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Perturbed Skew Diffusion Processes

Yingxu Tian and Haoyan Zhang *

College of Science, Civil Aviation University of China, Tianjin 300300, China; yxtian@cauc.edu.cn
* Correspondence: zhang-hy@cauc.edu.cn

Abstract: This work investigates whether there uniquely exists a solution to the perturbed skew diffusion process. We construct the solution by iteration and divide the whole time interval into parts on which we disperse the perturbed skew diffusion process into two tractable portions, one for perturbed diffusion process, the other for skew diffusion process. After this disposition, we only focus on the process in each time interval. Noticing the continuity on every time interval boundaries generalized by a sequence of stopping times, we acquire the main result of this paper as well as a time change for the perturbed skew process.

Keywords: skew diffusion process; perturbed diffusion process; perturbed skew diffusion process; local time; change in time

MSC: 60J55; 60J60

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1. Introduction

This work documents the properties of the solution to the perturbed skew diffusion process, which writes as

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \alpha d \max_{0\le s\le t} X_s + (2p-1)d\hat{L}_t^X(0), \tag{1}$$

where W denotes a standard Brownian motion on a filtered complete probability space (Ω, $\mathcal{F}$, P) with respect to a filtration $\{\mathcal{F}_t, t \geq 0\}$, μ, σ are supposed to be globally Lipschitz, σ satisfies additionally the Engelbert–Schimdt condition (Lemmas 2 and 3), $\alpha < 1$, $p \in]0,1[$, and the symmetric local time $\hat{L}^X(0)$ is an increasing process starting from $0 \in \mathbf{R}$ such that

$$\int_0^t 1_{\{X_s=0\}} d\hat{L}_s^X(0) = \hat{L}_t^X(0).$$

Brownian motion with skew point was first practiced by Itô and McKean [1] to describe certain stochastic dynamics related to Feller's classification. Later Walsh [2] researched the skew Brownian motion with a discontinuous local time. Afterwards, the classical stochastic differential equation (SDE) expression was established by Harrison and Shepp [3]. Then, Le Gall [4] solved the process as a solution of generalized SDE with local time. Recently, multiskewed Brownian motion was studied by Ramirez [5]. Before arriving at skew point, skew Brownian motion just behaves as a standard Brownian motion. Once hitting skew point, skew Brownian motion moves up and down with different probability, being p and $1-p$, respectively. Note that skew Brownian motion reduces to standard Brownian motion (reflected Brownian motion) when $p = 0.5$ ($p = 0$ or 1). Naturally, skew Brownian motion is more flexible than Brownian motion. The reader may consult more details on recent theoretical development and applications of skew Brownian motion in Lejay [6].

Based on this advantage, the skew diffusion process as a generalization of typical diffusion processes has diverse applications, ranging from mathematical finance in Decamps et al. [7] and Monte Carlo simulation schemes in Lejay and Martinez [8] to heterogeneous

media in Freidlin and Sheu [9]. In this paper, the skew diffusion process satisfies the following SDE
$$dZ_t = \mu(Z_t)dt + \sigma(Z_t)dW_t + (2p-1)d\hat{L}_t^Z(0). \tag{2}$$

Just as the skew diffusion process solving the SDE with the term of symmetric local time, the perturbed diffusion process arises with the term of maximum

$$dU_t = \mu(U_t)dt + \sigma(U_t)dW_t + \alpha d \max_{0\le s\le t} U_s. \tag{3}$$

This version of perturbed process has attracted a crowd of scholars who have devoted themselves to creating a rich literature (see e.g., Carmona, Petit, and Yor [10,11], Chaumont and Doney [12,13], Le Gall and Yor [14,15] and Perman and Werner [16]). Lately, the existence and pathwise uniqueness of the solution of the perturbed reflected process and the doubly perturbed jump-diffusion processes have been practiced by Doney and Zhang [17] and Hu and Ren [18], respectively.

In addition, classical models without skew point have recently failed to capture the actual dynamics caused by more and more world events. It is easy in Figure 1 to observe that from 2005 to 2017, the federal funds rate of America expresses a novel trend. The rate approaches zero from 2008, then stays near zero until 2015. Evidently, a special level governs such a trend and naturally should be taken into consideration in our setting. Hence, we introduce skew diffusion process in our setting to show the mean-reverting and bounded situation.

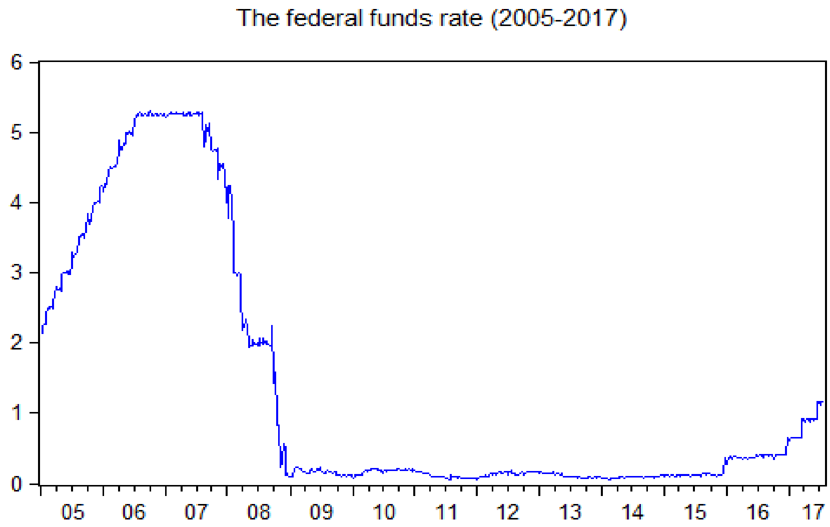

Figure 1. The federal funds rate from 2005 to 2017.

