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Asymmetric Equivalences in Fuzzy Logic

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Abstract: We introduce a new class of operations called asymmetric equivalences. Several properties of asymmetric equivalence operations have been investigated. Based on the asymmetric equivalence, quasi-metric spaces are constructed on [0, 1]. Finally, we discuss symmetrization of asymmetric equivalences.

Keywords: fuzzy logic; t-norm; fuzzy implication; asymmetric equivalence; quasi-metric

1. Introduction

In fuzzy logic, a fuzzy biconditional statement has the form of "p if and only if q", where p and q are fuzzy statements. This is often abbreviated "$p \leftrightarrow q$". The corresponding biconditional logical connective denoted using a double-headed arrow ($\leftrightarrow$) is an operation of equivalence. As we know, join, meet, multiplication and implication are basic operations in several fuzzy logic systems, such as monoidal t-norm based logic (MTL) [1], basic logic (BL) [2] and their extensions product logic (Π), Łukasiewicz logic (Ł) and Gödel logic (G). Then, the logical equivalence is a derived operation that is interpreted by the bi-implication

$$p \leftrightarrow q \overset{def}{=} (p \to q) \wedge (q \to p). \tag{1}$$

Based on this formula of equivalence, Georgescu [3] investigated the similarity of fuzzy choice functions. Jin et al. [4] and Dai et al. [5] investigated the robustness of fuzzy inference methods, and Wang et al. [6] and Duan et al. [7] discussed fuzzy logic metric structures. In addition, Dyba and Novák [8] developed a fuzzy equivalence based logic in which fuzzy equivalence is one of the basic connectives.

In the above-mentioned applications and logic systems, two fuzzy implications used in Formula (1) are consistent. However, there exist several fuzzy implications in one logic system. For example, Łukasiewicz implication and product implication are two of the basic connectives in ŁΠ [9,10], which is a complete fuzzy system joining Π and Ł logic. Moreover, Gödel implication also is a derived operation in ŁΠ logic. This generates some special kinds of equivalence. For example, a fuzzy equivalence can be defined by setting the first operator in Formula (1) be Łukasiewicz implication and the latter one be product implication. In addition, this kind of equivalence-constructing method based on two or more different implications have been well studied in quantum logic (expressed as orthomodular lattice) [11–13]. Enlightened by these, we can let the two implications in Formula (1) be different and then generalize a special class of equivalence. Since the new class of equivalence is asymmetrical, it is called asymmetric equivalence. Traditionally, equivalence corresponding to the biconditional logical connective is symmetric. Researchers naturally continue to study symmetric equivalence in the fuzzy logic systems. In our view, strategic use of the deliberate lack of symmetry is also a useful method to investigate and design fuzzy logic systems. In this paper, we show that the asymmetric equivalence has good expression of other operations (see Section 3) and a straightforward application in constructions of quasi-metrics (see Section 4).

The remainder of this paper is organized in the following way. In Section 2, we review necessary concepts and two lemmas related to this paper. In Section 3, we introduce the operations of asymmetric equivalence associated with two different fuzzy implications and give some of their properties. In Section 4, we study the quasi-metrics induced by asymmetric equivalences. In Section 5, we discuss the symmetrization of asymmetric equivalences. Conclusions are presented in Section 6.

2. Preliminaries

In this section, we recall some basic concepts of t-norms and their residua. For more details, see [2,14,15].

Definition 1. *In reference [15], a function $* : [0,1]^2 \to [0,1]$ is called a t-norm if it is commutative, associative and nondecreasing in each place and $1 * x = x$ for all $x \in [0,1]$.*

Definition 2. *In reference [14], a function $\to: [0,1]^2 \to [0,1]$ is called a fuzzy implication if it is non-increasing on the first component and non-decreasing on the second one, and $1 \to 0 = 0$, $0 \to 0 = 1 \to 1 = 1$.*

A t-norm $*$ is said to be left-continuous if it is left-continuous in the first component. With any left-continuous t-norm $*$, the residual implication (R-implication)$\to_*$ induced by $*$ is the function defined as, for any $a, b \in [0,1]$,

$$a \to_* b = \bigvee\{c \in [0,1] \mid a * c \leq b\}. \tag{2}$$

Obviously, the R-implication $\to_*$ induced by the given left-continuous t-norm $*$ is a fuzzy implication.

