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A Unified Local-Semilocal Convergence Analysis of Efficient Higher Order Iterative Methods in Banach Spaces

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Abstract: To deal with the estimation of the locally unique solutions of nonlinear systems in Banach spaces, the local as well as semilocal convergence analysis is established for two higher order iterative methods. The given methods do not involve the computation of derivatives of an order higher than one. However, the convergence analysis was carried out in earlier studies by using the assumptions on the higher order derivatives as well. Such types of assumptions limit the applicability of techniques. In this regard, the convergence analysis is developed in the present study by imposing the conditions on first order derivatives only. The central idea for the local analysis is to estimate the bounds on convergence domain as well as the error approximations of the iterates along with the formulation of sufficient conditions for the uniqueness of the solution. Based on the choice of initial estimate in the given domain, the semilocal analysis is established, which ensures the convergence of iterates to a unique solution in that domain. Further, some applied problems are tested to certify the theoretical deductions.

Keywords: convergence analysis; iterative methods; Fréchet-derivative; Banach spaces

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1. Introduction

Solving the problems in many branches of science and engineering, through the process of mathematical modeling, is one of the major challenges in the field of applied mathematics. Under some assumptions, a particular problem is modeled into a nonlinear equation, or more general, a system of nonlinear equations, which can be expressed mathematically as

$$F(x) = 0, \tag{1}$$

where $F : D \subset \mathcal{B} \to \mathcal{B}_1$ is a nonlinear mapping, $\mathcal{B}$ and $\mathcal{B}_1$ are Banach spaces, and D is an open convex subset of $\mathcal{B}$. It should be noted, however, that obtaining the analytical or closed form solutions of nonlinear systems is a challenging task. However, iterative methods based on the fixed point theory [1–5] provide the approximate solutions in numerical form up to the desired accuracy. Numerous iterative schemes have been developed in the given context (for example, see [5–8] and references therein) and further investigated for their convergence behavior in Banach spaces.

As a conventional approach, the convergence order of an iterative scheme is estimated by employing the Taylor series expansions, which involve the set of assumptions on higher order derivatives $(F^{(n)},\ n = 1, 2, ...)$. Such assumptions certainly limit the applicability of techniques, since most of these involve derivatives only up to the first order. As a matter of fact, consider an example of a real valued function $F : D \subseteq \mathbb{R} \to \mathbb{R},\ D = [-0.5\,, 1.5]$, which is defined by

$$F(x) = \begin{cases} x^3 \ln(x^2) + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Consequently,

$$F'(x) = 3x^2 \ln(x^2) + 5x^4 - 4x^3 + 2x^2,$$
$$F''(x) = 6x \ln(x^2) + 20x^3 - 12x^2 + 10x,$$

and

$$F'''(x) = 6 \ln(x^2) + 60x^2 - 24x + 22.$$

Apparently, $F'''(x)$ is unbounded in the given domain D, and therefore, expanding $F(x)$ as the Taylor series might not be the suitable approach to analyze the convergence behavior of an iterative technique.

Apart from the convergence order, the convergence behavior of an iterative method is significantly affected by the selection of the initial approximation in the neighborhood of the solution. An iterative scheme is classified as locally convergent if the convergence is bound to happen only in the case of a sufficient proximity of initial estimate to the desired solution, whereas the globally convergent schemes are not affected by the selection of a initial estimate for their convergence. In contrary to the Taylor series approach, in recent times, many authors have investigated the locally convergent techniques by imposing assumptions only on the first order derivatives. The most appropriate methodologies adopted in Banach spaces are local and semilocal convergence analysis [9–18]. Based on the information around the solution, the local analysis provides the bounds on the convergence domain and the error approximations of iterates along with the set of conditions for the uniqueness of solution in the given domain. On the other hand, the semilocal analysis is concerned about the convergence of an iterative scheme for a given initial estimate, which particularly involves the study of majorizing sequences [9] that are established to further formulate the sufficient conditions for the convergence of iterates. It is important to further discuss the importance as well as the difference between the local and semilocal analysis by stating:

(i) The local analysis requires information about the solution, whereas the semilocal analysis utilizes information about the initial guess.
(ii) The local convergence results are crucial to study the convergence behavior, since these results illustrate the degree of difficulty in choosing the initial guesses. However, the semilocal results require the initial guess to be close enough to the solution so that the sequence generated by the given method converges to it.
(iii) The local as well as semilocal analysis provide knowledge in advance about the number of steps required to achieve the desired error tolerance level. Moreover, these results provide the information about the uniqueness of the solution in the given domain.

From this perspective, we shall investigate the local as well as semilocal convergence of the sixth order iterative techniques, developed by Singh and Sharma [8], which are denoted by $\boldsymbol{\psi_1}$ and $\boldsymbol{\psi_2}$ as follows.

Method-1 ($\boldsymbol{\psi_1}$):

$$\left\{ \begin{array}{rcl} y^{(k)} & = & x^{(k)} - F'(x^{(k)})^{-1} F(x^{(k)}), \\ z^{(k)} & = & y^{(k)} - F'(x^{(k)})^{-1} F(y^{(k)}), \\ x^{(k+1)} & = & z^{(k)} - (3I - 3T_k + T_k^2) F'(x^{(k)})^{-1} F(z^{(k)}). \end{array} \right. \tag{2}$$

Method-2 ($\boldsymbol{\psi_2}$):

$$\left\{ \begin{array}{rcl} y^{(k)} & = & x^{(k)} + F'(x^{(k)})^{-1} F(x^{(k)}), \\ z^{(k)} & = & y^{(k)} - F'(x^{(k)})^{-1} F(y^{(k)}), \\ x^{(k+1)} & = & z^{(k)} - (3I - 3T_k + T_k^2) F'(x^{(k)})^{-1} F(z^{(k)}). \end{array} \right. \tag{3}$$

Here, $T_k = F'(x^{(k)})^{-1}F'(z^{(k)})$. For any $x \in D \subseteq \mathcal{B}$, $F'(x) \in \mathcal{L}(\mathcal{B}, \mathcal{B}_1)$ is the first Fréchet derivative [19] of the nonlinear operator defined by (1), where $\mathcal{L}(\mathcal{B}, \mathcal{B}_1)$ stands for the set of bounded linear mappings from $\mathcal{B}$ to $\mathcal{B}_1$. The given methods are investigated thoroughly in [8] for their convergence behavior and computational efficiency along with illustrating the benefits over the existing methods by testing on the variety of applied nonlinear problems. It is evident from the Equations (2) and (3) that both of the methods utilize the computation of derivatives of order only up to one, however, the sixth order convergence is proven in [8] by using the hypotheses on the existence of higher order derivatives. It is worth noticing that the convergence order is determined in [8] only for a special case of $\mathcal{B} = \mathcal{B}_1 = \mathbb{R}^m$, but it is imperative to study the convergence in a more general setting of a Banach space to extend their applicability to a wider section of problems.

Keeping in mind the above discussion, our objective here is to weaken the conditions given in [8], by developing the hypotheses on the first order derivatives only, which undoubtedly extends the applicability of the considered iterative schemes. In what follows, the local convergence analysis for both the methods is established in Section 2, and further, the semilocal analysis is developed in Section 3 by imposing conditions on the operators involved in the given methods. Some numerical applications are given in Section 4 to validate the theoretical deductions. Section 5 contains the concluding remarks.