To our knowledge, because there are no previous works concerning the perturbed skew diffusion process, we must handle with proving the properties of the solution defined in Equation (1). However, it seems not easy to prove the result when both perturbed item and skew item exist. To overcome this obstacle, we divide the whole time interval into many parts, hence we are able to focus on the perturbed skew diffusion process in these interval parts, instead of the whole interval. With this division, we disperse the perturbed skew process into two tractable elements, perturbed diffusion process and skew diffusion process, respectively. Then we give a clear proof by iteration on these time intervals. In the meantime, we check the continuity in each time interval, which are generated by a sequence of stopping times. Hence, the existence and uniqueness of solution to the perturbed skew diffusion process is proved.

The remainder of our work is arranged as follows. In Section 2, we provide the iteration lemma as well as prove existence and uniqueness of solution to the perturbed diffusion process by means of this lemma. Section 3 puts forward the relevant analysis about the solution to skew diffusion with the help of signed measure. Section 4 deduces the basic property of solution to the perturbed skew diffusion process and performs one time change version for this process.

2. Perturbed Diffusion Process

This section answers the question of whether the solution to the perturbed diffusion process Equation (3) uniquely exists.

Let $\{W_t, t \geq 0\}$ be a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We consider the following SDE

$$U_t = U_0 + \int_0^t \mu(U_s)ds + \int_0^t \sigma(U_s)dW_s + \alpha \max_{0 \leq s \leq t} U_s, \tag{4}$$

with the assumption that the coefficients of perturbed diffusion process satisfy the Lipschitz continuous condition, i.e., for an existing constant b, the following inequalities hold:

$$|\sigma(u) - \sigma(v)| \leq b|u - v|$$

and

$$|\mu(u) - \mu(v)| \leq b|u - v|.$$

Before obtaining the result of the solution to the perturbed diffusion process, we show a useful lemma.

Lemma 1. *Suppose that there are two continuous functions U_t and H_t. If*

$$U_t = H_t + \alpha \max_{0 \leq s \leq t} U_s,$$

then

$$U_t = H_t + \frac{\alpha}{1-\alpha} \max_{0 \leq s \leq t} H_s.$$

Proof. We write the equation into iteration for two steps,

$$\begin{aligned} U_t &= H_t + \alpha \max_{0 \leq s \leq t} U_s \\ &= H_t + \alpha \max_{0 \leq s \leq t} [H_s + \alpha \max_{0 \leq s_1 \leq s} U_{s_1}] \\ &= H_t + \alpha \max_{0 \leq s \leq t} [H_s + \alpha \max_{0 \leq s_1 \leq s} [H_{s_1} + \alpha \max_{0 \leq s_2 \leq s_1} U_{s_2}]]. \end{aligned}$$

In order to establish the equality, we need two procedures, as follows.

First, we check the upper bound. It is obvious to see the maximum of the sum is less than or equal to the sum of the maximum, that is

$$\begin{aligned} U_t &= H_t + \alpha \max_{0 \leq s \leq t} [H_s + \alpha \max_{0 \leq s_1 \leq s} [H_{s_1} + \cdots]] \\ &\leq H_t + \alpha \max_{0 \leq s \leq t} [H_s + \alpha \max_{0 \leq s_1 \leq t} [H_{s_1} + \cdots]] \\ &= H_t + \alpha \max_{0 \leq s \leq t} [H_s + \alpha \max_{0 \leq s \leq t} [H_s + \cdots]] \\ &= H_t + \alpha \max_{0 \leq s \leq t} H_s + \alpha^2 \max_{0 \leq s \leq t} H_s + \cdots \\ &= H(t) + \frac{\alpha}{1-\alpha} \max_{0 \leq s \leq t} H_s. \end{aligned}$$

Second, we check the lower bound, noticing the fact that for any two functions $P(t), Q(t)$,

$$\max_{0 \leq s \leq t} \{P(s) + \max_{0 \leq s_1 \leq s} Q(s_1)\} \geq \max_{0 \leq s \leq t} \{P(s) + Q(s)\},$$

Thus, we obtain

$$
\begin{aligned}
U_t &= H_t + \alpha \max_{0\le s\le t}[H_s + \alpha \max_{0\le s_1\le s}[H_{s_1} + \alpha \max_{0\le s_2\le s_1}[H_{s_2} + \cdots]]] \\
&\ge H_t + \alpha \max_{0\le s\le t}[H_s + \alpha[H_s + \alpha \max_{0\le s_2\le s}[H_{s_2} + \cdots]]] \\
&\ge H_t + \alpha \max_{0\le s\le t}[H_s + \alpha[H_s + \alpha[H_s + \cdots]]] \\
&= H(t) + \frac{\alpha}{1-\alpha} \max_{0\le s\le t} H_s.
\end{aligned}
$$

This proves the lemma. □

The following theorem states the existence and uniqueness of the solution to the perturbed diffusion process.

