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Some Applications of Fuzzy Sets in Residuated Lattices

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Abstract: Many papers have been devoted to applying fuzzy sets to algebraic structures. In this paper, based on ideals, we investigate residuated lattices from fuzzy set theory, lattice theory, and coding theory points of view, and some applications of fuzzy sets in residuated lattices are presented. Since ideals are important concepts in the theory of algebraic structures used for formal fuzzy logic, first, we investigate the lattice of fuzzy ideals in residuated lattices and study some connections between fuzzy sets associated to ideals and Hadamard codes. Finally, we present applications of fuzzy sets in coding theory.

Keywords: residuated lattice; fuzzy ideal; lattice; code

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1. Introduction

The notion of a residuated lattice, introduced in [1] by Ward and Dilworth, provides an algebraic framework for fuzzy logic. MV-algebras (or the equivalent Wajsberg algebras) and Boolean algebras are particular residuated lattices [1–6]. These algebras are important because of the role they play in fuzzy logic.

There are many real-life situations wherein the information we obtain is imprecise. The theory of fuzzy sets proposes techniques for analyzing these data (see [7–9]).

Managing certain and uncertain information is a priority of artificial intelligence, in an attempt to imitate human thinking. To make this possible, in [10], Zadeh introduced the notion of a fuzzy set, and many researchers applied this concept in branches of mathematics such as automata theory, lattice theory, group and ring theory, and topology.

Ideals and fuzzy ideals theory are important tools in the study of algebras arising from logic (see [11–13]).

In [12], the concept of a fuzzy set was applied to residuated lattices, and fuzzy ideals were introduced and characterized.

In this paper, we investigate residuated lattices from three points of view: lattice theory, fuzzy set theory, and coding theory, and we study some applications of fuzzy sets associated with ideals in residuated lattices.

Since fuzzy ideals are important in the study of residuated lattices, in Section 3, we extend the results from [12] and we give equivalent characterizations of fuzzy ideals. Also, we investigate their lattice structure and prove that fuzzy ideals in a residuated lattice form a Heyting algebra.

In Section 4, we find connections between fuzzy sets associated with ideals in particular residuated lattices and Hadamard codes.

2. Preliminaries

A residuated lattice is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ with an order $\preceq$ such that

(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;
(ii) $(L, \odot, 1)$ is a commutative monoid;
(iii) $x \odot z \preceq y$ if and only if $x \preceq z \rightarrow y$, for $x, y, z \in L$, see [1].

In this paper, L will be denoted a residuated lattice, unless otherwise stated.

A *Heyting algebra* [14] is a lattice $(L, \vee, \wedge)$ with 0 such that for every $a, b \in L$, there exists an element $a \rightarrow b \in L$ (called the *pseudocomplement of* a *with respect to* b) where $a \rightarrow b = \sup\{x \in L : a \wedge x \leq b\}$. Heyting algebras are divisible residuated lattices.

For $x, y \in L$, we define $x \boxplus y = x^* \rightarrow y^{**}$ and $x \uplus y = x^* \rightarrow y$, where $x^* = x \rightarrow 0$. We remark that $\boxplus$ is associative and commutative and $\uplus$ is only associative.

We recall some rules of calculus in residuated lattices, see [6,15]:

(1) $1 \rightarrow x = x$, $x \rightarrow y = 1$ if and only if $x \preceq y$;
(2) $x, y \preceq x \uplus y \preceq x \boxplus y$, $x \boxplus 0 = x^{**}$, $x \boxplus x^* = 1$, $x \boxplus 1 = 1$, $x \boxplus y = y \boxplus x$, $(x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$, $x \preceq y \Rightarrow x \boxplus z \preceq y \boxplus z$;
(3) $x \boxplus y = (x^* \odot y^*)^*$, $(x \boxplus y)^{**} = x \boxplus y = x^{**} \boxplus y^{**}$, for every $x, y, z \in L$.

An ideal in residuated lattices is a generalization of the similar notion from MV-algebras, see [3]. This concept is introduced in [12] using the operator $\uplus$, which is not commutative. An equivalent definition is given in [15] using $\boxplus$. We remark that $\boxplus$ is associative and commutative and $\uplus$ is only associative.

Definition 1 ([15])**.** *An* ideal *residuated lattice L is a subset $I \neq \emptyset$ of L such that*

(i_1) *For $x \leq i, x \in L, i \in I \Longrightarrow x \in I$;*
(i_2) *$i, j \in I \Longrightarrow i \boxplus j \in I$.*

Let A be a set. A fuzzy set in A is a function $\mu : A \longrightarrow [0, 1]$, see [10], where $[0, 1]$ is the real unit interval.

The notion of a fuzzy ideal in residuated lattices is introduced in [12], and some characterizations are obtained.

Definition 2 ([12])**.** *A* fuzzy ideal of *a residuated lattice L is a fuzzy set μ in L such that*

(fi_1) $x \preceq y \Longrightarrow \mu(x) \geq \mu(y)$;
(fi_2) $\mu(x \uplus y) \geq \min(\mu(x), \mu(y))$, *for every* $x, y \in L$.

Two equivalent definitions for fuzzy ideals are given in [12]:

A *fuzzy ideal of* L is a fuzzy set μ in L such that

(fi_3) $\mu(0) \geq \mu(x)$, for every $x \in L$;
(fi_4) $\mu(y) \geq \min(\mu(x), \mu((x^* \rightarrow y^*)^*)$, for every $x, y \in L \Leftrightarrow (fi_4')$ $\mu(y) \geq \min(\mu(x), \mu(x^* \odot y))$, for every $x, y \in L$.

We denote by $\mathcal{I}(L)$ the set of ideals and by $\mathcal{FI}(L)$ the set of fuzzy ideals of the residuated lattice L.

Obviously, the constant functions $\mathbf{0}, \mathbf{1} : L \rightarrow [0, 1]$, $\mathbf{0}(x) = 0$, and $\mathbf{1}(x) = 1$ for every $x \in L$ are fuzzy ideals of L.

There are two important fuzzy sets in a residuated lattice L : For $I \subseteq L$ and $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$ is defined $\widehat{\mu}_I : L \rightarrow [0, 1]$ by

$$\widehat{\mu}_I(x) = \begin{cases} \alpha, \text{ if } x \in I \\ \beta, \text{ if } x \notin I. \end{cases}$$

The fuzzy set $\widehat{\mu}_I$ is a generalization of the characteristic function of I, denoted μ_I. Moreover, in [12], it is proved that $I \in \mathcal{I}(L)$ if and only if $\widehat{\mu}_I \in \mathcal{FI}(L)$.

