $f(0) = 1$. (**c**) $f(0) = 2$.

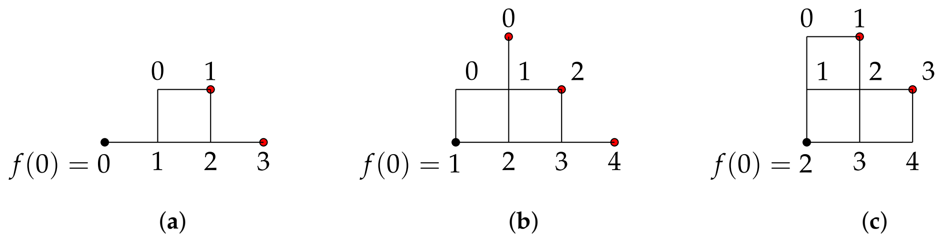


Figure 2. Square lattice presentations of all possible homomorphisms $f \in \mathrm{Hom}(P_4, P_5)$. (**a**) $f(0) = 0$. (**b**) $f(0) = 1$. (**c**) $f(0) = 2$.

Each homomorphism $f \in \mathrm{Hom}(P_m, P_n)|$ can be visualized using the square lattice, where movement from (i, j) to the next point is depicted as follows:

- To $(i+1, j)$ if $f(x+1) = f(x) + 1$.
- To $(i, j+1)$, if $f(x+1) = f(x) - 1$.

For example, if the images of successive vertices of $f \in |\mathrm{Hom}^3(P_{17}, P_{10})|$ are $3, 4, 5, 4, 5, 4, 3, 2, 1, 0, 1, 2, 3, 2, 3, 4$ and 5, then the homomorphism can be visualized as shown in Figure 3.

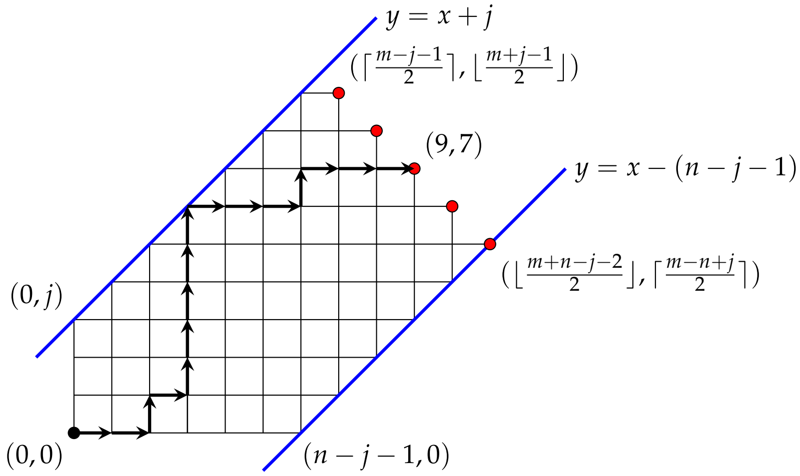


Figure 3. The shortest path from (0, 0) to (9, 7) that stays between lines $y = x + j$ and $y = x - n + j + 1$, where $j = 3$, $m = 17$ and $n = 10$.

In general, $|\mathrm{Hom}^j(P_m, P_n)|$ can be obtained from the number of shortest paths from $(0, 0)$ to any point $(i, n - i - 1)$ on the square lattice that stays between the lines $y = x + j$ and $y = x - n + j + 1$, where touching is allowed.

Lemma 2 ([5])**.** *The number of shortest paths from point $(0, 0)$ to any point $(i, n - i - 1)$ on the square lattice that stays between the lines $y = x + j$ and $y = x - (n - j - 1)$ is*

$$\sum_{|t| \leq \lfloor (m+n)/n \rfloor} \left(\binom{m-1}{i - t(n+1)} - \binom{m-1}{i + j - t(n+1) + 1} \right). \tag{3}$$

where $\binom{n}{k} = 0$ if $k > n$ or $k < 0$.

Hence, we obtain the following theorem.

Theorem 1. *Let m, n be positive integers and j be a non-negative integer. Let $\mathcal{L} = \max\{0, \lceil \frac{m-j-1}{2} \rceil\}$ and $\mathcal{U} = \min\{m-1, \lfloor \frac{m+n-j-2}{2} \rfloor\}$. Then,*

$$|\mathrm{Hom}^j(P_m, P_n)| = \sum_{i=\mathcal{L}}^{\mathcal{U}} \sum_{|t| \le \lfloor \frac{m+n}{n} \rfloor} \left(\binom{m-1}{i - t(n+1)} - \binom{m-1}{i+j-t(n+1)+1} \right) \tag{4}$$