We list several important left-continuous t-norms and their residua defined on [0, 1] as follows (see [2]).

(i) Minimum t-norm and its residuum, Gödel implication

$$a *_G b = a \wedge b,\ a \to_G b = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{if } a > b. \end{cases}$$

(ii) Łukasiewicz t-norm and its residuum, Łukasiewicz implication

$$a *_L b = 0 \vee (a + b - 1),\ a \to_L b = 1 \wedge (1 - a + b);$$

(iii) Product t-norm and its residuum, product implication

$$a *_P b = ab;\ a \to_P b = \begin{cases} 1, & \text{if } a \leq b, \\ b/a, & \text{if } a > b. \end{cases}$$

For any two t-norms $*_i$ and $*_j$, if $a *_i b \leq a *_j b$ holds for all $a, b \in [0,1]$, then we say that $*_i$ is weaker than $*_j$ or, equivalently, that $*_j$ is stronger than $*_i$, denoted $*_i \prec *_j$.

In [15], it is shown that

$$*_L \prec *_P \prec *_G. \tag{3}$$

Similarly, for any two implications $\rightarrow_i$ and $\rightarrow_j$, if $a \rightarrow_i b \leq a \rightarrow_j b$ holds for all $a, b \in [0,1]$, then we say that $\rightarrow_i$ is weaker than $\rightarrow_j$ or, equivalently, that $\rightarrow_j$ is stronger than $\rightarrow_i$, denoted $\rightarrow_i \prec \rightarrow_j$.

In [14], it is shown that

$$\rightarrow_G \prec \rightarrow_P \prec \rightarrow_L . \tag{4}$$

Definition 3. *In reference [14], a function* $\neg : [0,1] \rightarrow [0,1]$ *is called a fuzzy negation if it is decreasing and* $\neg 0 = 1, \neg 1 = 0$.

The negation operation $\neg$ associated with the given left-continuous t-norm $*$ is defined by

$$\neg a = a \rightarrow_* 0 = \bigvee \{c \in [0,1] | a * c = 0\}, \tag{5}$$

where $\rightarrow_*$ is an R-implication induced by $*$.

Lemma 1. *In reference [2,15], let* $\rightarrow$ *be an R-implication induced by* $*$*, for any* $a, b, c \in [0,1]$*, the following properties hold*

(i) $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$;
(ii) $a * (a \rightarrow b)) \leq a \wedge b$; *furthermore, if* $*$ *is continuous, then* $a * (a \rightarrow b)) = a \wedge b$;
(iii) $a \leq b \Leftrightarrow a \rightarrow b = 1$;
(iv) $1 \rightarrow a = a$;
(v) $(a \rightarrow b) * (b \rightarrow c) \leq a \rightarrow c$.

Lemma 2. *Let* $a_1, a_2, b_1, b_2, c_1, c_2$ *be real numbers,*

(i) if $a_1 + b_1 \leq c_1$*, and* $a_2 + b_2 \leq c_2$*, then* $a_1 \wedge a_2 + b_1 \wedge b_2 \leq c_1 \wedge c_2$.
(ii) if $a_1 + b_1 \geq c_1$*, and* $a_2 + b_2 \geq c_2$*, then* $a_1 \vee a_2 + b_1 \vee b_2 \geq c_1 \vee c_2$.

Proof.

(i) Since $a_1 \wedge a_2 + b_1 \wedge b_2 \leq a_1 + b_1 \leq c_1$ and $a_1 \wedge a_2 + b_1 \wedge b_2 \leq a_2 + b_2 \leq c_2$, then $a_1 \wedge a_2 + b_1 \wedge b_2 \leq c_1 \wedge c_2$.
(ii) Since $a_1 \vee a_2 + b_1 \vee b_2 \geq a_1 + b_1 \geq c_1$ and $a_1 \vee a_2 + b_1 \vee b_2 \geq a_2 + b_2 \geq c_2$, then $a_1 \vee a_2 + b_1 \vee b_2 \geq c_1 \vee c_2$.