2. Local Convergence Analysis

In this section, the local convergence analysis shall be developed for iterative methods $\boldsymbol{\psi_1}$ and $\boldsymbol{\psi_2}$, which are defined by Equation (2) and Equation (3), respectively. To establish the local convergence analysis of $\boldsymbol{\psi_1}$, we first define some real parameters and functions. For $M = [0, \infty)$, let the following suppositions (i–iv) hold.

(i) There exists a function $p_0 : M \to M$, which is non-decreasing and continuous, such that the equation
$$p_0(t) - 1 = 0,$$
has the smallest root $\rho_0 \in M - \{0\}$. Set $M_0 = [0,\ \rho_0)$.

(ii) There exists a function $p : M_0 \to M$, which is non-decreasing and continuous, such that the equation
$$h_1(t) - 1 = 0,$$
has the smallest root $\rho_1 \in M_0 - \{0\}$, where $h_1 : M_0 \to \mathbb{R}$ is defined as
$$h_1(t) = \frac{\int_0^1 p((1-\theta)t)d\theta}{1 - p_0(t)}.$$

(iii) The equation
$$h_2(t) - 1 = 0,$$
has the smallest root $\rho_2 \in M_0 - \{0\}$, where $h_2 : M_0 \to \mathbb{R}$ is defined as
$$h_2(t) = \frac{\int_0^1 p(t + \theta h_1(t)t)d\theta}{1 - p_0(t)} h_1(t).$$

(iv) The equation
$$h_3(t) - 1 = 0,$$
has the smallest root $\rho_3 \in M_0 - \{0\}$, where $h_3 : M_0 \to \mathbb{R}$ is defined as
$$h_3(t) = \left[\frac{\int_0^1 p(t + \theta h_2(t)t)d\theta}{1 - p_0(t)} + \left(\frac{p_0(h_2(t)t) + p_0(t)}{1 - p_0(t)} + \left(\frac{p_0(h_2(t)t) + p_0(t)}{1 - p_0(t)} \right)^2 \right) \right.$$
$$\left. \times \frac{1 + \int_0^1 p_0(\theta h_2(t)t)d\theta}{1 - p_0(t)} \right] h_2(t).$$

We shall aim to prove that

$$\rho_* = \min\{\rho_1, \rho_2, \rho_3\}, \tag{4}$$

is the convergence radius for the method $\boldsymbol{\psi_1}$. Setting $M_* = [0, \rho_*)$, and by the definition of ρ_*, we have for all $t \in M_*$,

$$0 \le p_0(t) < 1, \tag{5}$$

$$0 \le h_i(t) < 1, \tag{6}$$

for each $i = 1, 2, 3$.

For any $x \in D$, let us denote $S[x, r]$ as the closure of an open ball $S(x, r)$, having radius equal to 'r'. Before we proceed to the main result, it is required that the following conditions (C_1)–(C_4) hold.

(C_1) : The equation $F(x) = 0$ has a solution $x_* \in D$, such that the inverse operator $F'(x_*)^{-1} \in \mathcal{L}(\mathcal{B}_1, \mathcal{B})$.

(C_2) : For each $v \in D$,

$$\|F'(x_*)^{-1}(F'(v) - F'(x_*))\| \le p_0(\|v - x_*\|).$$

Let $D_1 = S(x_*, \rho_0) \cap D$.

(C_3) : For each $v_1, v_2 \in D_1$,

$$\|F'(x_*)^{-1}(F'(v_1) - F'(v_2))\| \le p(\|v_1 - v_2\|).$$

(C_4) : $S[x_*, \rho_*] \subset D$.

Next, the local convergence analysis is developed for method $\boldsymbol{\psi_1}$ using the conditions (C_1)–(C_4).

Theorem 1. *Under the conditions (C_1)–(C_4) and further choosing the initial estimate $x^{(0)} \in S(x_*, \rho_*) - \{x_*\}$ for the iterative method defined by Equation (2), the following assertions hold:*

$$x^{(k)} \in S(x_*, \rho_*), \quad \forall\, k = 0, 1, 2, \ldots, \tag{7}$$

$$\|y^{(k)} - x_*\| \le h_1(\|x^{(k)} - x_*\|)\, \|x^{(k)} - x_*\| < \|x^{(k)} - x_*\| < \rho_*, \tag{8}$$

$$\|z^{(k)} - x_*\| \le h_2(\|x^{(k)} - x_*\|)\, \|x^{(k)} - x_*\| < \|x^{(k)} - x_*\|, \tag{9}$$

$$\|x^{(k+1)} - x_*\| \le h_3(\|x^{(k)} - x_*\|)\, \|x^{(k)} - x_*\| < \|x^{(k)} - x_*\|, \tag{10}$$

and

$$\lim_{k \to \infty} x^{(k)} = x_*,$$

where the functions h_1, h_2 and h_3 are defined previously, and ρ_* is defined in Equation (4).

Proof. Assertions (7)–(10) shall be shown using the mathematical induction on 'k'. Let us choose an arbitrary point $u \in S(x_*,\ \rho_*) - \{x_*\}$. In view of (4) and (5), and the condition (C_1), we have in turn that

$$\|F'(x_*)^{-1}(F'(u) - F'(x_*))\| \le p_0(\|u - x_*\|) \le p_0(\rho_*) < 1. \tag{11}$$

The Banach lemma [19] on the invertible linear operators, together with Equation (11), imply that $F'(u)^{-1} \in \mathcal{L}(\mathcal{B}_1, \mathcal{B})$ and

$$||F'(u)^{-1}F'(x_*)|| \leq \frac{1}{1-p_0(||u-x_*||)}. \tag{12}$$

Therefore , the iterates $y^{(0)}$, $z^{(0)}$ and $x^{(1)}$, of the technique $\boldsymbol{\psi_1}$, are well defined by (12) for $u = x^{(0)}$. Then, we obtain

$$\begin{aligned} y^{(0)} - x_* &= x^{(0)} - x_* - F'(x^{(0)})^{-1}F(x^{(0)}) \\ &= F'(x^{(0)})^{-1}F'(x_*)\int_0^1 F'(x_*)^{-1}\Big[F'(x^{(0)}) - F'(x_* + \theta(x^{(0)} - x_*))\Big](x^{(0)} - x_*)d\theta. \end{aligned} \tag{13}$$

Using Equations (4), (6) and (12) (for $u = x^{(0)}$), and condition (C_2), the following estimate is obtained from Equation (13),

$$\begin{aligned} \|y^{(0)} - x_*\| &\leq \frac{\int_0^1 p((1-\theta)\|x^{(0)} - x_*\|)\, d\theta}{1 - p_0(\|x^{(0)} - x_*\|)}\|x^{(0)} - x_*\| \\ &\leq h_1(\|x^{(0)} - x_*\|)\|x^{(0)} - x_*\| < \|x^{(0)} - x_*\| < \rho_*, \end{aligned} \tag{14}$$

which proves that the iterate $y^{(0)} \in S(x_*, \rho_*)$, and therefore, assertion (8) holds for $k = 0$. Further, using the second sub-step of method $\boldsymbol{\psi_1}$ for $k = 0$, we can write

$$\begin{aligned} z^{(0)} - x_* &= y^{(0)} - x_* - F'(x^{(0)})^{-1}F(y^{(0)}) \\ &= F'(x^{(0)})^{-1}\left[F'(x^{(0)}) - \int_0^1 F'(x_* + \theta(y^{(0)} - x_*))d\theta\right](y^{(0)} - x_*). \end{aligned} \tag{15}$$