Example 1. *Using Theorem 1, we have*

$$\begin{aligned}
|\mathrm{Hom}^0(P_4, P_5)| &= \sum_{i=2}^{3} \sum_{t=-1}^{1} \left(\binom{3}{i-6t} - \binom{3}{i-6t+1} \right) \\
&= \sum_{i=2}^{3} \left(\binom{3}{i+6} - \binom{3}{i+7} + \binom{3}{i} - \binom{3}{i+1} + \binom{3}{i-6} - \binom{3}{i-5} \right) \\
&= \left(\binom{3}{2} - \binom{3}{3} \right) + \left(\binom{3}{3} \right) \\
&= 3,
\end{aligned}$$

$$\begin{aligned}
|\mathrm{Hom}^1(P_4, P_5)| &= \sum_{i=1}^{3} \sum_{t=-1}^{1} \left(\binom{3}{i-6t} - \binom{3}{i-6t+2} \right) \\
&= \sum_{i=1}^{3} \left(\binom{3}{i+6} - \binom{3}{i+8} + \binom{3}{i} - \binom{3}{i+2} + \binom{3}{i-6} - \binom{3}{i-4} \right) \\
&= \left(\binom{3}{1} - \binom{3}{3} \right) + \left(\binom{3}{2} \right) + \left(\binom{3}{3} \right) \\
&= 6,
\end{aligned}$$

and

$$\begin{aligned}
|\mathrm{Hom}^2(P_4, P_5)| &= \sum_{i=1}^{2} \sum_{t=-1}^{1} \left(\binom{3}{i-6t} - \binom{3}{i-6t+3} \right) \\
&= \sum_{i=1}^{2} \left(\binom{3}{i+6} - \binom{3}{i+9} + \binom{3}{i} - \binom{3}{i+3} + \binom{3}{i-6} - \binom{3}{i-3} \right) \\
&= \left(\binom{3}{1} \right) + \left(\binom{3}{2} \right) \\
&= 6.
\end{aligned}$$

which is in line with counting directly from Figure 2. By counting the paths in Figure 2a, we have $|\mathrm{Hom}^0(P_4, P_5)| = 3$ (see Figure 4). By counting the paths in Figure 2b, we have $|\mathrm{Hom}^1(P_4, P_5)| = 6$ (see Figure 5). By counting the paths in Figure 2c, we have $|\mathrm{Hom}^2(P_4, P_5)| = 6$. (see Figure 6).

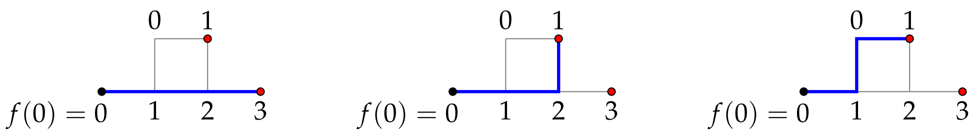


Figure 4. All possible presentations of homomorphisms $f \in \mathrm{Hom}^0(P_4, P_5)$ on a square lattice.

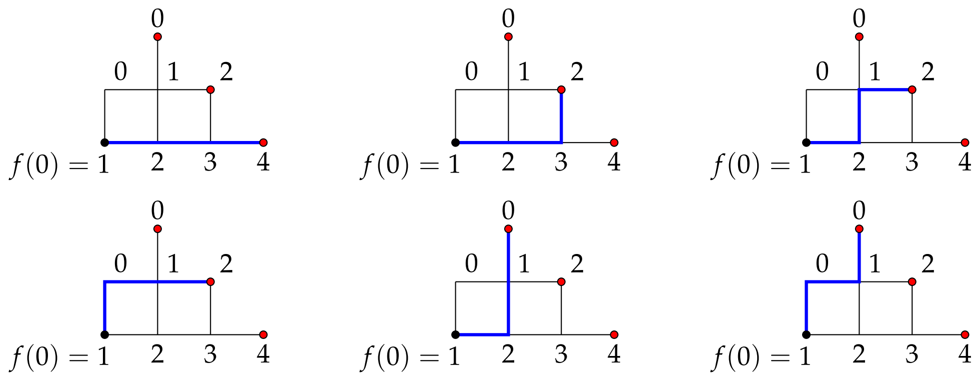


Figure 5. All possible presentations of homomorphisms $f \in \mathrm{Hom}^1(P_4, P_5)$ on a square lattice.

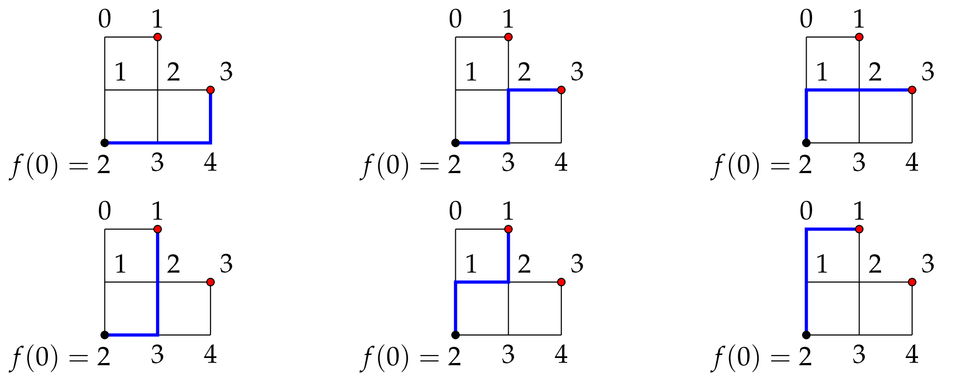


Figure 6. All possible presentations of homomorphisms $f \in \mathrm{Hom}^2(P_4, P_5)$ on a square lattice.