□

3. Asymmetric Equivalence Induced by R-Implication

Letting $\rightarrow_i, \rightarrow_j$ be two fuzzy implications, we define an asymmetric equivalence operation as follows:

$$a \leftrightarrow_{ij} b \stackrel{def}{=} (a \rightarrow_i b) \wedge (b \rightarrow_j a). \tag{6}$$

For an asymmetric equivalence $\leftrightarrow_{ij}$, its conjugate (or dual) asymmetric equivalence $\leftrightarrow_{ji}$ is defined for all $a, b \in [0,1]$ by

$$a \leftrightarrow_{ji} b \stackrel{def}{=} b \leftrightarrow_{ij} a. \tag{7}$$

Theorem 1. *Let* $\rightarrow_i, \rightarrow_j$ *be two R-implications induced by two continuous t-norms. If asymmetric equivalence* $\leftrightarrow_{ij}$ *and its conjugate asymmetric equivalence* $\leftrightarrow_{ji}$ *are induced by* $\rightarrow_i$ *and* $\rightarrow_j$*, then implications and negation operations can be expressed as follows:*

(i) $a \rightarrow_i b = a \leftrightarrow_{ij} (a \wedge b) = (a \wedge b) \leftrightarrow_{ji} a$;
(ii) $a \rightarrow_j b = (a \wedge b) \leftrightarrow_{ij} a = a \leftrightarrow_{ji} (a \wedge b)$;
(iii) $\neg_i a = a \leftrightarrow_{ij} 0$;
(iv) $\neg_j a = 0 \leftrightarrow_{ij} a$.

Proof.

(i) Since t-norm $*_i$ is continuous, from Lemma 1(ii), $a *_i (a \to_i b)) = a \wedge b$, then using Lemma 1(i), we obtain $a \to_i b = a \to_i a \wedge b$. In addition, from Lemma 1(iii), $a \wedge b \to_j a = 1$. Hence,

$$a \leftrightarrow_{ij} (a \wedge b) = (a \to_i a \wedge b) \wedge (a \wedge b \to_j a) = (a \to_i b) \wedge 1 = a \to_i b.$$

(ii) can be proven similarly. Properties (iii) and (iv) are special cases of (i) and (ii), respectively, by setting $b = 0$.

□

Theorem 1 shows the expression of the corresponding fuzzy implications and negations by asymmetric equivalence. Moreover, asymmetric equivalence has the following properties.

Theorem 2. *For any $a, b, c \in [0, 1]$, $\leftrightarrow_{ij}$ has the following properties*

(i) Reflexivity: $a \leftrightarrow_{ij} b$ if and only if $a = b$;
(ii) Left monotonicity: $(a \wedge b \wedge c) \leftrightarrow_{ij} a \geq (a \wedge b) \leftrightarrow_{ij} a$;
(iii) Right monotonicity: $a \leftrightarrow_{ij} (a \wedge b \wedge c) \leq a \leftrightarrow_{ij} (a \wedge b)$.

Proof.

(i) follows immediately from Lemma 1 (iii) and (iv).
(ii) and (iii) follow immediately from the monotonicity of fuzzy implications.

□

In classical and fuzzy logic [1,2,10], (symmetric) equivalences also satisfy the above properties. In addition, left and right monotonicity properties are the same, as the equivalences are symmetric. What is important is that the asymmetric equivalence does not have to satisfy the symmetry property. The following examples are operations of the asymmetric equivalence induced by two different implication operators.

Example 1. *If $\to_L$ is Łukasiewicz implication, and $\to_P$ is product implication, then $\leftrightarrow_{LP}$ is defined for all $a, b \in [0, 1]$ by*

$$a \leftrightarrow_{LP} b = \begin{cases} a/b, & \text{if } a < b, \\ 1 - a + b, & \text{if } a \geq b, \end{cases} \tag{8}$$

and its conjugate asymmetric equivalence by

$$a \leftrightarrow_{PL} b = \begin{cases} b/a, & \text{if } b < a, \\ 1 - b + a, & \text{if } b \geq a. \end{cases} \tag{9}$$

As shown in Figure 1, $a \leftrightarrow_{PL} b \neq b \leftrightarrow_{PL} a$, i.e., $a \leftrightarrow_{PL} b \neq a \leftrightarrow_{LP} b$. For example, $0.4 \leftrightarrow_{PL} 0.5 = 0.9$, but $0.4 \leftrightarrow_{LP} 0.5 = 0.8$. However, from the concept of conjugate asymmetric equivalence, we have $a \leftrightarrow_{PL} b = b \leftrightarrow_{LP} a$ for any $a, b \in [0, 1]$.