Lemma 1 ([12])**.** *For $\mu \in \mathcal{FI}(L)$, the following hold:*

(*i*) $\mu(x) = \mu(x^{**})$
(*ii*) $\mu(x \uplus y) = \min(\mu(x), \mu(y))$, *for every* $x, y \in L$.

For μ_1 and μ_2 two fuzzy sets in L is define the order relation $\mu_1 \subset \mu_2$ if $\mu_1(x) \leq \mu_2(x)$, for every $x \in L$.

Moreover, for a family $\{\mu_i : i \in I\}$ of fuzzy ideals of L, we define $\underset{i\in I}{\cup}\mu_i, \underset{i\in I}{\cap}\mu_i : L \to [0,1]$ by

$$(\underset{i\in I}{\cup}\mu_i)(x) = \sup\{\mu_i(x) : i \in I\} \text{ and } (\underset{i\in I}{\cap}\mu_i)(x) = \inf\{\mu_i(x) : i \in I\},$$

for every $x \in L$, see [10].

Obviously, $\underset{i\in I}{\cap}\mu_i \in \mathcal{FI}(L)$, but in general $\underset{i\in I}{\cup}\mu_i$ is not a fuzzy ideal of L, see [11].

We recall (see [14]) that a complete lattice $(\mathcal{A}, \vee, \wedge)$ is called Brouwerian if it satisfies the identity $a \wedge (\bigvee\limits_i b_i) = \bigvee\limits_i (a \wedge b_i)$ whenever arbitrary unions exist. An element $a \in \mathcal{A}$ is called compact if $a \leq \vee X$ for some $X \subseteq \mathcal{L}$ implies $a \leq \vee X_1$ for some finite $X_1 \subseteq X$.

Remark 1 ([14])**.** *Let A be a set of real numbers. We say that $l \in R$ is the supremum of A if*

1. *l is an upper bound for A;*
2. *l is the least upper bound: for every $\epsilon > 0$ there is $a_\epsilon \in A$ such that $a_\epsilon > l - \epsilon$, i.e., $l < a_\epsilon + \epsilon$.*

Remark 2. *If a, b are real numbers such that $a, b \in [0,1]$ and $a > b - \epsilon$, for every $\epsilon > 0$, then $a \geq b$. Indeed, if we suppose that $a < b$, then there is $\epsilon_0 > 0$ such that $b - a > \epsilon_0 > 0$, which is a contradiction with the hypothesis.*

3. The Lattice of Fuzzy Ideals in a Residuated Lattice L

In this section, we provide new characterizations for fuzzy ideals and investigate the properties of their lattice.

Proposition 1. *Let μ be a fuzzy set in L. Then, $\mu \in \mathcal{FI}(L)$ if and only if it satisfies the following conditions:*

(fi_1) $x \preceq y \Longrightarrow \mu(x) \geq \mu(y)$;
(fi_2') $\mu(x \boxplus y) \geq \min(\mu(x), \mu(y))$, *for every* $x, y \in L$.

Proof. If $\mu \in \mathcal{FI}(L)$, from Definition 2 and Lemma 1, (fi_1) and (fi_2') hold since $\mu(x \boxplus y) = \mu(x \uplus y^{**}) = \min(\mu(x), \mu(y^{**})) = \min(\mu(x), \mu(y))$, for every $x, y \in L$.

Conversely, assume that (fi_1) and (fi_2') hold and let $x, y \in L$. Since $x \uplus y \preceq x \boxplus y$, we obtain $\min(\mu(x), \mu(y)) \leq \mu(x \boxplus y) \leq \mu(x \uplus y)$, so (fi_2) holds. Thus, $\mu \in \mathcal{FI}(L)$. □

Proposition 2. *Let μ be a fuzzy set in L. Then, $\mu \in \mathcal{FI}(L)$ if and only if*

$$\mu(x \boxplus y) = \mu(x \vee y) = \min(\mu(x), \mu(y^{**})),$$

for every $x, y \in L$.

Proof. If $\mu \in \mathcal{FI}(L)$, then from Lemma 1, $\mu(x \boxplus y) = \mu(x \uplus y^{**}) = \min(\mu(x), \mu(y^{**})) = \min(\mu(x), \mu(y))$, for every $x, y \in L$.

Also, using [12], Corollary 3.3, $\mu(x \vee y) = \min(\mu(x), \mu(y))$, for every $x, y \in L$.

We conclude that $\mu(x \boxplus y) = \mu(x \vee y) = \min(\mu(x), \mu(y^{**}))$, for every $x, y \in L$.

Conversely, suppose that $\mu(x \boxplus y) = \mu(x \vee y) = \min(\mu(x), \mu(y^{**}))$, for every $x, y \in L$.

Thus, for $x = 0$, we obtain

$$\mu(y^{**}) = \mu(y),$$

for every $y \in L$.

If we consider $x, y \in L$ such that $x \preceq y$ then $\mu(y) = \mu(x \vee y) = \min(\mu(x), \mu(y^{**})) = \min(\mu(x), \mu(y))$; hence, $\mu(x) \geq \mu(y)$.

From (2), $x \vee y \preceq x \uplus y \preceq x \boxplus y$, so $\min(\mu(x), \mu(y^{**})) = \mu(x \boxplus y) \leq \mu(x \uplus y) \leq \mu(x \vee y) = \min(\mu(x), \mu(y^{**}))$, for every $x, y \in L$.

We deduce that

$$\mu(x \uplus y) = \min(\mu(x), \mu(y^{**})) = \min(\mu(x), \mu(y)),$$

for every $x, y \in L$.

Using Definition 2, we conclude that $\mu \in \mathcal{FI}(L)$. $\square$

Lemma 2. *Let $x, y, z \in L$. Then, $x^* \boxplus (y \boxplus z) = 1$ iff $x \preceq y \boxplus z$.*

Proof. If $x^* \boxplus (y \boxplus z) = 1$, then $1 = x^{**} \to (y \boxplus z)^{**} = x^{**} \to (y \boxplus z)$, so $x \preceq x^{**} \preceq y \boxplus z$.