Then, by Equations (4), (6) and (14), and condition (C_2), the Equation (15) yields the following estimate,

$$\begin{aligned} \|z^{(0)} - x_*\| &\leq \frac{\int_0^1 p(\|x^{(0)} - x_*\| + \theta\|y^{(0)} - x_*\|)d\theta}{1 - p_0(\|x^{(0)} - x_*\|)}\|y^{(0)} - x_*\| \\ &\leq h_2(\|x^{(0)} - x_*\|)\|x^{(0)} - x_*\| \\ &< \|x^{(0)} - x_*\|, \end{aligned} \tag{16}$$

which means that the iterate $z^{(0)} \in S(x_*, \rho_*)$ and the assertion (9) is true for $k = 0$. Furthermore, by the third sub-step of $\boldsymbol{\psi_1}$ for $k = 0$, we have

$$\begin{aligned} x^{(1)} - x_* = z^{(0)} - x_* - F'(x^{(0)})^{-1}F(z^{(0)}) - F'(x^{(0)})^{-1}(F'(x^{(0)}) - F'(z^{(0)})) \\ \times \Big[I + F'(x^{(0)})^{-1}(F'(x^{(0)}) - F'(z^{(0)}))\Big]F'(x^{(0)})^{-1}F(z^{(0)}), \end{aligned} \tag{17}$$

and then using the approximation

$$\begin{aligned} \|F'(x_*)^{-1}F(z^{(0)})\| &\leq \int_0^1 \|F'(x_*)^{-1}F'(x_* + \theta(z^{(0)} - x_*))\|\|z^{(0)} - x_*\|d\theta \\ &\leq \int_0^1 \|I + F'(x_*)^{-1}(F'(x_* + \theta(z^{(0)} - x_*)) - F'(x_*))\|\|z^{(0)} - x_*\|d\theta \\ &\leq \left[1 + \int_0^1 p_0(\theta\|z^{(0)} - x_*\|)d\theta\right]\|z^{(0)} - x_*\|, \end{aligned} \tag{18}$$

the following estimate is obtained,

$$\|x^{(1)} - x_*\| \leq \Bigg[\frac{\int_0^1 p(\|x^{(0)} - x_*\| + \theta\|z^{(0)} - x_*\|)d\theta}{1 - p_0(\|x^{(0)} - x_*\|)} + \frac{p_0(\|x^{(0)} - x_*\|) + p_0(\|z^{(0)} - x_*\|)}{1 - p_0(\|x^{(0)} - x_*\|)} \\ \times \left(1 + \frac{p_0(\|x^{(0)} - x_*\|) + p_0(\|z^{(0)} - x_*\|)}{1 - p_0(\|x^{(0)} - x_*\|)}\right) \frac{1 + \int_0^1 p_0(\theta\|z^{(0)} - x_*\|)d\theta}{1 - p_0(\|x^{(0)} - x_*\|)} \Bigg] \\ \times \|z^{(0)} - x_*\| \\ \leq h_3(\|x_0 - x_*\|)\|x_0 - x_*\| < \|x_0 - x_*\| < \rho_*,$$

which proves that the iterate $x^{(1)} \in S(x_*, \rho_*)$, and the assertion (10) holds for $k = 0$. The induction process on 'k', for the estimates (8), (9) and (10), is terminated if $x^{(0)}, y^{(0)}$ and $x^{(1)}$ are replaced by $x^{(k)}, y^{(k)}$ and $x^{(k+1)}$, respectively. Moreover, in view of the estimate

$$\|x^{(k+1)} - x_*\| < \beta\|x^{(k)} - x_*\| < \rho_*,$$

where $\beta = h_3(||x^{(0)} - x_*||) \in [0, 1)$, we eventually have that $x^{(k+1)} \in S(x_*, \rho_*)$ for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} x^{(k)} = x_*$. □

Further, we claim the uniqueness of solution through the following proposition.

Proposition 1. *Assume that:*

(i) The equation $F(x) = 0$ has a solution $x_* \in S(x_*, \bar{\rho}) \subset D$ for some $\bar{\rho} > 0$.
(ii) The conditions (C_1) and (C_2) hold on $S(x_*, \bar{\rho})$.
(iii) There exists $\tilde{\rho} \geq \bar{\rho}$, such that

$$\int_0^1 p_0(\theta\tilde{\rho})d\theta < 1.$$

Let $D_2 = S[x_*, \tilde{\rho}] \cap D$. Then, the equation $F(x) = 0$ is uniquely solvable by x_* in the region D_2.

Proof. Suppose $y_* \in D_2$ solves $F(x) = 0$. Define the linear operator

$$H = \int_0^1 F'(x_* + \theta(y_* - x_*))d\theta.$$

By applying the given conditions (i)–(iii), we obtain in turn that

$$\begin{aligned} \|F'(x_*)^{-1}(H - F'(x_*))\| &\leq \int_0^1 p_0(\theta\|y_* - x_*\|)d\theta \\ &\leq \int_0^1 p_0(\theta\tilde{\rho})d\theta < 1. \end{aligned}$$

Hence $x_* = y_*$ follows from $H^{-1} \in \mathcal{L}(\mathcal{B}_1, \mathcal{B})$ and $H(y_* - x_*) = F(y_*) - F(x_*) = 0$. □

Remark 1. *The conditions (C_3) and (C_4) are not used in Proposition 1, otherwise we can set $\bar{\rho} = \rho_*$.*

In what follows, the local convergence analysis of iterative technique $\boldsymbol{\psi_2}$, which is defined by Equation (3), and is shown along the same lines of the analysis of method $\boldsymbol{\psi_1}$ but with a couple of modifications in the definitions of function h_1 and convergence radius. The function h_1 is re-defined as

$$\bar{h}_1(t) = \frac{\int_0^1 p((1-\theta)t)d\theta + 2\left(1 + \int_0^1 p_0(\theta t)d\theta\right)}{1 - p_0(t)}. \tag{19}$$

Set $r_* = \min\{r_2, r_3\}$ and $R = \max\{r_*, \bar{h}_1(r_*)r_*\}$, where r_2 and r_3 are defined as the smallest zeros of functions h_2 and h_3, respectively. Moreover, instead of (C_4), we also assume the following condition,

$(C_4)'$: $\quad S[x_*, r_*] \subset D$.

Finally, we arrive at the corresponding result for the local convergence of $\boldsymbol{\psi_2}$.

Theorem 2. *Under the conditions (C_1)–(C_3) and $(C_4)'$, choose $x^{(0)} \in S(x_*, r_*) - x_*$. Then, $\lim_{k\to\infty} x^{(k)} = x_*$, where the sequence $\{x^{(k)}\}_{k\geq 1}$ is generated by the method defined by Equation (3). Moreover, the following assertions hold:*

$$\begin{aligned}
x^{(k)} &\in S(x_*, r_*) \quad \forall\, k = 0, 1, 2, \ldots, \\
\|y^{(k)} - x_*\| &\leq \bar{h}_1(\|x^{(k)} - x_*\|)\, \|x^{(k)} - x_*\| \leq R, \\
\|z^{(k)} - x_*\| &\leq h_2(\|x^{(k)} - x_*\|)\, \|x^{(k)} - x_*\| < \|x^{(k)} - x_*\|, \\
\|x^{(k+1)} - x_*\| &\leq h_3(\|x^{(k)} - x_*\|)\, \|x^{(k)} - x_*\| < \|x^{(k)} - x_*\|.
\end{aligned}$$

Further, the uniqueness result of Proposition 1 holds for the given method.