Moreover, we have

$$a \to_P b = a \leftrightarrow_{PL} (a \wedge b) = (a \wedge b) \leftrightarrow_{LP} a, \tag{10}$$
$$a \to_L b = (a \wedge b) \leftrightarrow_{PL} a = a \leftrightarrow_{LP} (a \wedge b), \tag{11}$$
$$\neg_P a = a \leftrightarrow_{PL} 0 = 0 \leftrightarrow_{LP} a, \tag{12}$$
$$\neg_L a = 0 \leftrightarrow_{PL} a = a \leftrightarrow_{LP} 0, \tag{13}$$

for any $a, b \in [0, 1]$.

The above connectives $\neg_L$ *and* $\neg_P$ *are Łukasiewicz negation and product negation, respectively. We also can obtain the expression of Gödel implication in the following way:* $\Delta a = \neg_P \neg_L a$, $a \vee b = (a \to_L b) \to_L b$, $a \to_G b = \Delta(a \to_L b) \vee b$ *(see ŁΠ logic [9,10]).*

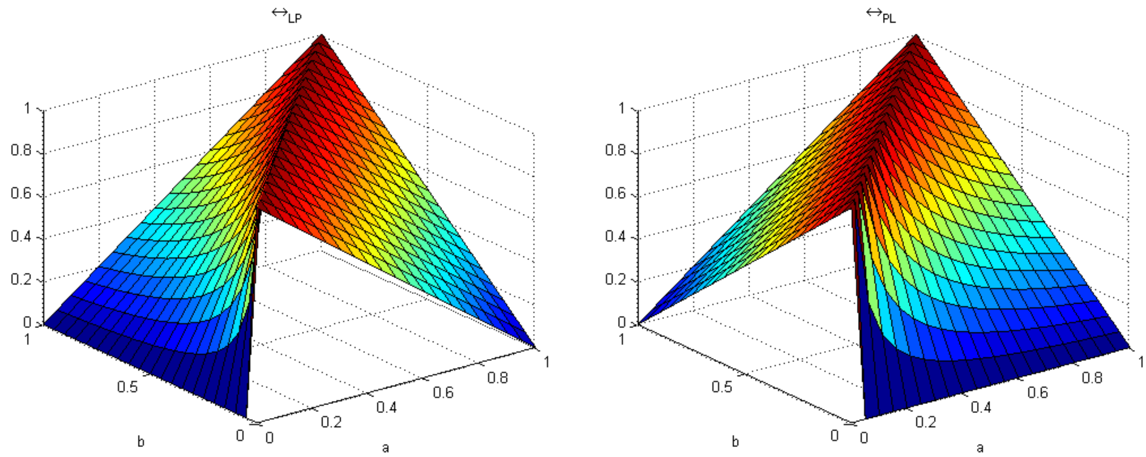

Figure 1. 3D plots of two asymmetric equivalences, $\leftrightarrow_{LP}$ and $\leftrightarrow_{PL}$.

Example 2. *If* $\to_L$ *and* $\to_G$ *are Łukasiewicz implication and Gödel implication, respectively, then* $\leftrightarrow_{GL}$ *is defined for all* $a, b \in [0,1]$ *by*

$$a \leftrightarrow_{GL} b = \begin{cases} 1 - b + a, & \text{if } a \leq b, \\ b, & \text{if } a > b, \end{cases} \tag{14}$$

and its conjugate asymmetric equivalence $\leftrightarrow_{LG}$ *by*

$$a \leftrightarrow_{LG} b = \begin{cases} a, & \text{if } a < b, \\ 1 - a + b, & \text{if } a \geq b. \end{cases} \tag{15}$$