For convenience, we compute $|\mathrm{Hom}^j(P_m, P_n)|$ for $2 \le m, n \le 9$ (Table 1).

Table 1. Numbers of homomorphisms $f \in \mathrm{Hom}^j(P_m, P_n)$ for $2 \le m, n \le 9$.

					n			
m	***j***	**2**	**3**	**4**	**5**	**6**	**7**	**8**
2	0	1	1	1	1	1	1	1
	1	1	2	2	2	2	2	2
	2	0	1	2	2	2	2	2
	3	0	0	1	2	2	2	2
3	0	1	2	2	2	2	2	2
	1	1	2	3	3	3	3	3
	2	0	2	3	4	4	4	4
	3	0	0	2	3	4	4	4
4	0	1	2	3	3	3	3	3
	1	1	2	5	6	6	6	6
	2	0	2	5	6	7	7	7
	3	0	0	3	6	7	8	8

Table 1. *Cont.*

		n						
m	***j***	**2**	**3**	**4**	**5**	**6**	**7**	**8**
5	0	1	4	5	6	6	6	6
	1	1	4	8	9	10	10	10
	2	0	4	8	12	13	14	14
	3	0	0	5	9	13	14	15
6	0	1	4	8	9	10	10	10
	1	1	8	13	18	19	20	20
	2	0	4	13	18	23	24	25
	3	0	0	8	18	23	28	29
7	0	1	8	13	18	19	20	20
	1	1	8	21	27	33	34	35
	2	0	8	21	36	42	48	49
	3	0	0	13	27	42	48	54
8	0	1	8	21	27	33	34	35
	1	1	16	34	54	61	68	69
	2	0	8	34	54	75	82	89
	3	0	0	21	54	75	96	103

3. The Number of Homomorphisms from Paths to Rectangular Grid Graphs

In this section, we provide the formulas for finding the number of homomorphisms from paths P_m to rectangular grid graphs $P_n \Box P_k$. We denote the set of homomorphisms from P_m to $P_n \Box P_k$, which maps 0 to (i,j), by $\mathrm{Hom}^{ij}(P_m, P_n \Box P_k)$.

For $0 \le i \le n-1$, $0 \le j \le k-1$, let

$$\mathrm{Hom}^{ij}(P_m, P_n \Box P_k) = \{f \in \mathrm{Hom}(P_m, P_n \Box P_k) \mid f(0) = (i,j)\}. \tag{5}$$

From the symmetry of $P_n \Box P_k$, we obtain the following lemma:

Lemma 3. *Let i and n be integers such that $0 \le j < n$, and let $m > 2$ be a positive integer.*

(1) $|\mathrm{Hom}^{ij}(P_m, P_n \Box P_k)| = |\mathrm{Hom}^{(n-i-1)j}(P_m, P_n \Box P_k)| = |\mathrm{Hom}^{i(k-j-1)}(P_m, P_n \Box P_k)|$
$= |\mathrm{Hom}^{(n-i-1)(k-j-1)}(P_m, P_n \Box P_k)|,$
for all $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{0, 1, \ldots, k-1\}$.

(2) $|\mathrm{Hom}(P_m, P_{2n} \Box P_{2k})| = 4 \sum_{i=0}^{n-1} \sum_{j=0}^{k-1} |\mathrm{Hom}^{ij}(P_m, P_{2n} \Box P_{2k})|.$

(3) $|\mathrm{Hom}(P_m, P_{2n+1} \Box P_{2k})| = 4 \sum_{i=0}^{n-1} \sum_{j=0}^{k-1} |\mathrm{Hom}^{ij}(P_m, P_{2n+1} \Box P_{2k})|$
$+2 \sum_{j=0}^{k-1} |\mathrm{Hom}^{nj}(P_m, P_{2n+1} \Box P_{2k})|.$

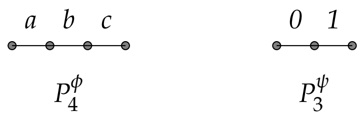
(4) $|\mathrm{Hom}(P_m, P_{2n} \Box P_{2k+1})| = 4 \sum_{i=0}^{n-1} \sum_{j=0}^{k-1} |\mathrm{Hom}^{ij}(P_m, P_{2n} \Box P_{2k+1})|$
$+2 \sum_{i=0}^{n-1} |\mathrm{Hom}^{ik}(P_m, P_{2n} \Box P_{2k+1})|.$