Proof. We simply need to provide the derivation of the estimate involving the new function $\bar{h}_1(t)$. With the definition given by Equation (19), the following estimate is obtained for the first sub-step of method $\boldsymbol{\psi_2}$,

$$\begin{aligned}
\|y^{(0)} - x_*\| =& \|x^{(0)} - x_* - F'(x^{(0)})^{-1}F(x^{(0)}) + 2F'(x^{(0)})^{-1}F(x^{(0)})\| \\
&\leq \frac{\int_0^1 p((1-\theta)\|x^{(0)} - x_*\|)d\theta + 2\Big(1 + \int_0^1 p_0(\theta\|x^{(0)} - x_*\|)d\theta\Big)}{1 - p_0(\|x^{(0)} - x_*\|)} \|x^{(0)} - x_*\| \\
&\leq \bar{h}_1(\|x^{(0)} - x_*\|)\|x^{(0)} - x_*\| \leq \bar{h}_1(r_*)r_* \leq R.
\end{aligned}$$

The derivation for the rest of assertions, being identical as presented in Theorem 1, are omitted. □

Remark 2. *It is worth noticing that the methodology in this section provides the computable error estimates on $\|x^{(k)} - x_*\|$ and the uniqueness results for both the techniques, which are not given in the previous studies [8].*

3. Semilocal Convergence Analysis

In this section, the semilocal convergence analysis is established for the iterative methods $\boldsymbol{\psi_1}$ and $\boldsymbol{\psi_2}$. To start with, let $M = [0, \infty)$ and assume that the following conditions hold.

(i) There exists a function $q_0 : M \to M$, which is non-decreasing and continuous such that the function $q_0(t) - 1$ has the smallest zero $r_0 \in M - \{0\}$. Let $M_1 = [0, r_0)$.

(ii) There exists a function $q : M_1 \to M$, which is continuous and non-decreasing.

Then, define the scalar sequences $\{t^{(k)}\}$, $\{s^{(k)}\}$ and $\{u^{(k)}\}$ for all $k = 0, 1, 2, \ldots$, such that $t^{(0)} = 0$, $s^{(0)} = d$ for some $d \geq 0$, and

$$u^{(k)} = s^{(k)} + \frac{\int_0^1 q(\theta(s^{(k)} - t^{(k)}))d\theta}{1 - q_0(t^{(k)})}(s^{(k)} - t^{(k)}), \tag{20}$$

$$t^{(k+1)} = u^{(k)} + \left[\frac{3(q_0(t^{(k)}) + q_0(u^{(k)}))}{(1 - q_0(t^{(k)}))^2} + \frac{(1 + q_0(u^{(k)}))^2}{(1 - q_0(t^{(k)}))^3}\right] \times \left[\int_0^1 q(s^{(k)} - t^{(k)} + \theta(u^{(k)} - s^{(k)}))d\theta\right](u^{(k)} - s^{(k)}), \tag{21}$$

$$s^{(k+1)} = t^{(k+1)} + \frac{1}{1 - q_0(t^{(k+1)})}\left[\left(1 + q_0(u^{(k)}) + \int_0^1 q(\theta(t^{(k+1)} - u^{(k)}))d\theta\right)(t^{(k+1)} - u^{(k)}) + \left(\int_0^1 q(s^{(k)} - t^{(k)} + \theta(u^{(k)} - s^{(k)}))d\theta\right)(u^{(k)} - s^{(k)})\right]. \tag{22}$$

In what follows, the sequences defined by Equations (20)–(22) are shown to be the majorizing for the sequence of iterates generated by technique $\boldsymbol{\psi_1}$. Let us recall the definition of the majorizing sequence [9].

Definition 1. *Let $\{x^{(k)}\}_{k\geq 0}$ be a sequence in a normed linear space and $\{\gamma^{(k)}\}_{k\geq 0}$ be a non-negative scalar sequence, then $\{\gamma^{(k)}\}$ is called the majorizing sequence of $\{x^{(k)}\}$, if*

$$\|x^{(k+1)} - x^{(k)}\| \leq \gamma^{(k+1)} - \gamma^{(k)}, \quad \text{for each } k = 0, 1, 2, \ldots.$$

It follows by the above definition that the sequence $\{\gamma^{(k)}\}$ is non-decreasing. Moreover, if it is bounded above by some $\bar{\gamma} \geq 0$, then it is convergent to some $\gamma_* \in [0, \bar{\gamma}]$. Furthermore, if the normed linear space is complete, it follows that the sequence $\{x^{(k)}\}$ is also convergent to some x_* and

$$\|x_* - x^{(k)}\| \leq \gamma_* - \gamma^{(k)}, \quad \text{for each } k = 0, 1, 2, \ldots.$$

This definition plays a crucial part in the convergence analysis of iterative methods. Before proceeding to the main theorem, let us first show a general convergence result through the following Lemma.

Lemma 1. *For all $k = 0, 1, 2, \ldots$, assume that:*

$$q_0(t^{(k)}) < 1, \tag{23}$$

and

$$t^{(k)} \leq \bar{r}_0, \quad \text{for some } \bar{r}_0 \leq r_0. \tag{24}$$

Then, the sequence $\{t^{(k)}\}$ is convergent monotonically to its unique least upper bound $t_* \in [0, \bar{r}_0]$.

Proof. It follows by the definition of sequence $\{t^{(k)}\}$ and the conditions given by Equations (23) and (24) that it is non-decreasing and bounded above by $\bar{r}_0$. Hence, it will converge to its unique least upper bound t_*. □

Remark 3. *By the definitions of sequences given in Equations (20)–(22), and as a consequence of Lemma 1, it follows that $t^{(k)} \leq s^{(k)} \leq u^{(k)} \leq t^{(k+1)} \leq \bar{r}_0$ for each $k = 0, 1, 2, \ldots$.*

The conditions used in the semilocal convergence analysis of method $\boldsymbol{\psi_1}$ are as follows. Assume that:

(H_1) : There exists a point $x^{(0)} \in D$ such that $F'(x^{(0)})^{-1} \in \mathcal{L}(\mathcal{B}_1, \mathcal{B})$ and

$$\|F'(x^{(0)})^{-1}F(x^{(0)})\| \leq d.$$

(H_2) : For each $v \in D$,

$$\|F'(x^{(0)})^{-1}(F'(v) - F'(x^{(0)}))\| \leq q_0(\|v - x^{(0)}\|).$$

Let $D_3 = S(x^{(0)}, r_0) \cap D$.

(H_3) : For each $v_1, v_2 \in D_3$,

$$\|F'(x^{(0)})^{-1}(F'(v_1) - F'(v_2))\| \leq q(\|v_1 - v_2\|).$$

(H_4) : Conditions of Lemma 1 hold.

(H_5) : $S[x^{(0)}, t_*] \subset D$.

The semilocal convergence of method $\boldsymbol{\psi_1}$ is as follows using the conditions (H_1)–(H_5) and the developed notation.