Example 3. *If* $\to_G$ *and* $\to_P$ *are Gödel implication and product implication, respectively, then* $\leftrightarrow_{PG}$ *is defined for all* $a, b \in [0,1]$ *by*

$$a \leftrightarrow_{PG} b = \begin{cases} a, & \text{if } a \leq b, \\ b/a, & \text{if } a > b, \end{cases} \tag{16}$$

and its conjugate asymmetric equivalence $\leftrightarrow_{GP}$ *by*

$$a \leftrightarrow_{GP} b = \begin{cases} b, & \text{if } a \geq b, \\ a/b, & \text{if } a < b. \end{cases} \tag{17}$$

4. Quasi-Metrics Induced by Asymmetric Equivalences

Definition 4. *In reference [16], a quasi-metric on a set* U *is a function* $d : U \times U \to R^+$ *having the following properties: for all* $a, b, c \in U$

(i) $d(a,a) = 0$;
(ii) $d(a,b) = d(b,a) = 0 \quad \Rightarrow \quad a = b$;
(iii) $d(a,b) + d(b,c) \geq d(a,c)$.

The axiom (iii) is the triangle inequality. The axiom (ii) is called the separation axiom. The pair (U, d) is called a quasi-metric space. If d also satisfies the symmetry axiom, $d(a, b) = d(b, a)$, then d is a metric and (U, d) is called a metric space. For more information about quasi-metric spaces, see [16].

Theorem 3. *Let $\leftrightarrow_{ij}$ be an asymmetric equivalence and $\leftrightarrow_{ji}$ be its conjugate asymmetric equivalence. If*

$$d(a, b) = 1 - a \leftrightarrow_{ij} b \tag{18}$$

is a quasi-metric on $[0, 1]$, then $d'(a, b) = 1 - a \leftrightarrow_{ji} b$ also is a quasi-metric on $[0, 1]$,

Proof. It is easy to prove that d' satisfies the axioms (i) and (ii). Now, we prove that d' satisfies the triangle inequality.

Since d satisfies the triangle inequality, for any $a, b, c \in [0, 1]$,

$$1 - a \leftrightarrow_{ij} b + 1 - b \leftrightarrow_{ij} c \geq 1 - a \leftrightarrow_{ij} c. \tag{19}$$

From the concept of conjugate asymmetric equivalence in Formula (2), we obtain

$$1 - b \leftrightarrow_{ji} a + 1 - c \leftrightarrow_{ji} b \geq 1 - c \leftrightarrow_{ij} a. \tag{20}$$

Thus, $d'(c, b) + d'(b, a) \geq d'(c, a)$. In consideration of the arbitrariness of $a, b, andc$, d' satisfies the triangle inequality. □

Theorem 4. *Let $*_i, *_j$ be two continuous t-norms with with residuum $\rightarrow_i$ and $\rightarrow_j$, respectively. If both $*_i$ and $*_j$ are stronger than Łukasiewicz t-norm $*_L$, then $d(a, b) = 1 - a \leftrightarrow_{ij} b$ is a quasi-metric on $[0, 1]$.*

Proof. For any $a, b, c \in [0, 1]$.

(i) Since $a \rightarrow a = 1$, $\rightarrow \in \{\rightarrow_i, \rightarrow_j\}$, then $d(a, a) = 1 - a \rightarrow_i a \wedge a \rightarrow_j a = 0$.
(ii) If $d(a, b) = 1 - a \leftrightarrow_{ij} b = 1 - a \rightarrow_i b \wedge b \rightarrow_j a = 0$, then we have $a \rightarrow_i b = 1$, $b \rightarrow_j a = 1$, from Lemma 1(iii), $a \leq b, b \leq a$, thus $a = b$. Similarly, we obtain $d(b, a) = 0 \Rightarrow a = b$.
(iii) Now, we prove that d satisfies the triangle inequality.