(5) $|\mathrm{Hom}(P_m, P_{2n+1} \Box P_{2k+1})| = 4 \sum_{i=0}^{n-1} \sum_{j=0}^{k-1} |\mathrm{Hom}^{ij}(P_m, P_{2n+1} \Box P_{2k+1})|$
$+2 \sum_{j=0}^{k-1} |\mathrm{Hom}^{nj}(P_m, P_{2n+1} \Box P_{2k+1})|$
$+2 \sum_{i=0}^{n-1} |\mathrm{Hom}^{ik}(P_m, P_{2n+1} \Box P_{2k+1})|$
$+|\mathrm{Hom}^{nk}(P_m, P_{2n+1} \Box P_{2k+1})|.$

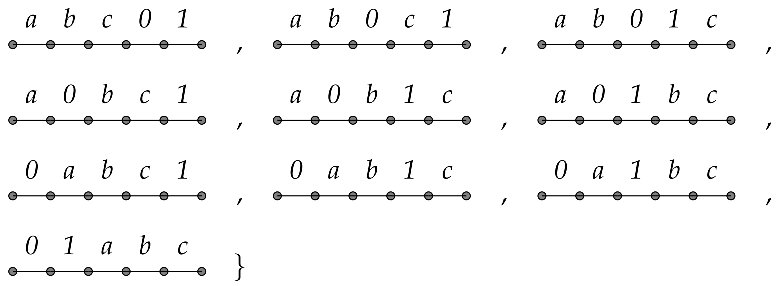
To prove the main theorem, we define a new operation for two paths with their edge labelings.

Definition 1. *Let P_m^{ϕ}, P_n^{ψ} be paths P_m, P_n with edge labelings ϕ and ψ. Define P_m^{ϕ} and P_n^{ψ}* ***entwined*** *or $P_m^{\phi} \between P_n^{\psi}$ as the set of all paths P_{m+n-1} with edge labels from ϕ and ψ that preserve the sequential order of ϕ and ψ.*

Example 2. *Consider paths P_4 and P_3 with injective edge labelings ϕ and ψ, as shown below.*



We have $P_4^{\phi} \between P_3^{\psi} = \{$



This leads to the following lemma:

Lemma 4. *Let P_m^{ϕ}, P_n^{ψ} be paths with edge labelings. Then,*

$$|P_m \between P_n| = \binom{m+n-2}{m-1}. \tag{6}$$

Proof. It is easy to see that the number of ways to label P_{m+n-1} is equal to the permutations of all $m+n-2$ edge labels with a fixed sequential order. $\square$

Next, we observe a simple example to visualize homomorphisms from paths to rectangular grid graphs on a square lattice.

Example 3 ($\mathrm{Hom}^{00}(P_4, P_4 \square P_5) = 18$). *All possible homomorphisms $f \in \mathrm{Hom}^{00}(P_4, P_4 \square P_5)$ are shown in Figure 7. The numbers on the top are elements of the domain set $V(P_4)$, and the tuples on the left are elements of the image set $V(P_4 \square P_5)$. The tuples with the same second elements are represented by circles of the same color.*

The mappings $f_1, f_2 \in \mathrm{Hom}^{00}(P_4, P_4 \square P_5)$ with $f_1(0) = (0,0), f_1(1) = (0,1), f_1(2) = (0,2), f_1(3) = (0,1)$ and $f_2(0) = (0,0), f_2(1) = (1,0), f_2(2) = (2,0), f_2(3) = (1,0)$ are represented by the red lines on the top and the black lines (see Figure 8). We note that the normal black lines represent the increment of the first coordinate, the dashed black lines represent the decrement of the first coordinate, the normal red lines represent the increment of the second coordinate, and the red lines represent the decrement of the second coordinate.

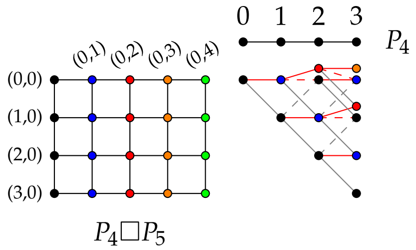


Figure 7. Graphical presentation of the domain and image of all possible homomorphisms $f \in \mathrm{Hom}^{00}$ $(P_4, P_4 \square P_5)$.