Theorem 3. *Assume that the conditions (H_1)–(H_5) hold. Then, the following assertions hold for the sequences generated by the method defined by Equation (2).*

$$x^{(k)} \in S(x^{(0)}, t_*) \quad \forall\, k = 0, 1, 2, \ldots, \tag{25}$$

$$\|y^{(k)} - x^{(k)}\| \leq s^{(k)} - t^{(k)}, \tag{26}$$

$$\|z^{(k)} - y^{(k)}\| \leq u^{(k)} - s^{(k)}, \tag{27}$$

$$\|x^{(k+1)} - z^{(k)}\| \leq t^{(k+1)} - u^{(k)}, \tag{28}$$

and further, $\lim_{k\to\infty} x^{(k)} = x_* \in S[x^{(0)}, t_*]$, *such that* $F(x_*) = 0$.

Proof. To show the assertions (25)–(28), the process of mathematical induction is used. Let $v \in S(x^{(0)}, t_*)$ be arbitrary, then by applying the conditions (H_1) and (H_2), we have

$$\|F'(x^{(0)})^{-1}(F'(v) - F'(x^{(0)}))\| \leq q_0(\|v - x^{(0)}\|) \leq q_0(t_*) < 1,$$

and therefore

$$||F'(v)^{-1}F'(x^{(0)})|| \leq \frac{1}{1 - q_0(||v - x^{(0)}||)}. \tag{29}$$

It follows by condition (H_1) and Equation (29) (for $v = x^{(0)}$) that the iterates $y^{(0)}$, $z^{(0)}$ and $x^{(1)}$ are well defined for the method $\boldsymbol{\psi_1}$. In view of the first sub-step of $\boldsymbol{\psi_1}$, we have

$$\|y^{(0)} - x^{(0)}\| = \|F'(x^{(0)})^{-1}F(x^{(0)})\| \leq d = s^{(0)} - t^{(0)} < t_*.$$

So, the iterate $y^{(0)} \in S(x^{(0)}, t_*)$ and the estimate (26) holds for $k = 0$. Further, the second sub-step of $\boldsymbol{\psi_1}$ along with the definition of sequence $\{t^{(k)}\}$ yields

$$\begin{aligned}
\|z^{(0)} - y^{(0)}\| &= \|F'(x^{(0)})^{-1}F(y^{(0)})\| \\
&= \|F'(x^{(0)})^{-1}(F(y^{(0)}) - F(x^{(0)}) - F'(x^{(0)})(y^{(0)} - x^{(0)}))\| \\
&\leq \frac{\int_0^1 q(\theta\|y^{(0)} - x^{(0)}\|)d\theta}{1 - q_0(\|x^{(0)} - x^{(0)}\|)}\|y^{(0)} - x^{(0)}\| \\
&\leq u^{(0)} - s^{(0)},
\end{aligned}$$

and therefore

$$\begin{aligned}\|z^{(0)} - x^{(0)}\| &= \|z^{(0)} - y^{(0)}\| + \|y^{(0)} - x^{(0)}\| \\ &\leq u^{(0)} - s^{(0)} + s^{(0)} - t^{(0)} < t_*.\end{aligned}$$

Thus, the iterate $z^{(0)} \in S(x^{(0)}, t_*)$ and the estimate (27) holds for $k = 0$.

Then, by the third sub-step of $\boldsymbol{\psi_1}$, we obtain in turn that

$$\begin{aligned}\|x^{(1)} - z^{(0)}\| &\leq \left[3\|I - F'(x^{(0)})^{-1}F'(z^{(0)})\| + \|F'(x^{(0)})^{-1}F'(z^{(0)})\|^2\right]\|F'(x^{(0)})^{-1}F(z^{(0)})\| \\ &\leq \left[3\|I - F'(x^{(0)})^{-1}F'(z^{(0)})\| + \|F'(x^{(0)})^{-1}F'(z^{(0)})\|^2\right] \\ &\qquad \times \|F'(x^{(0)})^{-1}(F(z^{(0)}) - F(y^{(0)}) - F'(x^{(0)})(z^{(0)} - y^{(0)}))\| \\ &\leq \left[\frac{3(q_0(\|x^{(0)} - x^{(0)}\|) + q_0(\|z^{(0)} - x^{(0)}\|))}{(1 - q_0(\|x^{(0)} - x^{(0)}\|))^2} + \frac{(1 + q_0(\|z^{(0)} - x^{(0)}\|))^2}{(1 - q_0(\|x^{(0)} - x^{(0)}\|))^3}\right] \\ &\qquad \times \left[\int_0^1 q(\|y^{(0)} - x^{(0)}\| + \theta\|z^{(0)} - y^{(0)}\|)d\theta\right]\|z^{(0)} - y^{(0)}\| \\ &\leq t^{(1)} - u^{(0)},\end{aligned}$$

and consequently

$$\begin{aligned}\|x^{(1)} - x^{(0)}\| &= \|x^{(1)} - z^{(0)}\| + \|z^{(0)} - x^{(0)}\| \\ &\leq t^{(1)} - u^{(0)} + u^{(0)} - t^{(0)} < t_*.\end{aligned}$$

Therefore, the iterate $x^{(1)} \in S(x^{(0)}, t_*)$ and the estimate (28) holds for $k = 0$. Now, by the first sub-step of method $\boldsymbol{\psi_1}$ for $k = 1$, the following estimate is obtained,

$$\begin{aligned}\|y^{(1)} - x^{(1)}\| &\leq \|F'(x^{(1)})^{-1}F'(x^{(0)})\|\|F'(x^{(0)})^{-1}F(x^{(1)})\| \\ &\leq \|F'(x^{(1)})^{-1}F'(x^{(0)})\|\|F'(x^{(0)})^{-1}(F(x^{(1)}) - F(z^{(0)}) + F(z^{(0)}))\| \\ &\leq \frac{1}{1 - q_0(\|x^{(1)} - x^{(0)}\|)}\left[\left(1 + q_0(\|z^{(0)} - x^{(0)}\|) + \int_0^1 q(\theta\|x^{(1)} - z^{(0)}\|)d\theta\right)\right. \\ &\qquad \left.\times\|x^{(1)} - z^{(0)}\| + \int_0^1 q(\|y^{(0)} - x^{(0)}\| + \theta\|z^{(0)} - y^{(0)}\|)d\theta\, \|z^{(0)} - y^{(0)}\|\right] \\ &\leq s^{(1)} - t^{(1)},\end{aligned}$$

and consequently

$$\begin{aligned}\|y^{(1)} - x^{(0)}\| &= \|y^{(1)} - x^{(1)}\| + \|x^{(1)} - x^{(0)}\| \\ &\leq s^{(1)} - t^{(1)} + t^{(1)} - t^{(0)} < t_*.\end{aligned}$$

Hence, the iterate $y^{(1)} \in S(x^{(0)}, t_*)$ and the estimate (26) holds for $k = 1$. By repeating the previous calculations but switching $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ and $x^{(1)}$ by $x^{(k)}$, $y^{(k)}$, $z^{(k)}$ and $x^{(k+1)}$, the induction process is completed for the assertions (25)–(28). In particular, we have the estimate

$$\begin{aligned}\|F'(x^{(0)})^{-1}F(x^{(k)})\| \leq& \left(1 + q_0(u^{(k)}) + \int_0^1 q(\theta(t^{(k+1)} - u^{(k)}))d\theta\right)(t^{(k+1)} - u^{(k)}) \\ &+ \left(\int_0^1 q(s^{(k)} - t^{(k)} + \theta(u^{(k)} - s^{(k)}))d\theta\right)(u^{(k)} - s^{(k)}). \qquad (30)\end{aligned}$$

Since the sequence $\{t^{(k)}\}$ is convergent by condition (H_4), and the sequence $\{x^{(k)}\}$ is fundamental in the Banach space, so there exists $x_* \in S(x^{(0)}, t_*)$, such that $\lim_{k\to\infty} x^{(k)} = x_*$.