Conversely, $x \preceq y \boxplus z \Rightarrow x^{**} \preceq (y \boxplus z)^{**} \Rightarrow x^{**} \to (y \boxplus z)^{**} = 1 \Rightarrow x^* \boxplus (y \boxplus z) = 1$. $\square$

Proposition 3. *Let μ be a fuzzy set in L. The following are equivalent:*

(i) $\mu \in \mathcal{FI}(L)$;
(ii) *For every $x, y, z \in L$, if $(x \boxplus y) \boxplus z^* = 1$, then $\mu(z) \geq \min(\mu(x), \mu(y))$;*
(iii) *For every $x, y, z \in L$, if $z \preceq x \boxplus y$, then $\mu(z) \geq \min(\mu(x), \mu(y))$.*

Proof. $(i) \Longrightarrow (ii)$. Let $x, y, z \in L$ such that $(x \boxplus y) \boxplus z^* = 1$. Then, $1 = (x \boxplus y)^* \longrightarrow z^*$ so, $(x \boxplus y)^* \preceq z^*$. Thus, using Lemma 1 and Proposition 1, we have $\mu(z) = \mu(z^{**}) \geq \mu((x \boxplus y)^{**}) = \mu(x \boxplus y) \geq \min(\mu(x), \mu(y))$.

$(ii) \Longrightarrow (i)$. Since $(x \boxplus x) \boxplus 0^* = 1$, by the hypothesis, we deduce (fi_3). Also, since $[x \boxplus (x^* \odot y)] \boxplus y^* = (x \boxplus y^*) \boxplus (x^* \odot y) = (x^* \odot y)^* \boxplus (x^* \odot y) = 1$, we obtain (fi'_4). Thus, $\mu \in \mathcal{FI}(L)$.

$(ii) \Leftrightarrow (iii)$. Using Lemma 2, $z \preceq x \boxplus y$ iff $(x \boxplus y) \boxplus z^* = 1$. $\square$

If μ is a fuzzy set in a residuated lattice L, we denote by $\overline{\mu}$ the smallest fuzzy ideal containing μ. $\overline{\mu}$ is called the fuzzy ideal generated by μ, and it is characterized in [12], Theorem 3.19 and [11], Theorem 5.

In the following, we show a new characterization:

Proposition 4. *Let L be a residuated lattice and $\mu, \mu' : L \to [0,1]$ be fuzzy sets in L such that*

$$\mu'(x) = \sup\{\min(\mu(x_1), ..., \mu(x_n)) : x \preceq x_1 \boxplus ... \boxplus x_n, n \in N, x_1, ..., x_n \in L\},$$

for every $x \in L$. Then, $\mu' = \overline{\mu}$.

Proof. First, using Proposition 3, we will prove that $\mu' \in \mathcal{FI}(L)$.

Let $x, y, z \in L$ such that $z \preceq x \boxplus y$ and $\epsilon > 0$ arbitrary.

By definition of μ', for $x, y \in L$ there are $n, m \in N$ and $x_1, ..., x_n, y_1, ..., y_m \in L$ such that

$$x \preceq x_1 \boxplus ... \boxplus x_n \text{ and } \mu'(x) < \epsilon + \min(\mu(x_1), ..., \mu(x_n))$$

and

$$y \preceq y_1 \boxplus ... \boxplus y_m \text{ and } \mu'(y) < \epsilon + \min(\mu(y_1), ..., \mu(y_m)).$$

Then, $x \boxplus y \preceq x_1 \boxplus ... \boxplus x_n \boxplus y_1 \boxplus ... \boxplus y_m$ and $\mu'(x \boxplus y) = \sup\{\min(\mu(t_1), ..., \mu(t_k)) : x \boxplus y \preceq t_1 \boxplus ... \boxplus t_k, k \in N, t_1, ..., t_k \in L\} \geq \min(\mu(x_1), ..., \mu(x_n), \mu(y_1), ..., \mu(y_m)) = \min(\min(\mu(x_1), ..., \mu(x_n)), \min(\mu(y_1), ..., \mu(y_m))) > \min(\mu'(x) - \epsilon, \mu'(y) - \epsilon) = \min(\mu'(x), \mu'(y)) - \epsilon$.

Since ϵ is arbitrary, using Remark 2, we deduce that $\mu'(x \boxplus y) \geq \min(\mu'(x), \mu'(y))$.

Similarly, for $x \boxplus y$, there are $p \in N$ and $s_1, ..., s_p \in L$ such that

$$x \boxplus y \preceq s_1 \boxplus ... \boxplus s_p \text{ and } \mu'(x \boxplus y) < \epsilon + \min(\mu(s_1), ..., \mu(s_p)).$$

Thus, $z \preceq s_1 \boxplus ... \boxplus s_p$, so $\mu'(z) = \sup\{\min(\mu(z_1), ..., \mu(z_r)) : z \preceq z_1 \boxplus ... \boxplus z_r, r \in N, z_1, ..., z_r \in L\} \geq \min(\mu(s_1), ..., \mu(s_p)) > \mu'(x \boxplus y) - \epsilon$.

We obtain $\mu'(z) \geq \mu'(x \boxplus y)$. Finally, we conclude that $\mu'(z) \geq \min(\mu'(x), \mu'(y))$, so $\mu' \in \mathcal{FI}(L)$.

Obviously, $\mu \subset \mu'$ since for every $x \in L, x \preceq x \boxplus x$, so $\mu'(x) \geq \min(\mu(x), \mu(x)) = \mu(x)$.

Also, if $\mu'' \in \mathcal{FI}(L)$ such that $\mu \subset \mu''$, then $\mu'(x) = \sup\{\min(\mu(x_1), ..., \mu(x_n)) : x \preceq x_1 \boxplus ... \boxplus x_n, n \in N, x_1, ..., x_n \in L\} \leq \sup\{\min(\mu''(x_1), ..., \mu''(x_n)) : x \preceq x_1 \boxplus ... \boxplus x_n, n \in N, x_1, ..., x_n \in L\} \leq \mu''(x)$, for every $x \in L$, since $x \preceq x_1 \boxplus ... \boxplus x_n \Rightarrow \mu''(x) \geq \mu''(x_1 \boxplus ... \boxplus x_n) = \min(\mu''(x_1), ..., \mu''(x_n))$.

Thus, $\mu' \subset \mu''$, so μ' is the least fuzzy ideal of L containing μ, i.e., $\mu' = \overline{\mu}$. □

Theorem 1. *The lattice $(\mathcal{FI}(L), \subset)$ is a complete Brouwerian lattice.*

Proof. If $(\mu_i)_{i \in I}$ is a family of fuzzy ideals of L, then the infimum of this family is $\underset{i \in I}{\sqcap} \mu_i = \underset{i \in I}{\cap} \mu_i$ and the supremum is $\underset{i \in I}{\sqcup} \mu_i = \overline{\underset{i \in I}{\cup} \mu_i}$.