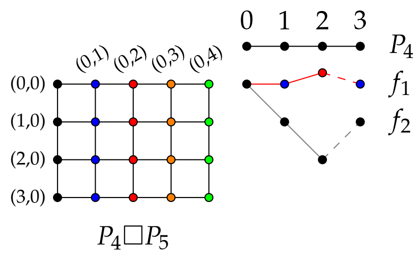
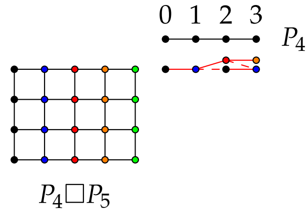
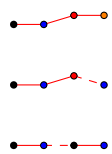
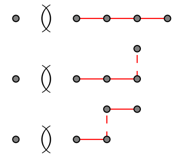
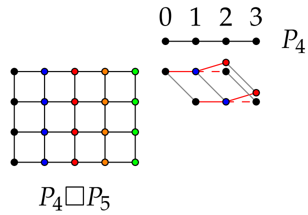
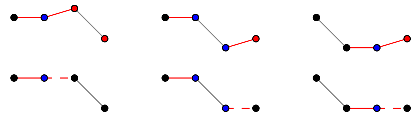
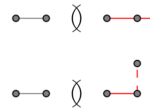
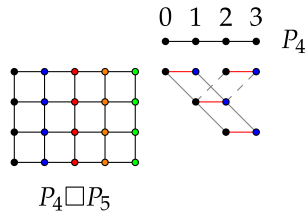
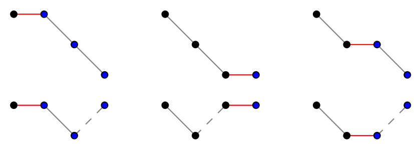
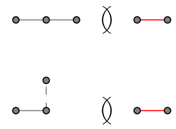
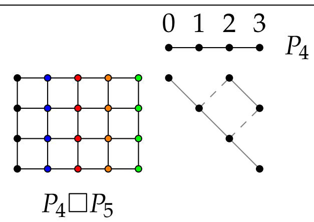
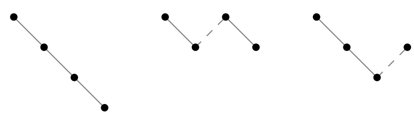
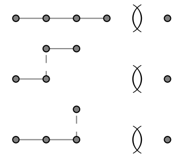


Figure 8. Square lattice presentation of f_1 and f_2.

We now divide all mappings in $\mathrm{Hom}^{00}(P_4, P_4 \square P_5)$ *into groups according to the number of change occurrences in the first coordinate h and rewrite each path as entwined black and red paths.*

h	$f \in \mathrm{Hom}^{00}(P_4, P_4 \square P_5)$ *with changes in the first coordinate h times*	*Paths represent each* $f \in \mathrm{Hom}^{00}(P_4, P_4 \square P_5)$ *(Expanded Diagram)*	$P_{h+1} \between P_{4-h}$
0			
1			
2			
3			

For each $h \in \{0,1,2,3\}$*, observe that out of the* 3 *edges of* P_4 *from* $P_{h+1} \between P_{4-h}$*, there are* $\binom{3}{h}$ *ways to place h edges from the black path and one way to place* $3-h$ *edges from the red path. Moreover, the black line* P_{h+1} *is the square lattice representation of* $f_1 \in \mathrm{Hom}^0(P_{h+1}, P_4)$*, while the red line* P_{4-h} *is the square lattice representation of* $f_2 \in \mathrm{Hom}^0(P_{4-h}, P_5)$*. Thus, there are* $\binom{3}{h}|\mathrm{Hom}^0(P_{h+1}, P_4)||\mathrm{Hom}^0(P_{4-h}, P_5)|$ *possible paths in* $P_{h+1} \between P_{4-h}$*. Hence,*

$$
\begin{aligned}
|\mathrm{Hom}^{00}(P_4, P_4\square P_5)| &= \binom{3}{0}|\mathrm{Hom}^{0}(P_1, P_4)||\mathrm{Hom}^{0}(P_4, P_5)| \\
&\quad + \binom{3}{1}|\mathrm{Hom}^{0}(P_2, P_4)||\mathrm{Hom}^{0}(P_3, P_5)| \\
&\quad + \binom{3}{2}|\mathrm{Hom}^{0}(P_3, P_4)||\mathrm{Hom}^{0}(P_2, P_5)| \\
&\quad + \binom{3}{3}|\mathrm{Hom}^{0}(P_4, P_4)|\mathrm{Hom}^{0}(P_1, P_5)| \\
&= 1(1)(3) + 3(1)(2) + 3(2)(1) + 1(3)(1) \\
&= 18.
\end{aligned}
$$

Example 4 ($|\mathrm{Hom}^{11}(P_4, P_4\square P_5)| = 47$ **).** *All possible homomorphisms* $f \in \mathrm{Hom}^{11}(P_4, P_4\square P_5)$ *are shown in Figure 9. The numbers on the top are elements of the domain set* $V(P_4)$*, and the tuples on the left are elements of the image set* $V(P_4\square P_5)$*. The tuples with the same second elements are represented by circles of the same color.*

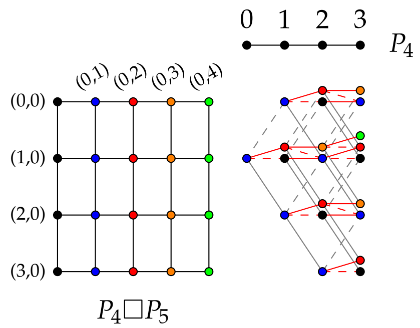


Figure 9. Graphical presentation of the domain and image of all possible homomorphisms $f \in \mathrm{Hom}^{11}$ $(P_4, P_4\square P_5)$.