From Lemma 1(v),

$$(a \rightarrow_i b) *_i (b \rightarrow_i c) \leq a \rightarrow_i c, \tag{21}$$
$$(b \rightarrow_j a) *_j (c \rightarrow_j b) \leq c \rightarrow_j a. \tag{22}$$

Since both $*_i$ and $*_j$ are stronger than Łukasiewicz t-norm $*_L$, hence

$$(a \rightarrow_i b) *_L (b \rightarrow_i c) \leq (a \rightarrow_i b) *_i (b \rightarrow_i c) \leq a \rightarrow_i c, \tag{23}$$
$$(b \rightarrow_j a) *_L (c \rightarrow_j b) \leq (b \rightarrow_j a) *_j (c \rightarrow_j b) \leq c \rightarrow_j a. \tag{24}$$

Then,

$$(a \rightarrow_i b) + (b \rightarrow_i c) \leq 1 + (a \rightarrow_i c), \tag{25}$$
$$(b \rightarrow_j a) + (c \rightarrow_j b) \leq 1 + (c \rightarrow_j a). \tag{26}$$

From Lemma 2(i), we obtain

$$(a \rightarrow_i b) \wedge (b \rightarrow_j a) + (b \rightarrow_i c) \wedge (c \rightarrow_j b) \leq 1 + (a \rightarrow_i c) \wedge (c \rightarrow_j a), \tag{27}$$

i.e.,

$$(a \leftrightarrow_{ij} b) + (b \leftrightarrow_{ij} c) \leq 1 + (a \leftrightarrow_{ij} c). \tag{28}$$

Letting 2 minus both sides of the inequality, we then obtain $d(a,b) + d(b,c) \geq d(a,c)$ immediately. □

The above theorem investigated the construction method of quasi-metrics based on asymmetric equivalences. A sufficient condition of d to be a metric on $[0,1]$ was proposed. In Ref. [15], $*_L \prec *_P \prec *_G$. Consequently, the following corollary shows some examples of quasi-metrics induced by asymmetric equivalences.

Corollary 1. *If* $\leftrightarrow_{ij} \in \{\leftrightarrow_{LG}, \leftrightarrow_{GL}, \leftrightarrow_{LP}, \leftrightarrow_{PL}, \leftrightarrow_{PG}, \leftrightarrow_{GP}\}$*, then* $d(a,b) = 1 - a \leftrightarrow_{ij} b$ *is a quasi-metric on* $[0,1]$.

Proof. It can be proven from $*_L \prec *_P \prec *_G$ and Theorem 4. □

5. Symmetrization of Asymmetric Equivalences

In this section, we discuss the symmetrization of asymmetric equivalences.

Theorem 5. *Let* $\rightarrow_i$ *and* $\rightarrow_j$ *be two R-implications induced by two left-continuous t-norms. If* $d(a,b) = 1 - a \leftrightarrow_{ij} b$ *is a quasi-metric on* $[0,1]$*, then*

$$D(a,b) = 1 - (a \leftrightarrow_{ij} b) \wedge (b \leftrightarrow_{ij} a) \tag{29}$$

is a metric on $[0,1]$.

Proof. It is easy to prove that D satisfies the axioms (i) and (ii).

Since, for any $a, b \in [0,1]$,

$$D(a,b) = 1 - (a \leftrightarrow_{ij} b) \wedge (b \leftrightarrow_{ij} a) = d(a,b) \vee d(b,a) = d(b,a) \vee d(a,b) = D(b,a), \tag{30}$$

then D satisfies the symmetry axiom.

Since d satisfies the triangle inequality, for any $a, b, c \in [0,1]$,

$$d(a,b) + d(b,c) \geq d(a,c). \tag{31}$$

From Theorem 3, $d'(a,b) = 1 - a \leftrightarrow_{ji} b$ also is a quasi-metric on $[0,1]$, for any $a, b, c \in [0,1]$,

$$d'(a,b) + d'(b,c) \geq d'(a,c). \tag{32}$$

Then, from Lemma 2(ii),

$$d(a,b) \vee d'(a,b) + d(b,c) \vee d'(b,c) \geq d(a,c) \vee d'(a,c). \tag{33}$$

Since, for any $a, b \in [0,1]$,

$$\begin{aligned} d(a,b) \vee d'(a,b) &= (1 - a \leftrightarrow_{ij} b) \vee (1 - a \leftrightarrow_{ji} b) \\ &= 1 - (a \leftrightarrow_{ij} b) \wedge (a \leftrightarrow_{ji} b) \\ &= 1 - (a \leftrightarrow_{ij} b) \wedge (b \leftrightarrow_{ij} a) \\ &= D(a,b). \end{aligned}$$