Theorem 1. *Let U_0 be a random variable which is independent of W and $E[|U_0|^2] < \infty$. Then, for any fixed $T > 0$, there uniquely exists a continuous solution U_t (adapted with respect to $\mathcal{F}_t$), $t \ge 0$ to (4) satisfying $E[\max_{0\le s\le T} |U_s|^2] < \infty$.*

Proof. Set

$$
U_t^0 = \frac{U_0}{1-\alpha}, \quad 0 \le t < \infty.
$$

Then, we denote by

$$
U_t^{n+1} = U_0 + \int_0^t \mu(U_s^n)ds + \int_0^t \sigma(U_s^n)dW_s + \alpha \max_{0\le s\le t} U_s^{n+1} \tag{5}
$$

the unique continuous adapted solution to perturbed diffusion process with $n \ge 0$. By Lemma 1, set $H_t = U_0 + \int_0^t \mu(U_s^n)ds + \int_0^t \sigma(U_s^n)dW_s$ in Equation (5), we have

$$
\begin{aligned}
U_t^{n+1} = {} & \frac{U_0}{1-\alpha} + \int_0^t \mu(U_s^n)ds + \int_0^t \sigma(U_s^n)dW_s \\
& + \frac{\alpha}{1-\alpha} \max_{0\le s\le t}[\int_0^s \mu(U_\eta^n)d\eta + \int_0^s \sigma(U_\eta^n)dW_\eta].
\end{aligned} \tag{6}
$$

With this iteration expression, as well as the Theorem 2.1 in Doney and Zhang [17], we complete this proof. □

Remark 1. *In fact, Theorem 1 is borrowed from Doney and Zhang [17]. Because Equation (6) in Doney and Zhang [17] is straightly provided, we present more details in Lemma 1.*

3. Skew Diffusion Process

This section answers the question of whether the solution to the skew diffusion process Equation (2) uniquely exists.

When $p = 1$ or 0 in Equation (2), the skew diffusion process degenerates into the reflected diffusion process, and the existence and uniqueness of the solution to this reflected diffusion has been studied by Lions and Sznitman [19]. Therefore, we only study the case $0 < p < 1$.

To begin with, consider the stochastic equation with generalized drift as follows

$$
X_t = X_0 + \int_R \hat{L}_t^X(y)\nu(dy) + \int_0^t b(X_s)dB_s, \tag{7}
$$

in which B denotes a Brownian motion, $\hat{L}^X$ is the symmetric local time of the unknown process X, and $\nu(dy)$ signifies a signed measure. To help us prove the result, we need the description in Engelbert and Schmidt [20]. Set $N_f = \{y \in \mathbf{R} : f(y) = 0\}$ and $E_f = \{y \in \mathbf{R} :$

$\int_G f^{-2}(z)dz = +\infty\}$, where G is any open set including y. Here, we introduce two useful lemmas without proving them.

Lemma 2 (Theorem 4.35 in Engelbert and Schmidt [20])**.** *There exist three equivalent conclusions as follows:*
(a) There exists a fundamental solution X to Equation (7).
(b) There exists a solution X to Equation (7).
(c) $E_b \subseteq N_b$.

Lemma 3 (Theorem 4.37 in Engelbert and Schmidt [20])**.** *(a) The fundamental solution to Equation (7) is unique.*
(b) The solution to Equation (7) is unique if and only if the following condition is satisfied:
If $E_b \subseteq N_b$ *then* $E_b = N_b$.

Remark 2. *On page 153 in Engelbert and Schmidt [20], it states that "Firstly, we establish the existence of a unique solution to Equation (7) which spends minimal time at the zeros of the diffusion coefficient b. We call it fundamental solution. This solution is a strong MARKOV continuous semimartingale up to the explosion time."*

What follows next is the theorem to obtain the existence and uniqueness of the solution to the skew diffusion process.

Theorem 2. *Assume* $0 < p < 1$ *if the coefficient* $\sigma \neq 0$ *in Equation (2) satisfies the bounded condition, i.e., there exists a constant* $M < +\infty$ *such that for all open sets G containing x,* $\int_G 1/\sigma^2(x)dx \leq M$ *and both coefficients* μ, σ *satisfy the Lipschitz condition. Then, there exists a unique solution to Equation (2).*