Obviously, the lattice $(\mathcal{FI}(L), \subset)$ is complete.

To prove that $\mathcal{FI}(L)$ is a Brouwerian lattice, we show that for every fuzzy ideal μ and every family $(\mu_i)_{i \in I}$ of fuzzy ideals, $\mu \sqcap (\underset{i \in I}{\sqcup} \mu_i) = \underset{i \in I}{\sqcup} (\mu \sqcap \mu_i)$. Clearly, $\underset{i \in I}{\sqcup} (\mu \sqcap \mu_i) \subset \mu \sqcap (\underset{i \in I}{\sqcup} \mu_i)$, so we prove only that $\mu \sqcap (\underset{i \in I}{\sqcup} \mu_i) \subset \underset{i \in I}{\sqcup} (\mu \sqcap \mu_i)$.

For this, let $x \in L$ and $\epsilon > 0$ arbitrary.

Since $(\underset{i \in I}{\sqcup} \mu_i)(x) = \sup\{\min((\underset{i \in I}{\cup} \mu_i)(z_1), ..., (\underset{i \in I}{\cup} \mu_i)(z_m)) : x \preceq z_1 \boxplus ... \boxplus z_m, m \in N, z_1, ..., z_m \in L\}$, there are $n \in N$ and $x_1, ..., x_n \in L$ such that

$$x \preceq x_1 \boxplus ... \boxplus x_n \text{ and } (\underset{i \in I}{\sqcup} \mu_i)(x) < \epsilon + \min((\underset{i \in I}{\cup} \mu_i)(x_1), ..., (\underset{i \in I}{\cup} \mu_i)(x_n)).$$

Using the definition of $\underset{i \in I}{\cup} \mu_i$, for every $k = 1, ..., n$ there is $i_k \in N$ such that

$$(\underset{i \in I}{\cup} \mu_i)(x_k) < \epsilon + \mu_{i_k}(x_k).$$

Thus,

$$(\underset{i \in I}{\sqcup} \mu_i)(x) < \epsilon + \min(\epsilon + \mu_{i_1}(x_1), ..., \epsilon + \mu_{i_n}(x_n)).$$

Then,

$$(\mu \sqcap (\underset{i \in I}{\sqcup} \mu_i))(x) < 2\epsilon + \min(\mu(x), \mu_{i_1}(x_1), ..., \mu_{i_n}(x_n)).$$

We consider $y_1, ..., y_n \in L$ such that

$$y_1^* = (y_2 \boxplus ... \boxplus y_n) \boxplus x^*$$

$$y_n^* = (x_1 \boxplus ... \boxplus x_{n-1}) \boxplus x^*$$

and for every $t = 2, ..., n-1$

$$y_t^* = (x_1 \boxplus ... \boxplus x_{t-1}) \boxplus (y_{t+1} \boxplus ... \boxplus y_n) \boxplus x^*.$$

Obviously, for every $t = 1, ..., n$, $y_t^* \boxplus x = 1$, so, $y_t^{**} \preceq x^{**}$ and $\mu(x) = \mu(x^{**}) \leq \mu(y_t^{**}) = \mu(y_t)$.

Moreover, $(y_1 \boxplus ... \boxplus y_n) \boxplus x^* = y_1 \boxplus y_1^* = 1$, so using Lemma 2, we deduce that

$$x \preceq y_1 \boxplus ... \boxplus y_n.$$

Also, by Lemma 2, since $x \preceq x_1 \boxplus ... \boxplus x_n$, we have that $y_n^* \boxplus x_n = (x_1 \boxplus ... \boxplus x_n) \boxplus x^* = 1$ and for every $t = 1, ..., n-1$, $y_t^* \boxplus x_t = [(x_1 \boxplus ... \boxplus x_t) \boxplus (y_{t+2} \boxplus ... \boxplus y_n \boxplus x^*] \boxplus y_{t+1} = y_{t+1}^* \boxplus y_{t+1} = 1$.
So,

$$y_t \preceq x_t, \text{ for every } t = 1, ..., n.$$

Thus, we deduce that

$$\mu_{i_k}(x_k) \leq \mu_{i_k}(y_k), \text{ for every } k = 1, ..., n.$$

We conclude that

$$\min(\mu(x), \mu_{i_k}(x_k)) \leq \min(\mu(y_k), \mu_{i_k}(y_k)) = (\mu \sqcap \mu_{i_k})(y_k), \text{ for every } k = 1, ..., n.$$

Thus,

$$(\mu \sqcap (\bigsqcup_{i \in I} \mu_i))(x) < 2\epsilon + \min((\mu \sqcap \mu_{i_1})(y_1), ...(\mu \sqcap \mu_{i_n})(y_n)).$$

Since $(\mu \sqcap \mu_{i_k})(y_k) \leq (\bigsqcup_{i \in I} (\mu \sqcap \mu_i))(y_k)$, for every $k = 1, ..., n$, using the fact that $x \preceq y_1 \boxplus ... \boxplus y_n$, we obtain

$$(\mu \sqcap (\bigsqcup_{i \in I} \mu_i))(x) < 2\epsilon + \min((\bigsqcup_{i \in I} (\mu \sqcap \mu_i))(y_1), ..., (\bigsqcup_{i \in I} (\mu \sqcap \mu_i))(y_n)) < 2\epsilon + (\bigsqcup_{i \in I} (\mu \sqcap \mu_i))(x).$$

But ϵ is arbitrary, so from Remark 2,

$$(\mu \sqcap (\bigsqcup_{i \in I} \mu_i))(x) \leq (\bigsqcup_{i \in I} (\mu \sqcap \mu_i))(x).$$

□

By [14] and Theorem 1, we deduce that

Proposition 5. *If $\mu_1, \mu_2 \in \mathcal{FI}(L)$, then*

(i) $\mu_1 \rightsquigarrow \mu_2 = \sup\{\mu \in \mathcal{FI}(L) : \mu_1 \sqcap \mu \subset \mu_2\} = \sqcup\{\mu \in \mathcal{FI}(L) : \mu_1 \sqcap \mu \subset \mu_2\} \in \mathcal{FI}(L)$;
(ii) *If $\mu \in \mathcal{FI}(L)$, then $\mu_1 \sqcap \mu \subset \mu_2$ if and only if $\mu \sqsubset \mu_1 \rightsquigarrow \mu_2$.*

Corollary 1. *$(\mathcal{FI}(L), \sqcap, \sqcup, \rightsquigarrow, \mathbf{0})$ is a Heyting algebra.*

4. Applications of Fuzzy Sets in Coding Theory

4.1. Symmetric Difference of Ideals in a Finite Commutative and Unitary Ring

In this section, we present an application of fuzzy sets on some special cases of residuated algebras, namely, Boolean algebras. We find connections between the fuzzy sets associated to ideals in particular residuated lattices and Hadamard codes.