Moreover, we have $d(b,c) \vee d'(b,c) = D(b,c)$ and $d(a,c) \vee d'(a,c) = D(a,c)$. Thus, Formula (33) is $D(a,b) + D(b,c) \geq D(a,c)$. □

Theorem 6. *Let $\rightarrow_i$ and $\rightarrow_j$ be two R-implications induced by two left-continuous t-norms. If $\rightarrow_i$ is stronger than $\rightarrow_j$, then*

$$a \leftrightarrow_j b = (a \leftrightarrow_{ij} b) \wedge (b \leftrightarrow_{ij} a); \tag{34}$$

$$a \leftrightarrow_i b = (a \leftrightarrow_{ij} b) \vee (b \leftrightarrow_{ij} a). \tag{35}$$

where $a \leftrightarrow_k b = (a \rightarrow_k b) \wedge (b \rightarrow_k a)$, $k \in \{i, j\}$.

Proof.

Formula (3) can be proven from $\rightarrow_j \prec \rightarrow_i$ and the commutative law of $\wedge$.

Formula (4) can be proven from $\rightarrow_j \prec \rightarrow_i$ and distributivity of $\vee$ over $\wedge$. □

The above theorem investigated the method for symmetrization of asymmetric equivalences. Then, the following corollary shows some examples of symmetric equivalences using the symmetrisation methods above.

Corollary 2. *We have*

$$a \leftrightarrow_G b = (a \leftrightarrow_{LG} b) \wedge (b \leftrightarrow_{GL} a) = (a \leftrightarrow_{PG} b) \wedge (b \leftrightarrow_{GP} a); \tag{36}$$

$$a \leftrightarrow_P b = (a \leftrightarrow_{LP} b) \wedge (b \leftrightarrow_{PL} a) = (a \leftrightarrow_{PG} b) \vee (b \leftrightarrow_{GP} a); \tag{37}$$

$$a \leftrightarrow_L b = (a \leftrightarrow_{LG} b) \vee (b \leftrightarrow_{GL} a) = (a \leftrightarrow_{LP} b) \vee (b \leftrightarrow_{PL} a). \tag{38}$$

Proof. It can be proven from $\rightarrow_G \prec \rightarrow_P \prec \rightarrow_L$ and Theorem 6. □

6. Conclusions

In this paper, we introduced the operation of asymmetric equivalence that is interpreted by two different fuzzy implications. We used this operation to express other connectives, such as implication and negation. The quasi-metric structures induced by asymmetric equivalence, $d(a, b) = 1 - a \leftrightarrow_{ij} b$, were investigated. We proposed a sufficient condition of d to be a quasi-metric on $[0, 1]$ (see Theorem 4).

EQ-logic [8] is an attempt to develop fuzzy logic by starting with equivalence ($\sim$) instead of implication ($\rightarrow$). Then, $\sim$ is one of the basic operations and $\rightarrow$ is a derived operation. Since $\sim$ has the symmetry, we can call it symmetric equivalence. Unlike symmetric equivalence, asymmetric equivalence can derive two different implication operations. As we know, ŁΠ logic starts with two different implication operations: Łukasiewicz implication and product implication. Thus, it is interesting to develop fuzzy logic by starting with asymmetric equivalence. Theorem 1 and Example 1 gave expressions of other operations by asymmetric equivalence. Indeed, simplification of formal systems is an important topic of fuzzy logic [17,18]. Can we give simplified formal systems for fuzzy logic that contain fewer axioms and rules by using asymmetric equivalence operation? It will be meaningful to discuss further.

In Section 4, the constructions of quasi-metrics induced by asymmetric equivalences were investigated in detail. Quasi-metrics are common in control and theoretical computer science. Our results obtained in this section may be useful for some aspects of fuzzy theory and applications, such as upper and lower semicontinuity of fuzzy reasoning and stabilization of fuzzy control systems.

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