Proof. Define a signed measure by

$$\nu(A) \triangleq \int_A \frac{\mu(x)}{\sigma^2(x)}dx + (2p-1)\delta_0(A),$$

where $\delta_0(\cdot)$ is the Delta function. By the occupation time formula in Equation (A3) (see Appendix A), Equation (7) becomes

$$\begin{aligned}
&Z_0 + \int_R \hat{L}_t^Z(y)\nu(dy) + \int_0^t \sigma(Z_s)dW_s \\
=&Z_0 + \int_R \hat{L}_t^Z(y)\frac{\mu(y)}{\sigma^2(y)}dy + \int_R \hat{L}_t^Z(y)(2p-1)\delta_0(dy) + \int_0^t \sigma(Z_s)dW_s \\
=&Z_0 + \int_0^t \frac{\mu(Z_s)}{\sigma^2(Z_s)}d\langle Z\rangle_s + (2p-1)\hat{L}_t^Z(0) + \int_0^t \sigma(Z_s)dW_s \\
=&Z_0 + \int_0^t \mu(Z_s)dZ_s + (2p-1)\hat{L}_t^Z(0) + \int_0^t \sigma(Z_s)dW_s \\
=&Z_t.
\end{aligned}$$

Noticing that let $M \in [1, +\infty)$, we have a finite signed measure ν defined on $\mathcal{B}([-M, M])$. As the fundamental assumption in Engelbert and Schmidt [20], $|\nu(\{x\})| \in [0, 1)$ holds. On the other hand, because the coefficient σ satisfies the bounded condition, we know that the set $E_\sigma = \{x \in \mathbb{R} : \int_G \frac{1}{\sigma^2(x)}dx = \infty\} = \emptyset$. Note that $N_\sigma = \{x \in \mathbf{R} : \sigma(x) = 0\} = \emptyset$, then by Lemmas 2 and 3, we prove the existence and uniqueness of solution to Equation (2), and the solution (fundamental solution) is also a strong Markov process. □

4. Perturbed Skew Diffusion Process

With the last two sections devoted to obtaining the results of solutions to the perturbed diffusion process and skew diffusion process, respectively, this section answers the question of whether the solution to the perturbed skew diffusion process in Equation (8) uniquely exists.

$$X_t = x + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s + (2p-1)\hat{L}_t^X(0) + \alpha \max_{0\le s\le t} X_s, \tag{8}$$

where the coefficients μ, σ, p, α, and $W, \hat{L}^X(0)$ are the same in (1). Because there exists a big difference under case $x = 0$ and $x \neq 0$, we study them separately. What follows is the main result.

Theorem 3. *Let $x(\neq 0)$ be a random variable which is independent of W and $E(|x|^2) < \infty$. There exists a unique continuous solution to Equation (8).*

Proof. Iterative technique is adopted to prove the solution property parallel to Le Gall and Yor [15] and Doney and Zhang [17].

Set U^0 to be the unique solution to the following equation:

$$U_t^0 = x + \int_0^t \mu(U_s^0)ds + \int_0^t \sigma(U_s^0)dW_s^0 + \alpha \max_{0\le s\le t} U_s^0.$$

Obviously, such a unique solution to the perturbed diffusion process exists from Section 2. Then, put $T_1 = \inf\{t \geq 0, U_t^0 = 0\}$; we know $T_1 > 0$, as $x \neq 0$. Define $X_t = U_t^0$ and $\hat{L}_t^X(0) = \hat{L}_t^U(0) = 0$ for $t \in [0, T_1]$ with $W_t^1 = W_{t+T_1} - W_{T_1}$ for $t \in [0, +\infty)$, as it is known to all that W_t^1 denotes a standard Brownian motion independent of $\mathcal{F}_{T_1}$.

Next let us focus on the skew diffusion process

$$\begin{cases} V_t^1 = \int_0^t \mu(V_s^1)ds + \int_0^t \sigma(V_s^1)dW_s^1 + (2p-1)\hat{L}_t^{V,1}(0), \\ V_0^1 = 0, \\ \hat{L}_0^{V,1}(0) = 0, \int_0^t 1_{\{V_s^1=0\}} d\hat{L}_s^{V,1}(0) = \hat{L}_t^{V,1}(0). \end{cases} \tag{9}$$

It is known that such a unique solution exists from Section 3.

In general, assume that X has been defined in the time interval $t \in [0, T_{2n-1}]$. We establish X for $T_{2n-1} \leq t \leq T_{2n+1}$, $n \geq 1$, as follows.

First, suppose V^{2n-1} to be the solution to the skew diffusion process:

$$\begin{cases} V_t^{2n-1} = \int_0^t \mu(V_s^{2n-1})ds + \int_0^t \sigma(V_s^{2n-1})dW_s^{2n-1} + (2p-1)\hat{L}_t^{V,2n-1}(0), \\ V_0^{2n-1} = 0, \\ \hat{L}_0^{V,2n-1}(0) = 0, \int_0^t 1_{\{V_s^{2n-1}=0\}} d\hat{L}_s^{V,2n-1}(0) = \hat{L}_t^{V,2n-1}(0), \end{cases}$$

where $W_t^{2n-1} = W_{t+T_{2n-1}} - W_{T_{2n-1}}$. It should be noted that $\hat{L}^{V,2n-1}(0)$ stands for the symmetric local time of V^{2n-1} at 0 for all $n \geq 1$. Put $T_{2n} = \inf\{t > T_{2n-1}; V_{t-T_{2n-1}}^{2n-1} = \max\limits_{0\le s\le T_{2n-1}} X_s\}$ and for $T_{2n-1} \leq t \leq T_{2n}$, define

$$\begin{cases} X_t = V_{t-T_{2n-1}}^{2n-1}, \\ \hat{L}_t^X(0) = \hat{L}_{T_{2n-1}}^X(0) + \hat{L}_{t-T_{2n-1}}^{V,2n-1}(0). \end{cases} \tag{10}$$