$$
\begin{aligned}
|\mathrm{Hom}^{11}(P_4, P_4\square P_5)| &= \binom{3}{0}|\mathrm{Hom}^{1}(P_1, P_4)||\mathrm{Hom}^{1}(P_4, P_5)| \\
&\quad + \binom{3}{1}|\mathrm{Hom}^{1}(P_2, P_4)||\mathrm{Hom}^{1}(P_3, P_5)| \\
&\quad + \binom{3}{2}|\mathrm{Hom}^{1}(P_3, P_4)||\mathrm{Hom}^{1}(P_2, P_5)| \\
&\quad + \binom{3}{3}|\mathrm{Hom}^{1}(P_4, P_4)|\mathrm{Hom}^{1}(P_1, P_5)| \\
&= 1(1)(6) + 3(2)(3) + 3(3)(2) + 1(5)(1) \\
&= 47.
\end{aligned}
$$

Lemma 5. *Let* m, n *and* k *be positive integers and let* i, j *be non-negative integers, such that* $i < \dfrac{n}{2} - 1$ *and* $j < \dfrac{k}{2} - 1$*. It follows that*

$$|\mathrm{Hom}^{ij}(P_m, P_n\square P_k)| = \sum_{h=0}^{m-1} \binom{m-1}{h} |\mathrm{Hom}^{i}(P_{h+1}, P_n)||\mathrm{Hom}^{j}(P_{m-h}, P_k)|. \tag{7}$$

Proof. Let $f \in \mathrm{Hom}^{ij}(P_m, P_n\square P_k)$. For each $x \in \{0, 1, m-2\}$ in the domain, either $f(x+1) = f(x) \pm (1,0)$ or $f(x+1) = f(x) \pm (0,1)$. Assume changes in the first coordinate appear h times. Then, changes in the second coordinate appear $m-1-h$ times. The sequence of changes in the first coordinate form a homomorphism $f_1 \in Hom^{i}(P_{h+1}, P_n)$. Similarly, the sequence of changes in the second coordinate form a homomorphism $f_2 \in Hom^i(P_{m-1-h+1}, P_k)$. Thus, the corresponding path graph of f can be obtained from path graphs of f_1 and f_2 entwined. Hence, $|\mathrm{Hom}^{ij}(P_m, P_n \Box P_k)| = \sum_{h=0}^{m-1} \binom{m-1}{h} |\mathrm{Hom}^i(P_{h+1}, P_n)||\mathrm{Hom}^j(P_{m-h}, P_k)|$. $\square$

From Theorem 1, Lemma 3 and Lemma 5, we get the theorem below.

Theorem 2. *Let m, n and k be positive integers. The cardinalities $|\mathrm{Hom}(P_m, P_n \Box P_k)|$ of homomorphisms from paths P_m to rectangular grid graphs $P_n \Box P_k$ are*

$$\begin{aligned}|\mathrm{Hom}(P_m, P_n \Box P_k)| = {} & 4\sum_{i=0}^{\lfloor n/2 \rfloor -1} \sum_{j=0}^{\lfloor k/2 \rfloor -1} |\mathrm{Hom}^{ij}(P_m, P_n \Box P_k)| \\ & +(1-(-1)^n)\sum_{j=0}^{\lfloor k/2 \rfloor -1} |\mathrm{Hom}^{\lfloor n/2 \rfloor j}(P_m, P_n \Box P_k)| \\ & +(1-(-1)^k)\sum_{i=0}^{\lfloor n/2 \rfloor -1} |\mathrm{Hom}^{i\lfloor k/2 \rfloor}(P_m, P_n \Box P_k)| \\ & +(1/4)(1-(-1)^n)(1-(-1)^k)|\mathrm{Hom}^{\lfloor n/2 \rfloor \lfloor k/2 \rfloor}(P_m, P_n \Box P_k)|\end{aligned}$$

where $|\mathrm{Hom}^{ij}(P_m, P_n \Box P_k)| = \sum_{h=0}^{m-1} \binom{m-1}{h} |\mathrm{Hom}^i(P_{h+1}, P_n)||\mathrm{Hom}^j(P_{m-h}, P_k)|$ *and*

$$|\mathrm{Hom}^j(P_m, P_n)| = \sum_{i=\mathcal{L}}^{\mathcal{U}} \sum_{|t| \le \lfloor \frac{m+n}{n} \rfloor} \left(\binom{m-1}{i-t(n+1)} - \binom{m-1}{i+j-t(n+1)+1} \right),$$

where $\mathcal{L} = \max\{0, \lceil \frac{m-j-1}{2} \rceil\}$ *and* $\mathcal{U} = \min\{m-1, \lfloor \frac{m+n-j-2}{2} \rfloor\}$.

For convenience, we compute $|\mathrm{Hom}(P_m, P_n \Box P_k)|$ for $2 \le m, n, k \le 8$. The results are presented in Table 2.