Finally, letting $k \to \infty$ in the estimate (30) and using the continuity of operator F, we conclude that $F(x_*) = 0$. $\square$

The uniqueness result follows from the following proposition.

Proposition 2. *Assume that:*

(i) *The equation $F(x) = 0$ has a solution $x_* \in S(x^{(0)}, \bar{t}) \subseteq D$ for some $\bar{t} > 0$ and $F'(x^{(0)})^{-1} \in \mathcal{L}(\mathcal{B}_1, \mathcal{B})$.*
(ii) *The conditions (H_1) and (H_2) hold.*
(iii) *There exists $\tilde{t} \geq \bar{t}$, such that*

$$\int_0^1 q_0(\theta\tilde{t} + (1-\theta)\bar{t})d\theta < 1. \tag{31}$$

Let $D_4 = S[x^{(0)}, \tilde{t}\,] \cap D$. Then, x_* solves uniquely the equation $F(x) = 0$ in the region D_4.

Proof. Consider $y_* \in D_4$ with $F(y_*) = 0$, and let the operator H be defined as in Proposition 1. By applying the conditions (H_1), (H_2), and using Equation (31), we obtain

$$\begin{aligned}
\|F'(x^{(0)})^{-1}(H - F'(x^{(0)}))\| &\leq \int_0^1 q_0(\theta\|y_* - x^{(0)}\| + (1-\theta)\|x_* - x^{(0)}\|)d\theta \\
&\leq \int_0^1 q_0(\theta\tilde{t} + (1-\theta)\bar{t})d\theta < 1.
\end{aligned}$$

Thus, we deduce that $x_* = y_*$. $\square$

Remark 4. *The limit point t_* can be replaced in the condition (H_4) by the point r_0.*

Remark 5. *The semilocal convergence analysis of the iterative technique $\boldsymbol{\psi_2}$ is given analogously to the analysis of technique $\boldsymbol{\psi_1}$ but re-defining the sequence $\{u^{(k)}\}$ by*

$$u^{(k)} = t^{(k)} + \frac{1 + q_0(t^{(k)}) + \int_0^1 q(\theta(s^{(k)} - t^{(k)}))d\theta}{1 - q_0(t^{(k)})}(s^{(k)} - t^{(k)}). \tag{32}$$

The definition of sequences $\{t^{(k)}\}$ and $\{s^{(k)}\}$ are same as given by Equations (21) and (22), respectively. The results of Lemma 1, Theorem 3, Proposition 2 and Remark 4 also hold for the analysis of method $\boldsymbol{\psi_2}$. However, note that the estimate (27) is reformulated as

$$\|z^{(k)} - x^{(k)}\| \leq u^{(k)} - t^{(k)},$$

wherein $z^{(0)} \in S(x^{(0)}, t_*)$ by the definition (32) and due to the following estimate,

$$\begin{aligned}
\|z^{(0)} - x^{(0)}\| &= \|F'(x^{(0)})^{-1}(F(y^{(0)}) - F(x^{(0)}))\| \\
&\leq \frac{1 + q_0(\|x^{(0)} - x^{(0)}\|) + \int_0^1 q(\theta\|y^{(0)} - x^{(0)}\|)d\theta}{1 - q_0(\|x^{(0)} - x^{(0)}\|)}\|y^{(0)} - x^{(0)}\| \\
&\leq u^{(0)} - t^{(0)} < t_*.
\end{aligned}$$

4. Numerical Results

In coherence with the theoretical results, we provide here the parameters and functions, defined in Sections 2 and 3, for each of the following numerical examples.

Example 1. *Consider the domain $\mathbb{R}^m$, for any integer $m \geq 2$, which is equipped with the norm, $\|x\| = \max_{1 \leq i \leq m} |x_i|$ for each $x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m$ and the corresponding matrix norm is* *given by* $\|A\| = \max_{1\le i\le m} \sum_{j=1}^{m} |a_{ij}|$ *for any* $A = (a_{ij})_{1\le i,j\le m} \in \mathcal{L}(\mathbb{R}^m)$*. Define the two-point boundary value problem on the closed interval* $[0,1]$ *as*

$$\begin{aligned} x''(t) &= -x(t)^2, \\ x(0) &= x(1) = 0. \end{aligned} \tag{33}$$

To transform the given equation into a finite dimensional problem, consider the uniform partitioning of interval $[0,1]$ *having a sub-interval length* $h = 1/k$ *as*

$$0 = t_0 < t_1 < t_2 < \ldots < t_{k-1} < t_k = 1.$$

Denoting $x_i = x(t_i)$ *for each i, and using the following divided difference approximations:*

$$x_i'' \approx \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}, \ \ \textit{for each } i = 1, 2, \ldots, k-1,$$

the Equation (33) *reduces into the system of nonlinear equations,* $F : D \subseteq \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$*, which is given by*

$$x_{i+1} - 2x_i + h^2 x_i^2 + x_{i-1} = 0, \ \ i = 1, 2, \ldots, k-1, \tag{34}$$

where $x_0 = x_k = 0$*. The Fréchet derivative at any point* $x = (x_1, \ldots, x_{k-1})^T \in D$ *is given by*

$$F'(x) = \begin{bmatrix} 2h^2x_1 - 2 & 1 & 0 & \cdots & 0 \\ 1 & 2h^2x_2 - 2 & 1 & \cdots & 0 \\ 0 & 1 & 2h^2x_3 - 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2h^2x_{k-1} - 2 \end{bmatrix}.$$

To estimate the parameters defined in the Sections 2 and 3, we take $k = 26$ *in particular so that the system* (34) *reduces to a system of* 25 *equations satisfying the solution* $x_* = (0, \overset{25}{\cdots\cdots}, 0)^T$*. Further, considering the domain as an open ball,* $D = S(x_*, c)$ *for some positive constant c, we select the initial estimate as* $x^{(0)} = (\frac{1}{2}, \overset{25}{\cdots\cdots}, \frac{1}{2})^T \in D$*. Then, for any* $x, y \in D$*, we can obtain that*

$$\|F'(x_*)^{-1}(F'(x) - F'(y))\| \le L_0\|x - y\|,$$

and

$$\|F'(x^{(0)})^{-1}(F'(x) - F'(y))\| \le L_1\|x - y\|,$$

where $L_0 = 0.25$ *and* $L_1 = 0.27903$*. Moreover,*

$$\|F'(x^{(0)})^{-1}F(x^{(0)})\| \le L_2,$$

where $L_2 = 0.53488$*.*

In view of the above approximations, the parameters defined in Section 2 under conditions (C_1)–(C_4) *for the local convergence analysis are chosen as*