Second, suppose U^{2n} be the solution to the perturbed diffusion process:

$$U_t^{2n} = (1-\alpha)X_{T_{2n}} + \int_0^t \mu(U_s^{2n})ds + \int_0^t \sigma(U_s^{2n})dW_s^{2n} + \alpha \max_{0\le s\le t} U_s^{2n},$$

with $W_t^{2n} = W_{t+T_{2n}} - W_{T_{2n}}$. It should be pointed out that $X_{T_{2n}}$ in the above perturbed diffusion process is known and is exactly equal to $\max\limits_{0\le s\le T_{2n-1}} X_s$. Put $T_{2n+1} = \inf\{t > T_{2n}; U_{t-T_{2n}}^{2n} = 0\}$ and for $T_{2n} \le t \le T_{2n+1}$, define

$$\begin{cases} X_t = U_{t-T_{2n}}^{2n}, \\ \hat{L}_t^X(0) = \hat{L}_{T_{2n}}^{U,2n}(0). \end{cases} \tag{11}$$

By this construction, we obtain a line of nondecreasing stopping times T_n, $n \ge 0$. We put $0 = T_0$ and $T = \lim\limits_{n\to\infty} T_n$. Then, X is a continuous process defined on the time interval $[0, T]$ and T is also a stopping time. At this rate, our aim is to show that X defined by the above construction satisfies Equation (8) for $T_{2n} \le t \le T_{2n+1}$, $n = 0, 1, 2, ...,$. First, we show the continuity for X on the time interval boundary. To see this, we show the following equalities.

If X satisfies Equation (10), we obtain

$$\begin{cases} X_{T_{2n-1}} = V_{T_{2n-1}-T_{2n-1}}^{2n-1} = V_0^{2n-1} = 0, \\ X_{T_{2n}} = V_{T_{2n}-T_{2n-1}}^{2n-1} = \max\limits_{0\le s\le T_{2n-1}} X_s. \end{cases}$$

If X satisfies Equation (11), we have

$$\begin{cases} X_{T_{2n+1}} = U_{T_{2n+1}-T_{2n}}^{2n} = 0, \\ X_{T_{2n}} = U_{T_{2n}-T_{2n}}^{2n} = U_0^{2n} = X_{T_{2n}} = \max\limits_{0\le s\le T_{2n-1}} X_s. \end{cases}$$

Noticing the value of X at the time interval boundaries $(T_n, n \ge 0)$, the continuity property follows.

Now, we show that X is the unique solution to Equation (8). For $n = 0$, e.g., $0 \le t \le T_1$, we know that X is defined by

$$X_t = U_t^0.$$

Recall the definition that $\hat{L}_t^X(0) = \hat{L}_t^U(0) = 0$ for $0 \le t \le T_1$, hence

$$\begin{aligned} X_t &= (1-\alpha)X_0 + \int_0^t \mu(U_s^0)ds + \int_0^t \sigma(U_s^0)dW_s^0 + \alpha \max_{0\le s\le t} U_s^0 \\ &= (1-\alpha)U_0^0 + \int_0^t \mu(U_s^0)ds + \int_0^t \sigma(U_s^0)dW_s^0 + \alpha \max_{0\le s\le t} U_s^0 + (2p-1)\hat{L}_t^X(0) \\ &= x + \int_0^t \mu(U_s^0)ds + \int_0^t \sigma(U_s^0)dW_s + \alpha \max_{0\le s\le t} U_s^0 + (2p-1)\hat{L}_t^X(0). \end{aligned} \tag{12}$$

We conclude that X is the solution to the perturbed skew diffusion process in $0 \le t \le T_1$. Then, for $n = 1$, i.e., $T_1 \le t \le T_2$, X is defined by

$$X_t = V_{t-T_1}^1.$$

It is easy to derive

$$\begin{aligned} X_t &= \int_0^{t-T_1} \mu(V_s^1)ds + \int_0^{t-T_1} \sigma(V_s^1)dW_s^1 + (2p-1)\hat{L}_{t-T_1}^{V,1}(0) \\ &= X_{T_1} + \int_0^{t-T_1} \mu(V_s^1)ds + \int_0^{t-T_1} \sigma(V_s^1)dW_s^1 + (2p-1)\hat{L}_{t-T_1}^{V,1}(0) \\ &= x + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s + (2p-1)\hat{L}_t^X(0) + \alpha \max_{0\le s\le T_1} X_s \\ &= x + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s + (2p-1)\hat{L}_t^X(0) + \alpha \max_{0\le s\le t} X_s, \end{aligned}$$

where we use the following facts: $\max_{0\leq s\leq T_1} X_s = \max_{0\leq s\leq t} X_s$ for $T_1 \leq t \leq T_2$, $X_{T_1} = 0$, substitute T_1 for t in Equation (12), and the definition by Equation (10). Furthermore, if $T_1 \leq t \leq T_2$, we also derive

$$\int_0^t 1_{\{X_s=0\}} d\hat{L}_s^X(0) = \int_{T_1}^t 1_{\{X_s=0\}} d\hat{L}_{s-T_1}^{V,1}(0) = \int_0^{t-T_1} 1_{\{V_u^1=0\}} d\hat{L}_u^{V,1}(0) = \hat{L}_{t-T_1}^{V,1}(0) = \hat{L}_t^X(0).$$