Table 2. Numbers of homomorphisms $f \in (P_m, P_n \Box P_k)$ for $2 \le m \le n, k \le 8$.

		k						
m	***n***	**2**	**3**	**4**	**5**	**6**	**7**	**8**
2	2	8	14	20	26	32	38	44
	3	14	24	34	44	54	64	74
	4	20	34	48	62	76	90	104
	5	26	44	62	80	98	116	134
	6	32	54	76	98	120	142	164
	7	38	64	90	116	142	168	194
	8	44	74	104	134	164	194	224
3	3	34	68	102	136	170	204	238
	4	52	102	152	202	252	302	352
	5	70	136	202	268	334	400	466
	6	88	170	252	334	416	498	580
	7	106	204	302	400	498	596	694
	8	124	238	352	466	580	694	808
4	4	136	308	488	668	848	1028	1208
	5	190	424	668	912	1156	1400	1644
	6	244	540	848	1156	1464	1772	2080
	7	298	656	1028	1400	1772	2144	2516
	8	352	772	1208	1644	2080	2516	2952

Table 2. *Cont.*

		k						
m	**n**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
5	5	518	1330	2226	3132	4038	4944	5850
	6	680	1726	2876	4038	5200	6362	7524
	7	842	2122	3526	4944	6362	7780	9198
	8	1004	2518	4176	5850	7524	9198	10872
6	6	1900	5528	9788	14172	18568	22964	27360
	7	2386	6880	12138	17544	22964	28384	33804
	8	2872	8232	14488	20916	27360	33804	40248
7	7	6774	22360	41884	62454	83196	103952	124708
	8	8232	26976	50384	75020	99856	124708	149560
8	8	23628	88496	175476	269596	365328	461288	557264

4. The Algorithm

In this section, we provide algorithms used to calculate $|\mathrm{Hom}^i(P_m, P_n)|$, $|\mathrm{Hom}^{ij}(P_m, P_n \square P_k)|$ and $|\mathrm{Hom}(P_m, P_n \square P_k)|$ with the aforementioned theorems.

Algorithms 1–3 are implementations of Theorem 1, Lemma 5 and Theorem 2, respectively.

Algorithm 1 LOCALPATH2PATH: Number of Homomorphisms from P_m to P_n with $f(0) = j$

Input:
- m: the size of the domain
- n: the size of the range
- Fixed value j where $f(0) = j$ (with $0 \leq j \leq n-1$)

Output: number of homomorphisms from P_m to P_n with $f(0) = j$

$\mathcal{L} \leftarrow \max\{0, \left\lceil \frac{m-j-1}{2} \right\rceil\}$
$\mathcal{U} \leftarrow \min\{m-1, \left\lfloor \frac{m+n-j-2}{2} \right\rfloor\}$
if $\mathcal{L} > \mathcal{U}$ **then**
    return 0
end if
$homj \leftarrow 0$
for $i = \mathcal{L}$ to $\mathcal{U}$ **do**
    for $t = -\left\lfloor \frac{m+n}{n} \right\rfloor$ to $\left\lfloor \frac{m+n}{n} \right\rfloor$ **do**
        $homj \leftarrow homj + \binom{m-1}{i-t(n+1)} - \binom{m-1}{i+j-t(n+1)+1}$
    end for
end for
return $homj$

Algorithm 2 LocalPath2Grid: Number of Homomorphisms from P_m to $P_n \square P_k$ with $f(0) = (i, j)$

Input:
- m: the size of the domain
- n, k: the dimensions of the grid representing the range
- Fixed value i, j where $f(0) = (i, j)$ (with $0 \leq i \leq n-1$ and $0 \leq j \leq k-1$)

Output: number of homomorphisms from P_m to $P_n \square P_k$ with $f(0) = (i, j)$

$homij \leftarrow 0$
for $h = 0$ to $m - 1$ **do**
  $c_i \leftarrow$ LocalPath2Path$(h+1, n, i)$
  $c_j \leftarrow$ LocalPath2Path$(m-h, k, j)$
  $homij \leftarrow homij + \binom{m-1}{h} c_i c_j$
end for
return homij

Algorithm 3 Path2Grid: Number of Homomorphisms from P_m to $P_n \square P_k$

Input:
- m: the size of the domain
- n, k: the dimensions of the grid representing the range

Output: number of homomorphisms from P_m to $P_n \square P_k$

$homgrid \leftarrow 0$
$sum \leftarrow 0$
for $i = 0$ to $\lfloor n/2 \rfloor - 1$ **do**
  for $j = 0$ to $\lfloor k/2 \rfloor - 1$ **do**
    $sum \leftarrow sum +$ LocalPath2Grid(m, n, k, i, j)
  end for
end for
$homgrid \leftarrow homgrid + sum * 4$
$sum \leftarrow 0$
for $j = 0$ to $\lfloor k/2 \rfloor - 1$ **do**
  $sum \leftarrow sum +$ LocalPath2Grid$(m, n, k, \lfloor n/2 \rfloor, j)$
end for
$homgrid \leftarrow homgrid + (1 - (-1)^n) * sum$
$sum \leftarrow 0$
for $i = 0$ to $\lfloor n/2 \rfloor - 1$ **do**
  $sum \leftarrow sum +$ LocalPath2Grid$(m, n, k, i, \lfloor k/2 \rfloor)$
end for
$homgrid \leftarrow homgrid + (1 - (-1)^k) * sum$
$homgrid \leftarrow homgrid + \frac{1}{4}(1 - (-1)^n)(1 - (-1)^k)$LocalPath2Grid$(m, n, k, \lfloor n/2 \rfloor, \lfloor k/2 \rfloor)$
return $homgrid$

Lemma 6. *Algorithm* Path2Grid *has time-complexity* $O(n \cdot m \cdot k \cdot \max(n, m, k))$.