We recall that if A is a nonempty set and $B \subset A$ is a nonempty subset of A, then the map $\mu_B : A \to [0, 1]$,

$$\mu_B(x) = \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases},$$

is called the *characteristic function* of the set B.

For two nonempty sets, A, B, we define the *symmetric difference* of the sets A, B,

$$A \Delta B = (A - B) \cup (B - A) = (A \cup B) - (B \cap A)$$

Proposition 6. *We consider A and B as two nonempty sets.*

(*i*) *We have* $\mu_{A\Delta B} = 0$ *if and only if* $A = B$*;*
(*ii*) *([16], p. 215). The following relation is true*

$$\mu_{A\Delta B} = \mu_A + \mu_B - 2\mu_A\mu_B.$$

(*iii*) *Let* $A_i, i \in \{1, 2, ..., n\}$ *be n nonempty sets. The following relation is true*

$$\begin{aligned}\mu_{A_1\Delta A_2\Delta...\Delta A_n} &= \sum_{i\in\{1,2,...,n\}} \mu_{A_i} - 2\sum_{i\neq j}\mu_{A_i}\mu_{A_j} + 2^2 \sum_{i\neq j\neq k} \mu_{A_i}\mu_{A_j}\mu_{A_k} - ... + \\ &+(-1)^{n-1}2^{n-1}\mu_{A_1}\mu_{A_2}...\mu_{A_n}.\end{aligned}$$

Remark 3. *Let* $(R, +, \cdot)$ *be a unitary and a commutative ring and* $I_1, I_2, ..., I_s$ *be ideals in R.*

(*i*) *For* $i \neq j$*, we have* $I_i\Delta I_j$ *is not an ideal in R. Indeed,* $0 \notin I_i\Delta I_j$*; therefore,* $I_i\Delta I_j$ *is not an ideal in R;*
(*ii*) *In general,* $I_1\Delta I_2\Delta...\Delta I_n$*, for* $n \geq 2$*, is not an ideal in R. Indeed, if* $n \geq 3$ *and* $x, y \in I_1\Delta I_2\Delta...\Delta I_n$*, supposing that* $x \in I_j$ *and* $y \in I_k$*, we have that* $xy \in I_j$ *and* $xy \in I_k$*; therefore,* $xy \in I_j \cap I_k$*. We obtain that* $\mu_{I_1\Delta I_2\Delta...\Delta I_n}(xy) = \mu_{I_j}(xy) + \mu_{I_k}(xy) - 2\mu_{I_j}\mu_{I_k}(xy) = 0$*, then* $xy \notin I_1\Delta I_2\Delta...\Delta I_n$ *and* $I_1\Delta I_2\Delta...\Delta I_n$ *is not an ideal in R.*

Definition 3. *If* $A = \{a_1, a_2, ..., a_n\}$ *is a finite set with n elements and B is a nonempty subset of A, we consider the vector* $c_B = (c_i)_{i\in\{1,2,...,n\}}$*, where* $c_i = 0$ *if* $a_i \notin B$ *and* $c_i = 1$ *if* $a_i \in B$*. The vector* c_B *is called the codeword attached to the set B. We can represent* c_B *as a string* $c_B = c_1c_2...c_n$*.*

4.2. Linear Codes

We consider p a prime number and $\mathbf{F}_{p^n}$ a finite field of characteristic p. $\mathbf{F}_{p^n}$ is a vector space over the field $\mathbb{Z}_p$. A *linear code* $\mathcal{C}$ of length n and dimension k is a vector subspace of the vector space $\mathbf{F}_{p^n}$. If $p = 2$, we call this code a binary linear code. The elements of $\mathcal{C}$ are called *codewords*. The *weight* of a codeword is the number of its elements that are nonzero, and the *distance* between two codewords is the *Hamming distance* between them, which represents the number of elements by which they differ. The distance d of the linear code is the minimum weight of its nonzero codewords or, equivalently, the minimum distance between distinct codewords. A linear code of length n, dimension k, and distance d is called an $[n, k, d]$ code (or, more precisely, an $[n, k, d]_p$ code). *The rate* of a code is $\frac{k}{n}$, which means it is an amount such that for each k bits of transmitted information, the code generates n bits of data, in which $n - k$ are redundant. Since $\mathcal{C}$ is a vector subspace of dimension k, it is generated by bases of k vectors. The elements of such a basis can be represented as a rows of a matrix G, named the *generating matrix* associated with the code $\mathcal{C}$. This matrix is a matrix of $k \times n$ type (see [17]). The codes of the type $[2^t, t, 2^{t-1}]_2$, $t \geq 2$, are called *Hadamard codes*. Hadamard codes are a class of error-correcting codes (see [18], p. 183). Named after french mathematician Jacques Hadamard, these codes are used for error detection and correction when transmitting messages over noisy or unreliable channels. Usually, Hadamard codes are constructed by using Hadamard matrices of Sylvester's type, but there are Hadamard codes using an arbitrary Hadamard matrix that are not necessarily of the above type (see [19]). As we can see, Hadamard codes have a good distance property, but the rate is of a low level (see [17]).

Remark 4 ([17], Definition 16)**.** *The generating matrix of a Hadamard code of the type* $[2^t, t, 2^{t-1}]_2$*,* $t \geq 2$*, has as columns all t-bit vectors over* $\mathbb{Z}_2$ *(vectors of length t).*

4.3. Connections between Boolean Algebras and Hadamard Codes

In the following, we present a particular case of residuated lattices, named *MV-algebras, and their connections to Hadamard codes.*

Definition 4 ([2])**.** *An abelian monoid* $(X, \theta, \oplus)$ *is called an MV-algebra if and only if we have an operation "′" such that*

(i) $(x')' = x$;
(ii) $x \oplus \theta' = \theta'$;
(iii) $(x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x$, *for all* $x, y \in X$. *We denote it by* $(X, \oplus, ', \theta)$.