Thus, we obtain the solution to the perturbed skew diffusion process for $0 \leq t \leq T_2$. In general, assume that we establish the solution X to Equation (8) at the time interval $[0, T_{2n}]$. When $T_{2n} \leq t \leq T_{2n+1}$, from the definition, X takes the form of

$$X_t = U_{t-T_{2n}}^{2n},$$

which leads to

$$\begin{aligned}
X_t &= (1-\alpha)X_{T_{2n}} + \int_0^{t-T_{2n}} \mu(U_s^{2n})ds + \int_0^{t-T_{2n}} \sigma(U_s^{2n})dW_s^{2n} + \alpha \max_{0\leq s\leq t-T_{2n}} U_s \\
&= (1-\alpha)(x + \int_0^{T_{2n}} \mu(X_s)ds + \int_0^{T_{2n}} \sigma(X_s)dW_s + (2p-1)\hat{L}_{T_{2n}}^X(0) + \alpha \max_{0\leq s\leq T_{2n}} X_s) \\
&\quad + \int_0^{t-T_{2n}} \mu(U_s^{2n})ds + \int_0^{t-T_{2n}} \sigma(U_s^{2n})dW_s^{2n} + \alpha \max_{0\leq s\leq t-T_{2n}} U_s \\
&= x + \int_0^{T_{2n}} \mu(X_s)ds + \int_0^{T_{2n}} \sigma(X_s)dW_s + (2p-1)\hat{L}_{T_{2n}}^X(0) + \alpha \max_{0\leq s\leq T_{2n}} X_s - \alpha X_{T_{2n}} \\
&\quad + \int_0^{t-T_{2n}} \mu(U_s^{2n})ds + \int_0^{t-T_{2n}} \sigma(U_s^{2n})dW_s^{2n} + \alpha \max_{0\leq s\leq t-T_{2n}} U_s \\
&= x + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s + (2p-1)\hat{L}_t^X(0) + \alpha \max_{0\leq s\leq t} X_s,
\end{aligned}$$

Note that $X_{T_{2n}} = \max_{0\leq s\leq t-T_{2n-1}} X_s$, and $\max_{0\leq s\leq T_{2n}} X_s = \max_{0\leq s\leq t-T_{2n-1}} X_s$, the expression $\max_{T_{2n}\leq s\leq t} X_s = \max_{0\leq s\leq t} X_s$ holds.

Then, X satisfies the equation Equation (8) for $T_{2n} \leq t \leq T_{2n+1}$. In addition, from the definition of stopping times T_n, we see $X_t \neq 0$, $t \in [T_{2n}, T_{2n+1})$, implying

$$\hat{L}_t^X(0) = \int_0^t 1_{\{X_s=0\}} d\hat{L}_s^X(0) = \int_0^{T_{2n}} 1_{\{X_s=0\}} d\hat{L}_s^X(0) = \hat{L}_{T_{2n}}^X(0).$$

In a similar way, we can show the solution X for $T_{2n+1} \leq t \leq T_{2n+2}$ as well.

Lastly, we prove $T = \infty$, *a.s.* With the definition of T_{2n+1}, we have

$$X_{T_{2n+1}} = 0.$$

We can also write $X_{T_{2n+1}}$ by

$$\begin{aligned}
X_{T_{2n+1}} &= X_{T_{2n+1}} + X_{T_{2n}} - X_{T_{2n}} \\
&= \max_{0\leq s\leq T_{2n}} X_s + \int_{T_{2n}}^{T_{2n+1}} \mu(X_s)ds + \int_{T_{2n}}^{T_{2n+1}} \sigma(X_s)dW_s \\
&\quad + \alpha(\max_{0\leq s\leq T_{2n+1}} X_s - \max_{0\leq s\leq T_{2n}} X_s) + (2p-1)(\hat{L}_{T_{2n+1}}(0) - \hat{L}_{T_{2n}}(0)).
\end{aligned}$$

Suppose $T < \infty$ *a.s.* with positive probability and let $n \to \infty$; we have $\max_{0\leq s\leq T} X_s = X_{T_{2n+1}} = 0$, which contradicts the definition of $X_0 = \frac{x}{1-\alpha} \neq 0$.