Proof. It is easy to see that the complexity of the algorithm depends on the first loop, which is also nested with $O(n \cdot k)$ rounds. Each round consists of an execution of LocalPath2Grid, which is essentially a loop with $O(m)$ rounds. Each of these deeper rounds calls LocalPath2Path twice.

To see the runtime for LocalPath2Path given parameters m and n, we first see the complexity of the outer loop:

$$
\begin{aligned}
O(\mathcal{U}-\mathcal{L}) = O\Bigg(\min\Bigg\{ & m-1, m-1-\left\lceil\frac{m-j-1}{2}\right\rceil, \left\lfloor\frac{m+n-j-2}{2}\right\rfloor, \\
& \left\lfloor\frac{m+n-j-2}{2}\right\rfloor - \left\lceil\frac{m-j-1}{2}\right\rceil\Bigg\}\Bigg) \\
\leq\ & O(\min\{m, m+n, n\}) = O(\min\{m,n\})
\end{aligned} \tag{8}
$$

Then, we consider the following scenarios:

1. $m < n$: In this case $\lfloor (m+n)/n \rfloor = 1$; hence, the inner loop has fixed rounds. Therefore, the complexity is at most $O(\min\{m,n\})$.
2. $m >= n$: In this case, the complexity of the inner loop is $O(\lfloor (m+n)/n \rfloor)$. Together, we have the overall complexity:

$$
\begin{aligned}
O(\min\{m,n\})O(\lfloor (m+n)/n \rfloor) &= O(n)O(\lfloor (m+n)/n \rfloor) \\
&\leq O(n)O(m/n) \\
&\leq O(m)
\end{aligned} \tag{9}
$$

Therefore, the overall complexity of LocalPath2Path is $O(\min\{m,n\})$. Since each round of LocalPath2Grid calls LocalPath2Path twice, respectively with parameters $(h+1, n)$ and $(m-h, k)$, we have its complexity as:

$$
\begin{aligned}
& O(\min\{h+1, n\}) + O(\min\{m-h, k\}) \\
& \leq O(\min\{m,n\}) + O(\min\{m,k\}) \\
& \leq O(\max\{m,n\}) + O(\max\{m,k\}) \\
& \leq O(\max\{m,n,k\})
\end{aligned} \tag{10}
$$

Together, the total complexity of Path2Grid is $O(m \cdot n \cdot k \cdot \max\{m,n,k\})$. □

Author Contributions: Conceptualization, S.P.; methodology, S.P.; software, H.Y.; validation, H.Y.; investigation, P.R.; writing—original draft, P.R.; writing—review and editing, H.Y. and S.P.; visualization, H.Y. and P.R.; project administration, S.P.; funding acquisition, S.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Chiang Mai University.

Data Availability Statement: Data sharing not applicable.

Acknowledgments: We thank the reviewers for their insightful comments and suggestions that improved the quality of our manuscript. We wish to express our thanks to Pham Hoang Viet for several helpful comments concerning algorithm analyzing. This research was supported by Faculty of Science, Chiang Mai University, and Chiang Mai University, Thailand.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hell, P.; Nestril, J. *Graphs and Homomorphisms*; Oxford University Press: Oxford, UK , 2004.
2. Knauer, U.; Knauer, K. *Algebraic Graph Theory: Morphisms, Monoids and Matrices*; Walter de Gruyter: Berlin, Germany , 2011.
3. Arworn, S. An algorithm for the numbers of endomorphisms on paths. *Discret. Math.* **2009**, *309*, 94–103. [CrossRef]
4. Arworn, S.; Wojtylak, P. An algorithm for the number of path homomorphisms. *Discret. Math.* **2009**, *309*, 5569–5573. [CrossRef]
5. Lina, Z.; Zeng, J. On the number of congruence classes of paths. *Discret. Math.* **2012**, *312*, 1300–1307. [CrossRef]
6. Michels, M.A.; Knauer, U.H. The congruence classes of paths and cycles. *Discret. Math.* **2009**, *309*, 5352–5359. [CrossRef]
7. Eggleton, R.B.; Morayne, M. A Note on Counting Homomorphisms of Paths. *Graphs Comb.* **2014**, *30*, 159–170. [CrossRef]
8. Knauer, U.; Pipattanajinda, N. A formula for the number of weak endomorphisms on paths. *Algebra Discret. Math.* **2018**, *26*, 270–279.

9. Pomsri, T.; Wannasit, W.; Panma, S. Finding the Number of Weak Homomorphisms of Paths. *J. Math.* **2022**, *2022*, 2153927. [CrossRef]
10. Keshavarz-Kohjerdi, F.; Bagheri, A. Finding Hamiltonian cycles of truncated rectangular grid graphs in linear time. *Appl. Math. Comput.* **2023**, *436*, 127513. [CrossRef]