Definition 5 ([3], Definition 4.2.1)**.** *An algebra* $(W, \circ, ^{-}, 1)$ *of type* $(2, 1, 0)$ *is called a Wajsberg algebra (or W-algebra) if and only if for every* $x, y, z \in W$ *we have*

(i) $1 \circ x = x$;
(ii) $(x \circ y) \circ [(y \circ z) \circ (x \circ z)] = 1$;
(iii) $(x \circ y) \circ y = (y \circ x) \circ x$;
(iv) $(\overline{x} \circ \overline{y}) \circ (y \circ x) = 1$.

Remark 5 ([3], Lemma 4.2.2 and Theorem 4.2.5).

(i) *If* $(W, \circ, ^{-}, 1)$ *is a Wajsberg algebra, defining the following multiplications*

$$x \odot y = \overline{(x \circ \overline{y})}$$

and

$$x \oplus y = \overline{x} \circ y,$$

for all $x, y \in W$, *we obtain that* $(W, \oplus, \odot, ^{-}, 0, 1)$ *is an MV-algebra.*

(ii) *If* $(X, \oplus, \odot, ', \theta, 1)$ *is an MV-algebra, defining on* X *the operation*

$$x \circ y = x' \oplus y,$$

it results that $(X, \circ, ', 1)$ *is a Wajsberg algebra.*

Definition 6 ([5])**.** *If* $(W, \circ, ^{-}, 1)$ *is a Wajsberg algebra, on* W, *we define the following binary relation*

$$x \leq y \text{ if and only if } x \circ y = 1. \tag{3.2.}$$

This relation is an order relation, called the natural order relation on W.

Definition 7 ([4])**.** *Let* $(X, \oplus, ', \theta)$ *be an MV-algebra. The nonempty subset* $I \subseteq X$ *is called an ideal in* X *if and only if the following conditions are satisfied:*

(i) $\theta \in I$, *where* $\theta = \overline{1}$;
(ii) $x \in I$ *and* $y \leq x$ *implies* $y \in I$;
(iii) *If* $x, y \in I$, *then* $x \oplus y \in I$.

We remark that the concept of ideals in residuated lattices is a generalization for the notion of ideals in MV-algebras.

Definition 8 ([3], p. 13)**.** *An ideal* P *of the MV-algebra* $(X, \oplus, ', \theta)$ *is a prime ideal in* X *if and only if for all* $x, y \in P$ *we have* $(x' \oplus y)' \in P$ *or* $(y' \oplus x)' \in P$.

Definition 9 ([20], p. 56)**.** *Let* $(W, \circ, ^{-}, 1)$ *be a Wajsberg algebra and let* $I \subseteq W$ *be a nonempty subset.* I *is called an ideal in* W *if and only if the following conditions are fulfilled:*

(i) $\theta \in I$, *where* $\theta = \overline{1}$;
(ii) $x \in I$ *and* $y \leq x$ *implies* $y \in I$;
(iii) *If* $x, y \in I$, *then* $\overline{x} \circ y \in I$.

Definition 10. *Let $(W, \circ, ^{-}, 1)$ be a Wajsberg algebra and $P \subseteq W$ be a nonempty subset. P is called a prime ideal in W if and only if for all $x, y \in P$ we have $(x \circ y)' \in P$ or $(y \circ x)' \in P$.*

Definition 11. *The algebra $(B, \vee, \wedge, \partial, 0, 1)$, equipped with two binary operations $\vee$ and $\wedge$ and a unary operation ∂, is called a Boolean algebra if and only if $(B, \vee, \wedge)$ is a distributive and a complemented lattice with*

$$x \vee \partial x = 1,$$

$$x \wedge \partial x = 0,$$

for all elements $x \in B$. The elements 0 and 1 are the least and the greatest elements from the algebra B.

Remark 6.

(i) *Boolean algebras represent a particular case of MV-algebras. Indeed, if $(B, \vee, \wedge, \partial, 0, 1)$ is a Boolean algebra, then it can be easily checked that $(B, \vee, \partial, 0)$ is an MV-algebra;*
(ii) *A Boolean ring $(B, +, \cdot)$ is a unitary and commutative ring such that $x^2 = x$ for each $x \in B$;*
(iii) *To a Boolean algebra $(B, \vee, \wedge, \partial, 0, 1)$, we can associate a Boolean ring $(B, +, \cdot)$, where*

$$\begin{array}{rcl} x + y & = & (x \vee y) \wedge \partial(x \wedge y), \\ x \cdot y & = & x \wedge y, \end{array}$$

for all $x, y \in B$. Conversely, if $(B, +, \cdot)$ is a Boolean ring, we can associate a Boolean algebra $(B, \vee, \wedge, \partial, 0, 1)$, where

$$\begin{array}{rcl} x \vee y & = & x + y + xy, \\ x \wedge y & = & xy, \\ \partial x & = & 1 + x; \end{array}$$

(iv) *Let $(I, +, \cdot)$ be an ideal in a Boolean ring $(B, +, \cdot)$; therefore, I is an ideal in the Boolean algebra $(B, \vee, \wedge, \partial, 0, 1)$. The converse is also true.*

Remark 7.

(i) *If X is an MV-algebra and I is an ideal (prime ideal) in X, then on the Wajsberg algebra structure, obtained as in Remark 3.7. (ii), we have that the same set I is an ideal (prime ideal) in X as a Wajsberg algebra. The converse is also true.*
(ii) *Finite MV-algebras of order 2^t are Boolean algebras.*
(iii) *Between ideals in a Boolean algebra and ideals in the associated Boolean ring it is a bijective correspondence, which means that if I is an ideal in a Boolean algebra, the same set I, with the corresponding multiplications, is an ideal in the associated Boolean ring. The converse is also true.*

Proposition 7. *Let $(R, +, \cdot)$ be a finite, commutative, unitary ring and I, J be two ideals. If c_I and c_J are the codewords attached to these sets (as in Definition 3), then*

(i) *To the set $I \Delta J$ corresponds the codeword $c_I + c_J = c_I \oplus c_J$, where $\oplus$ is the XOR-operation;*
(ii) *If $I_1, I_2, ..., I_q$ are ideals in the ring R and $c_{I_1}, c_{I_2}, ..., c_{I_q}$ are the attached codewords, then the vectors $c_{I_1}, c_{I_2}, ..., c_{I_q}$ are linearly independent vectors.*

Proof.