$$p_0(t) = L_0 t, \ \ p(t) = L_0 t,$$

and consequently for the method $\boldsymbol{\psi_1}$*,*

$$\rho_* = \min\{\rho_1, \rho_2, \rho_3\} = \min\{2.6667, 2.2112, 1.6232\} = 1.6232,$$

whereas for the method $\boldsymbol{\psi_2}$*,*

$$r_* = \min\{r_2, r_3\} = \min\{0.57629, 0.60542\} = 0.57629.$$

Moreover, the parameters defined in Section 3 under conditions (H_1)*–*(H_5) *for the semilocal convergence analysis are chosen as*

$$q_0(t) = L_1 t,\ q(t) = L_1 t, \text{ and } d = L_2,$$

and consequently, we obtain the majorizing sequence $\{t^{(k)}\}$ *for method* $\boldsymbol{\psi_1}$ *as*

$$\{t_k\}_{k\geq 1} = \{0.58608...,0.60925...,0.60925...,\cdots\cdots\},$$

which converges to $t_* \approx 0.60926 < r_0 = 3.58389$*, whereas for method* $\boldsymbol{\psi_2}$*, the sequence is obtained as*

$$\{t_k\}_{k\geq 1} = \{0.58608...,0.61853...,0.61919...,\cdots\cdots\},$$

which converges to $t_* \approx 0.61919 < r_0 = 3.58389$*. It can be easily verified from the above estimations that the conditions of Sections 2 and 3 hold for each of the methods for the considered example.*

Example 2. *Let* $C[0,1]$ *represents the space of continuous functions defined on the domain as closed unit interval* $[0,1]$ *and equipped with the norm* $\|x\| = \sup_{0\leq t\leq 1} |x(t)|$ *for each* $x \in C[0,1]$*. Let* $D = \{x \in C[0,1] \mid \|x\| < 1\}$ *and define the nonlinear mapping (see [12])* $F : D \to C[0,1]$ *as*

$$F(x)(t) = x(t) - \mu \int_0^1 \kappa(s,t)x(s)^3 ds,\ \ t \in [0,1],\ \ x \in D, \tag{35}$$

where $\mu \in \mathbb{R}$ *is a parameter, and the kernel* $\kappa(s,t)$ *is defined as*

$$\kappa(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ (1-t)s, & s \leq t, \end{cases}$$

which satisfies the following approximation,

$$\left\| \int_0^1 \kappa(s,t) ds \right\| \leq \frac{1}{8}.$$

Moreover, the Fréchet derivative of the mapping (35) is given by

$$F'(x)\xi(t) = \xi(t) - 3\mu \int_0^1 \kappa(s,t)x(s)^2\xi(s)ds,\ \ \xi \in D.$$

Note that $x_* = 0$ *is the solution of (35), which also satisfies that* $F'(x_*) = I$*. Then, for any* $x, y \in D$*, we have the following approximation,*

$$\begin{aligned} \|F'(x_*)^{-1}(F'(x) - F'(y))\| &\leq 3|\mu| \left\| \int_0^1 \kappa(s,t)\Big(x(s)^2 - y(s)^2\Big)\xi(s)ds \right\| \\ &\leq L_0\|x - y\|, \end{aligned}$$

where $L_0 = \frac{3|\mu|}{4}$*. Moreover, for the initial approximation* $x^{(0)} \in D$ *which is defined as* $x^{(0)}(t) = \frac{t}{2}$*,* $t \in [0,1]$*, and using the estimation*

$$\|I - F'(x^{(0)})\| \leq 3|\mu| \left\| \int_0^1 \kappa(s,t)x^{(0)}(s)^2\xi(s)ds \right\| \leq \frac{3|\mu|}{32},$$

it is obtained that $\|F'(x^{(0)})^{-1}\| \leq \frac{32}{32-3|\mu|}$*, provided* $|\mu| < \frac{32}{3}$*. Therefore, for any* $x, y \in D$*, we have obtained that*

$$\|F'(x^{(0)})^{-1}(F'(x) - F'(y))\| \leq L_1\|x - y\|,$$

and

$$\|F'(x^{(0)})^{-1}F(x^{(0)})\| \leq L_2,$$

where $L_1 = \frac{24|\mu|}{32-3|\mu|}$ *and* $L_2 = \left(1 + \frac{|\mu|}{32}\right)\frac{16}{32-3\mu}$.

We particularly set $\mu = \frac{1}{2}$*, in above approximations, for the estimation of parameters defined in Sections 2 and 3. The parameters used in the conditions* (C_1)–(C_4) *are defined as*

$$p_0(t) = L_0 t, \ \ p(t) = L_0 t,$$

and therefore, for the convergence of method $\boldsymbol{\psi_1}$*, we have*

$$\rho_* = \min\{\rho_1, \rho_2, \rho_3\} = \min\{1.7778, 1.4741, 1.0821\} = 1.0821,$$

whereas for the method $\boldsymbol{\psi_2}$*,*

$$r_* = \min\{r_2, r_3\} = \min\{0.38419, 0.40362\} = 0.38419.$$

Further, the parameters defined in the conditions (H_1)–(H_5) *are chosen as*

$$q_0(t) = L_1 t, \ q(t) = L_1 t, \text{ and } d = L_2,$$

and consequently, we obtain the majorizing sequence $\{t^{(k)}\}$ *for method* $\boldsymbol{\psi_1}$ *as*

$$\{t_k\}_{k\geq 1} = \{0.61587..., 0.67771..., 0.67799..., \cdots\cdots\},$$

which converges to $t_* \approx 0.678 < r_0 = 2.54167$*, whereas for method* $\boldsymbol{\psi_2}$*, the sequence is given by*

$$\{t_k\}_{k\geq 1} = \{0.61587..., 0.72464..., 0.75542..., \cdots\cdots\},$$

which converges to $t_* \approx 0.75913 < r_0 = 2.54167$*. With these estimations, the conditions of Sections 2 and 3 hold for both the considered methods.*

Example 3. *Now consider a nonlinear equation due to Kepler [20]:*

$$F(x) = x - \beta\sin(x) - \mu = 0, \tag{36}$$

where $0 \leq \beta < 1$ *and* $0 \leq \mu \leq \pi$*. Different choices for the values of* β *and* μ *are given in [20]. In particular, for* $\beta = \frac{1}{4}$ *and* $\mu = \frac{1}{10}$*, the approximate solution of (36) is given as,* $x_* \approx 0.13320215$*. Let* $D = S(x_*, c)$*, where c is a positive constant, such that the initial approximation* $x^{(0)} = \frac{3}{4} \in D$*. Moreover, we have*

$$F'(x) = 1 - \beta\cos(x).$$

Thus, for any $x, y \in D$*, we have the following approximation,*

$$\begin{aligned}|F'(x_*)^{-1}(F'(x) - F'(y))| &= \frac{|\beta(\cos(x) - \cos(y))|}{|1 - \beta\cos(x_*)|} \\ &= \frac{2|\beta||\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})|}{|1 - \beta\cos(x_*)|} \\ &\leq L_0|x - y|,\end{aligned}$$

and similarly

$$|F'(x^{(0)})^{-1}(F'(x) - F'(y))| \leq L_1|x - y|,$$

where $L_0 = \frac{|\beta|}{|1-\beta\cos(x_*)|} \approx 0.332352$ *and* $L_1 = \frac{|\beta|}{|1-\beta\cos(x^{(0)})|} \approx 0.305968$. *moreover,*

$$|F'(x^{(0)})^{-1}F(x^{(0)})| = \frac{|x^{(0)} - \beta\sin(x^{(0)}) - \mu|}{|1-\beta\cos(x^{(0)})|} = L_2,$$

where $L_2 \approx 0.586958$.