On the other hand, the construction of the processes in every time intervals gives the uniqueness of the solution, and such a solution is unique in the whole time interval. We complete the proof. □

Theorem 4. *Suppose $x = 0$, if $0 \leq \alpha < \frac{1}{2}$ and the coefficient σ satisfies the bounded condition in the previous section (Theorem 2, Section 3), then there exists a unique continuous solution to Equation (8).*

Proof. We want to use the iteration scheme. Set $X_t^0 = 0$, and $\{X_t^{n+1}\}_{n\geq 0}$ satisfies

$$X_t^{n+1} = \int_0^t \mu(X_s^n)ds + \int_0^t \sigma(X_s^n)dW_s + (2p-1)\hat{L}_t^{X,n+1}(0) + \alpha \sup_{0\leq s\leq t} X_s^{n+1}. \tag{13}$$

By the reflection principle,

$$\hat{L}_t^{X,n+1}(0) = -\inf_{s\leq t}\Big\{\Big(\int_0^s \mu(X_u^n)du + \int_0^s \sigma(X_u^n)dW_u + \alpha \sup_{0\leq u\leq s} X_u^{n+1}\Big) \wedge 0\Big\}. \tag{14}$$

Now, Equations (13) and (14) lead to

$$\begin{aligned}|X_t^{n+1} - X_t^n| \leq{}& \Big|\int_0^t (\mu(X_s^n) - \mu(X_s^{n-1}))ds\Big| + \Big|\int_0^t (\sigma(X_s^n) - \sigma(X_s^{n-1}))dW_s\Big| \\ &+ \alpha\Big|\sup_{0\leq s\leq t} X_s^{n+1} - \sup_{0\leq s\leq t} X_s^n\Big| + (2p-1)|\hat{L}_t^{X,n+1}(0) - \hat{L}_t^{X,n}(0)|.\end{aligned}$$

As a result,

$$\begin{aligned}\sup_{s\leq t}|X_s^{n+1} - X_s^n| \leq{}& \frac{2p}{1-2\alpha p}\sup_{s\leq t}\Big|\int_0^s (\mu(X_u^n) - \mu(X_u^{n-1}))du\Big| \\ &+ \frac{2p}{1-2\alpha p}\sup_{s\leq t}\Big|\int_0^s \sigma(X_u^n) - \sigma(X_u^{n-1})dW_u\Big|.\end{aligned}$$

By Burkholder's inequality,

$$\begin{aligned}E\Big[\sup_{s\leq t}|X_s^{n+1} - X_s^n|^2\Big] &\leq C_{\alpha,p}E\Big[\int_0^t (\sigma(X_s^n) - \sigma(X_s^{n-1}))^2 ds\Big] \\ &\leq C_{\alpha,p}E\Big[\int_0^t (X_s^n - X_s^{n-1})^2 ds\Big].\end{aligned}$$

Thus, we deduce that for any fixed $T > 0$,

$$E\Big[\sup_{s\leq T}|X_s^{n+1} - X_s^n|^2\Big] \leq \frac{(C_{\alpha,p,T})^n}{n!},$$

which yields

$$P\Big[\max_{0\leq s\leq T}|X_s^{n+1} - X_s^n| > \frac{1}{2^n}\Big] \leq \frac{(4C_{\alpha,p,T})^n}{n!}.$$

With the lemma of Borel–Cantelli, we can deduce that $X^n \to X$ on $[0,T]$ *a.s.* We can also obtain the convergence property of $\int_0^t \mu(X_s^n)ds + \int \sigma(X_s^n)dW_s$ *a.s.* Accordingly, in Equation (13), it is seen that $\hat{L}^{X,n}(0)$ performs the convergence property to some nondecreasing process $\hat{L}^X(0)$. Lastly, we have the fact that

$$\hat{L}_t^X(0) = \lim_{n\to\infty}\hat{L}_t^{X,n}(0) = \lim_{n\to\infty}\int_0^t 1_{\{X_s^n=0\}}d\hat{L}_s^{X,n}(0) = \int_0^t 1_{\{X_s=0\}}d\hat{L}_s^X(0).$$

Indeed, for any $f \in C_0(0,+\infty)$, $\int_0^t f(X_s) d\hat{L}_s^X(0) = \lim_{n\to\infty} \int_0^t f(X_s^n) d\hat{L}_s^{X,n}(0)$ holds. Then, we turn to proving the uniqueness of the solution. Using a similar way as the one above, let X^1 and X^2 be two solutions to Equation (13); we find

$$E[\sup_{s\le t} |X_s^1 - X_s^2|^2] \le C_{\alpha,p} E[\int_0^t (X_u^1 - X_u^2)^2 du].$$

By Gronwall's inequality, it is obvious that $X_t^1 = X_t^2$. Thus, we complete the proof. □

To further explore the time change in the perturbed skew process, we provide the next corollary.