(i) It is clear, by straightforward computations.
(ii) Let R have n elements. We work on the vector space $V = \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times ... \times \mathbb{Z}_2}_{n-time}$ over the field $\mathbb{Z}_2$. We consider $\alpha_1 c_{I_1} + ... \alpha_q c_{I_q} = 0$, where $\alpha_1, ... \alpha_q \in \mathbb{Z}_2$. Supposing that

$\alpha_1 = ... = \alpha_q = 1$, we have that $\alpha_1 c_{I_1} + ... \alpha_q c_{I_q} = 0$ implies that $I_1 \Delta I_2 \Delta ... \Delta I_q = \emptyset$. Without losing the generality, since symmetric difference is associative, from here we have that $I_1 \Delta I_2 \Delta ... \Delta I_{q-1} = I_q$, which is false since I_q has an ideal structure and $I_1 \Delta I_2 \Delta ... \Delta I_{q-1}$ is not an ideal, from Remark 3.

□

We consider a matrix M_C, with rows the codewords associated to the ideals $I_1, I_2, ..., I_q$,

$$M_C = \begin{pmatrix} c_{I_1} \\ c_{I_2} \\ ... \\ c_{I_q} \end{pmatrix}.$$

Since these rows are linearly independent vectors, the matrix M_C can be considered as a generating matrix for a code, called *the code associated to the ideals* $I_1, I_2, ..., I_q$, denoted $\mathcal{C}_{I_1 I_2, ... I_q}$.

Theorem 2. *Let* $(B, \vee, \wedge, \partial, 0, 1)$ *be a finite Boolean algebra of order* 2^n. *The following statements are true:*

(i) *The algebra B has n ideals of order* 2^{n-1};

(ii) *The code associated with the above ideals generates a Hadamard code of the type* $[2^n, n, 2^{n-1}]_2$, $n \geq 2$.

Proof.

(i) It is clear since ideals in the Boolean algebra structure are ideals in the associated Boolean ring and vice-versa.

(ii) Let $I_1, I_2, ..., I_n$ be the ideals of order 2^{n-1}. We consider a matrix M_C, with rows the codewords associated with these ideals:

$$M_C = \begin{pmatrix} c_{I_1} \\ c_{I_2} \\ ... \\ c_{I_n} \end{pmatrix}.$$

Due to the correspondence between the ideals in the Boolean algebra structure, the ideals in the associated Boolean ring, and Proposition 7, we have that the rows of the matrix M_C are linearly independent vectors. Since $I_1, I_2, ..., I_n$ are the ideals of order 2^{n-1}, the associated codewords have 2^{n-1} nonzero elements; therefore, the Hamming distance is $d_H = 2^{n-1}$. From here, we have that M_C is a generating matrix for the code $\mathcal{C}_{I_1 I_2, ... I_n}$, which is a Hadamard code of the type $[2^n, n, 2^{n-1}]_2$, $n \geq 2$.

□

Remark 8. *A generating matrix* M_C *of a Hadamard code* $\mathcal{C}$ *of the type* $[2^n, n, 2^{n-1}]_2$, $n \geq 2$, *has* $2^{n-1}n$ *elements equal to* 1. *If the matrix has the following form, namely, on row i, we have the first* 2^{n-i} *elements equal to* 1, *the next* 2^{n-i} *elements equal to* 0, *and so on, for* $i \geq 1$, *we call this form the Boolean form of the generating matrix of the Hadamard code* $\mathcal{C}$, *and we denote it* M_B.

Remark 9.

(i). *If G, a* $r \times s$ *matrix over a field K is a generating matrix for a linear code* $\mathcal{C}$, *then any matrix that is row equivalent to G is also a generating matrix for the code* $\mathcal{C}$. *Two row equivalent matrices of the same type have the same row space. The row space of a matrix is the set of all possible linear combinations of its row vectors, which means that it is a vector subspace of the space* K^s, *with dimension the rank of the matrix G, rankG. From here, we have that two* *matrices are row equivalent if and only if one can be deduced to the other by a sequence of elementary row operations.*

(ii). *If G is a generating matrix for a linear code $\mathcal{C}$, then from the above notations, we have that M_C and M_B are row equivalent; therefore, these matrices generate the same Hadamard code $\mathcal{C}$ of the type $[2^n, n, 2^{n-1}]_2$, $n \geq 2$.*

Theorem 3. *With the above notations, let M_B be the Boolean form of a generating matrix of the Hadamard code of the type $[2^n, n, 2^{n-1}]_2$, $n \geq 2$. We can construct a Boolean algebra $\mathcal{B}$ of order 2^n, which has n ideals of order 2^{n-1}, with associated codewords being the rows of a matrix M_B.*

Proof. We consider the set $B_i = \{0_i, 1_i\}$, with $0_i \leq_i 1_i$, $i \in \{1, 2, ..., n\}$. On B_i, we define the following multiplication: $\begin{array}{c|cc} \circ_i & 0_i & 1_i \\ \hline 0_i & 1_i & 1_i \\ 1_i & 0_i & 1_i \end{array}$.

It is clear that $(B_i, \circ_i,', 1_i)$, where $0_i' = 1_i$ and $1_i' = 0_i$, is a Wajsberg algebra of order 2. On B_i, we have the following partial order relation $x_i \leq_i y_i$ if and only if $x_i \circ_i y_i = 1_i$.

Therefore, on the Cartesian product $\mathcal{B} = B_1 \times B_2 \times ... \times B_n$, we define a component-wise multiplication, denoted $\diamond$. From here, we have that $(\mathcal{B}, \diamond,', \mathbf{1})$, where $(x_1, x_2, ..., x_n)' = (x_1', x_2', ..., x_n')$ and $\mathbf{1} = (1, 1, ..., 1)$, is a Wajsberg algebra of order 2^n. We write and denote the elements of $\mathcal{B}$ in the lexicographic order. The element $(0_1, 0_2, ..., 0_n)$, denoted $(0, 0, ..., 0)$ or $\mathbf{0}$, is the first element in $\mathcal{B}$. With $\mathbf{1}$, we denote $(1, 1, ..., 1) = (1_1, 1_2, ..., 1_n)$, which is the last element in $\mathcal{B}$. From Definition 3.8, on $\mathcal{B}$, we have the following partial order relation