The above approximations lead to the estimation of parameters used in the conditions of Section 2 as well as the conditions of Section 3. The parameters used in conditions (C_1)–(C_4) *are given as*

$$p_0(t) = L_0 t, \;\; p(t) = L_0 t,$$

and consequently for the method $\boldsymbol{\psi_1}$, *we have*

$$\rho_* = \min\{\rho_1, \rho_2, \rho_3\} = \min\{2.0059, 1.6633, 1.2210\} = 1.2210,$$

whereas for the convergence of method $\boldsymbol{\psi_2}$,

$$r_* = \min\{r_2, r_3\} = \min\{0.43350, 0.45541\} = 0.43350.$$

Moreover, the parameters defined in the conditions (H_1)–(H_5) *are chosen as*

$$q_0(t) = L_1 t, \; q(t) = L_1 t, \text{ and } d = L_2,$$

and consequently, the majorizing sequence $\{t^{(k)}\}$ *for method* $\boldsymbol{\psi_1}$ *is obtained as*

$$\{t_k\}_{k\geq 1} = \{0.65961..., 0.70232..., 0.70236..., \cdots\cdots\},$$

which converges to $t_* \approx 0.70237 < r_0 = 3.26831$, *whereas the sequence for method* $\boldsymbol{\psi_2}$ *is given by*

$$\{t_k\}_{k\geq 1} = \{0.65961..., 0.72546..., 0.73050..., \cdots\cdots\},$$

which converges to $t_* \approx 0.73054 < r_0 = 3.26831$.

5. Conclusions

Two sixth-order iterative techniques are analyzed comprehensively for their local as well semilocal convergence in Banach Spaces. On the contrary to the usual approach using Taylor series expansions, the generalized results for their convergence are established using the assumptions only on first order derivatives. The presented analysis provides a new avenue for the study of the convergence of iterative methods, since it is based only on the operators involved in the given iterative methods. However, the earlier studies make use of the higher order derivatives, which are not appearing in the given methods and, in fact, these higher order derivatives may not even always exist. Thus, the previous results do not assure the convergence of these methods, although convergence may happen. Effectively, the applicability of given methods is extended to the wider section of problems. Further, testing the developed results on some applied problems satisfactorily favor the presented analysis. An important observation is that the technique, which is utilized in the analysis, can also be applied in general to the other methods in order to extend their applicability in a similar way. Furthermore, as a future work, the applicability of given iterative methods can be investigated for the solution of problems considered in [21–23].

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References

1. Debnath, P.; Konwar, N.; Radenovic, S. *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Sciences*; Springer: Singapore, 2021.
2. Debnath, P.; Srivastava, H.M.; Kumam, P.; Hazarika, B. *Fixed Point Theory and Fractional Calculus: Recent Advances and Applications*; Springer: Singapore, 2022.
3. Kantorovich, L.V.; Akilov, G.P. *Functional Analysis*; Pergamon Press: Oxford, UK, 1982.
4. Kirk, W.; Shahzad, N. *Fixed Point Theory in Distance Spaces*; Springer International Publishing: Cham, Switzerland, 2014.
5. Traub, J.F. *Iterative Methods for the Solution of Equations*; Chelsea Publishing Company: New York, NY, USA, 1982.
6. Amat, S.; Bermúdez, C.; Hernández-Verón, M.A.; Martínez, E. On an efficient k-step iterative method for nonlinear equations. *J. Comput. Appl. Math.* **2016**, *302*, 258–271. [CrossRef]
7. Ostrowski, A.M. *Solution of Equation and Systems of Equations*; Academic Press: New York, NY, USA, 1960.
8. Singh, H.; Sharma, J.R. Reduced cost numerical methods of sixth-order convergence for systems of nonlinear models. *Rev. Real Acad. Cienc. Exactas Fis. Nat. A Mat.* **2022**, *116*, 144. [CrossRef]
9. Argyros, I.K. Unified convergence criteria for iterative Banach space valued methods with applications. *Mathematics* **2021**, *9*, 1942. [CrossRef]
10. Argyros, I.K. *The Theory and Applications of Iteration Methods*, 2nd ed.; CRC Press: New York, NY, USA, 2022.
11. Cordero, A.; Ezquerro, J.A.; Hernández, M.A.; Torregrosa, J.R. On the local convergence of fifth-order iterative method in Banach spaces. *Appl. Math. Comput.* **2015**, *251*, 396–403. [CrossRef]
12. Gutiérrez, J.M.; Magreñán, Á.A.; Romero, N. On the semilocal convergence of Newton–Kantorovich method under center-Lipschitz conditions. *Appl. Math. Comput.* **2013**, *221*, 79–88. [CrossRef]
13. Hernández-Verón, M.A.; Martínez, E.; Teruel, C. Semilocal convergence of a k-step iterative process and its application for solving a special kind of conservative problems. *Numer. Algorithms* **2017**, *76*, 309–331. [CrossRef]
14. Jaiswal, J.P. Semilocal convergence of a computationally efficient iterative method in Banach spaces under weak condition. *J. Anal.* **2020**, *28*, 141–154. [CrossRef]
15. Liu, T.; Qin, X.; Wang, P. Local convergence of a family of iterative methods with sixth and seventh order convergence under weak conditions. *Int. J. Comput. Methods* **2019**, *16*, 1850120. [CrossRef]
16. Martínez, E.; Singh, S.; Hueso, J.L.; Gupta, D.K. Enlarging the convergence domain in local convergence studies for iterative methods in Banach spaces. *Appl. Math. Comput.* **2016**, *281*, 252–265. [CrossRef]
17. Sharma, J.R.; Argyros, I.K.; Singh, H.; Argyros, M.I. On the local convergence of a $(p+1)$-step method of order $2p+1$ for solving equations. *Foundations* **2022**, *2*, 242–250. [CrossRef]
18. Singh, S.; Martínez, E.; Maroju, P.; Behl, R. A study of the local convergence of a fifth order iterative method. *Indian J. Pure Appl. Math.* **2020**, *51*, 439–455. [CrossRef]
19. Ortega, J.M.; Rheinboldt, W.C. *Iterative Solution of Nonlinear Equations in Several Variables*; Academic Press: New York, NY, USA, 1970.
20. Danby, J.M.A.; Burkardt, T.M. The solution of Kepler's equation, I. *Celest. Mech.* **1983**, *31*, 95–107. [CrossRef]
21. Ramos, H.; Rufai, M.A. An adaptive one-point second-derivative Lobatto-type hybrid method for solving efficiently differential systems. *Int. J. Comput. Math.* **2022**, *99*, 1687–1705. [CrossRef]
22. Rufai, M.A.; Ramos, H. Numerical solution of Bratu's and related problems using a third derivative hybrid block method. *Comput. Appl. Math.* **2020**, *39*, 322. [CrossRef]
23. Rufai, M.A.; Ramos, H. Numerical integration of third-order singular boundary-value problems of Emden–Fowler type using hybrid block techniques. *Commun. Nonlinear Sci. Numer. Simul.* **2022**, *105*, 106069. [CrossRef]