Corollary 1. *Set $T_0 = 0$ and $X_0 = x$. Define $T_i(t) = \inf\{t > T_{i-1}, X_t > X_{T_{i-1}}\}$, $i \ge 1$ with $T_\infty(t) = \infty$. Define a new process $Y_t = X_{T_i(t)}$, then Y is a continuous skew diffusion process. Furthermore, $\{T_i(t), i \ge 0\}$ form a sequence of stopping times.*

Proof. It is obvious to see that $\{T_i(t), i \ge 0\}$ are stopping times depending on t, and for each time interval $[T_{i-1}(t), T_i(t)]$ it follows that

$$\max_{T_{i-1}(t)\le s\le T_i(t)} X_s = \max_{0\le s\le T_i(t)} X_s = X_{T_i(t)},$$

where the last equality comes from the apparent relationships

$$0 = T_0(t) < T_1(t) < \cdots < T_\infty(t) = \infty,$$

and

$$x = X_0 < X_{T_1(t)} < \cdots < X_{T_\infty(t)} = \infty.$$

Then, rewrite Equation (8) by

$$(1-\alpha)X_{T_i(t)} = x + \int_0^{T_i(t)} \mu(X_s) ds + \int_0^{T_i(t)} \sigma(x_s) dW_s + (2p-1)\hat{L}_{T_i(t)}^X(0).$$

We conclude that $Y_t = (1-\alpha)X_{T_i(t)}$ is a continuous-time skew diffusion after this time change, and we prove this corollary. □

5. Conclusions and Summary

In this work, we consider a novel dynamic called the perturbed skew diffusion process. Such a process contains perturbed and skew phenomena. Perturbed phenomenon means that the model will reflect the maximum of the model in the past time, while the skew phenomenon reflects different probabilities of the upward and downward movement by p and $1-p$, respectively. We first prove the existence and uniqueness of the solution to the perturbed skew diffusion process. The idea is to disperse the perturbed skew diffusion process into the perturbed diffusion and skew diffusion processes, respectively. To learn more construction about this model, we study the relation between the perturbed skew diffusion process and skew diffusion process. In the future, perturbed skew diffusion may be applied to lookback options. Lookback options are options where the return depends not only on the strike price of the underlying asset but also on the highest or lowest price of the underlying asset over the life of the option. As with many exotic options, the payoff structure of a lookback option is related to the maximum or minimum value reached by the underlying asset price during the term of the contract. Lookback options are also path-dependent options, which enable the holder to execute an option at the most beneficial price of the underlying dynamic during the term of an option. A benefit from this exotic option is that the investor can "look back" or have retrospect on the underlying setting of such an option after getting into a long or short position, and then they can seek to maximize the value of the options. In summary, perturbed phenomena may help investors know the past maximum dynamics, hence maximizing their benefits.

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Appendix A

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Denote by a continuous semimartingale $\{X_t, \mathcal{F}_t; 0 \leq t \leq \infty\}$ an adapted process which can uniquely be expressed by

$$X_t = X_0 + M_t + V_t.$$

M_t writes on a continuous martingale ($M_0 = 0$), and V_t is an adapted bounded variation process with continuous sample trajectory ($V_0 = 0$).

We borrow a similar expression of local times with respect to X in Protter [21]. Denote by $\mathrm{sign}(y)$ the sign function as follows

$$\mathrm{sign}(y) = \begin{cases} 1, & y > 0, \\ -1, & y \leq 0, \end{cases}$$

and provide the local time with respect to X defined by

$$\frac{1}{2} L_t^X(\beta) = (X_t - \beta)^+ - (X_0 - \beta)^+ - \int_0^t 1_{(\beta,+\infty)}(X_s) dX_s,$$

$$\frac{1}{2} L_t^X(\beta) = (X_t - \beta)^- - (X_0 - \beta)^- + \int_0^t 1_{(-\infty,\beta]}(X_s) dX_s,$$

$$L_t^X(\beta) = |X_t - \beta| - |X_0 - \beta| - \int_0^t \mathrm{sign}(X_s - \beta) dX_s.$$

Denote by $L_t^X(\beta)$ (resp. $L_t^X(\beta-)$) the local time (resp. left local time) for X_t, where $L^X(\beta-) = \lim_{c \to \beta, c < \beta} L^X(c)$. Then, the symmetric local time for X at the point β takes the following form

$$\hat{L}_t^X(\beta) = \frac{L_t^X(\beta) + L_t^X(\beta-)}{2}, \tag{A1}$$

$\mathbf{P}^\beta$-a.s. for every $\beta \in \mathbf{R}$. It is a continuous increasing process in t and is constant on any interval on which $X_t \neq \beta$.

Suppose that f is a function whose left and right derivatives are denoted by f'_+ and f'_- and the second derivative measure μ (μ can be viewed as a general second differential measure of f ($\mu = f''$ in much of the literature)). Then, from the Meyer–Tanaka formula (see also Salins and Spiliopoulos [22]), we have:

$$f(X_t) = f(X_0) + \frac{1}{2} \int_0^t [f'_+(X_s) + f'_-(X_s)] dX_s + \frac{1}{2} \int_{\mathbb{R}} \hat{L}_t^X(y) \mu(dy). \tag{A2}$$

Here, μ has the properties $\mu((a,b]) = f'_+(b) - f'_+(a)$ and $\mu(\{x\}) = \lim_{a\uparrow x} \mu((a,x])$.

We have almost surely, for every $t \geq 0$ and every non-negative measurable function φ on $\mathbb{R}$,

$$\int_0^t \varphi(X_s) d\langle X, X\rangle_s = \int_{\mathbb{R}} \varphi(a) L_t^X(a) da. \tag{A3}$$

For more details on symmetric local time, we refer the reader to Karatzas and Shreve [23] and Revuz and Yor [24]).

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