$$x \leq_{\mathcal{B}} y \text{ if and only if } x \diamond y = \mathbf{1}.$$

It is clear that on $\mathcal{B}$, we have that $x \leq_{\mathcal{B}} y$ if and only if $x_i \leq_i y_i$, for $i \in \{1, 2, ..., n\}$. From the Wajsberg algebra structure, we obtain the MV-algebra structure on $\mathcal{B}$, which is a Boolean algebra structure, with the multiplication $x \oplus y = x' \diamond y$ ($\oplus$ which is the component-wise XOR-sum). The ideals of order 2^{n-1} in this Boolean algebra of order 2^n are generated by the maximal elements with respect to the order relation $\leq_{\mathcal{B}}$. These elements have $n-1$ "nonzero" components. The first maximal element in the lexicographic order is $m_1 = (0, 1, 1, ..., 1)$. This element generates an ideal of order 2^{n-1}, containing all elements x_j equal to or less than m_1 with respect to the order relation $\leq_{\mathcal{B}}$. Indeed, all these elements x_j are maximum $n-2$ nonzero components, and $x_{ji} \leq_i m_{1i}$, $i \in \{1, 2, ..., n\}$, $j \in \{1, 2, ..., 2^{n-1}\}$, with the first component always zero. We denote with J_1 the set all elements equal to or less than m_1. It results that J_1 with the multiplication $\oplus$ is isomorphic to the vector space $\mathbb{Z}_2^{n-1}$; therefore, J_1 is an ideal in $\mathcal{B}$. The codeword corresponding to this ideal is $(1, 1, ..., 1, 0, 0, ..., 0)$ in which the first 2^{n-1} positions are equal to 1 and the next 2^{n-1} are 0 and represent the first row of the matrix $M_{\mathcal{B}}$. The next maximal element in lexicographic order is $m_2 = (1, 0, 1, ..., 1)$, with zero in the second position and 1 in the rest. This element generates an ideal J_2 of order 2^{n-1}, containing all elements x_j equal to or less than m_2 with respect to the order relation $\leq_{\mathcal{B}}$. All these elements x_j are maximum $n-2$ nonzero components and $x_{ji} \leq_i m_{1i}$, $i \in \{1, 2, ..., n\}$, $j \in \{1, 2, ..., 2^{n-1}\}$, with the second component always zero. With the same reason as above, we have that J_2, with the multiplication $\oplus$, is isomorphic to the vector space $\mathbb{Z}_2^{n-1}$; therefore, J_2 is an ideal in $\mathcal{B}$. The codeword corresponding to this ideal is $(1, 1, ..., 1, 0, 0, ..., 0, 1, 1, ..., 0, ...)$, with the first 2^{n-2} positions equal to 1, the next 2^{n-2} are 0 and so on. This codeword represents the second row of the matrix $M_{\mathcal{B}}$, etc. $\square$

Example 1. *In [21], the authors described all Wajsberg algebras of order less than or equal to 9. In the following, we provide some examples of codes associated to these algebras.*

Case $n = 4$. *We have two types of Wajsberg algebras of order 4. The first type is a totally ordered set that has no proper ideals, and the second type is a partially ordered Wajsberg algebra, $W = \{0, a, b, 1\}$. This algebra has the multiplication given by the following table:*

$\circ$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

This algebra has two proper ideals $I = \{0, a\}$ and $J = \{0, b\}$. The associated MV-algebra of this algebra is a Boolean algebra. We consider $c_I = (1, 1, 0, 0)$ and $c_J = (1, 0, 1, 0)$ the codewords attached to the ideals I and J. The matrix

$$M_C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

is the generating matrix for the Hadamard code of the type $\left(2^2, 2, 2\right)$. As in Remark 4, this matrix has as columns all 2-bit vectors over $\mathbb{Z}_2 : \{11, 10, 01, 00\}$.

Case $n = 8$. *We consider the partially ordered Wajsberg algebra, $W = \{0, a, b, c, d, e, f, 1\}$ with the multiplication given by the following table:*

$\circ$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	f	1	f	1	f	1	f	1
b	e	e	1	1	e	e	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	b	c	b	c	f	1	f	1
f	a	a	c	c	e	e	1	1
1	0	a	b	c	d	e	f	1

All proper ideals of the form $I_1 = \{0, a\}$, $I_2 = \{0, b\}$, $I_3 = \{0, d\}$, $I_4 = \{0, a, b, c\}$, $I_5 = \{0, a, d, e\}$, $I_6 = \{0, b, d, f\}$ are also prime ideals. This algebra has three ideals of order three I_4, I_5, I_6. The associated MV-algebra of this algebra is a Boolean algebra. We consider $c_{I_4} = (1, 1, 1, 1, 0, 0, 0, 0)$, $c_{I_5} = (1, 1, 0, 0, 1, 1, 0, 0), c_{I_6} = (1, 0, 1, 0, 1, 0, 1, 0)$ the codewords attached to the ideals I_4, I_5, I_6. The matrix

$$M_C = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

is the generating matrix for the Hadamard code $\left(2^3, 2, 2^2\right)$. As in Remark 4, this matrix has as columns all 3-bit vectors over $\mathbb{Z}_2$, namely, $\{111, 110, 101, 100, 011, 010, 001, 000\}$.

Example 2 ([21], case $n = 9$)**.** *If a finite Wajsberg algebra has an even number of proper ideals, we can consider their associated codewords as above. The obtained matrix generates a linear code with* *a Hamming distance* ≥ 3. *Indeed, for* $n = 9$, *we consider the partially ordered Wajsberg algebra,* $W = \{0, a, b, c, d, e, f, g, 1\}$ *with the multiplication given by the following table:*

$\circ$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	a	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

.

All proper ideals are $I_1 = \{0, a, b\}$, $I_2 = \{0, c, f\}$ *and are also prime ideals. We consider* $c_{I_1} = (1, 1, 1, 0, 0, 0, 0, 0, 0)$ *and* $c_{I_2} = (1, 0, 0, 1, 0, 0, 1, 0, 0)$ *the codewords attached to the ideals* I_1, I_2. *The matrix*

$$M_C = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

is the generating matrix for the linear code of the form $(9, 2, 3), \mathcal{C}_{I_1 I_2}$. *The even numbers of ideals assure us that the rows in the generating matrix are linear independent vectors.*

5. Conclusions

Ideals and fuzzy ideals theory are tools in the study of algebras of logic.

In this paper, based on ideals, we investigated residuated lattices from three points of view: fuzzy set theory, lattice theory, and coding theory. To identify the properties of fuzzy ideals that are useful for the study of residuated lattices, we analyzed their lattice structure and proved that they form a Heyting algebra. We also found connections between the fuzzy sets associated to ideals in particular residuated lattices and Hadamard codes.

In further research, we will investigate fuzzy congruences in residuated lattices to embed the lattice of fuzzy ideals into the lattice of fuzzy congruences. Another direction is to study other connections between fuzzy sets and some types of logic algebras.

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