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Two Special Types of Curves in Lorentzian α-Sasakian 3-Manifolds

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Abstract: In this paper, we focus on the research and analysis of the geometric properties and symmetry of slant curves and contact magnetic curves in Lorentzian α-Sasakian 3-manifolds. To do this, we define the notion of Lorentzian cross product. From the perspectives of the Legendre and non-geodesic curves, we found the ratio relationship between the curvature and torsion of the slant curve and contact magnetic curve in the Lorentzian α-Sasakian 3-manifolds. Moreover, we utilized the property of the contact magnetic curve to characterize the manifold as Lorentzian α-Sasakian and to find the slant curve type of the Frenet contact magnetic curve. Furthermore, we found an example to verify the geometric properties of the slant curve and contact magnetic curve in the Lorentzian α-Sasakian 3-manifolds.

Keywords: Lorentzian cross product; slant curves; Lorentzian α-Sasakian manifold; magnetic curves

MSC: 53C25; 53B30

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Citation: Chen, X.; Liu, H. Two Special Types of Curves in Lorentzian α-Sasakian 3-Manifolds. *Symmetry* **2023**, *15*, 1077. https://doi.org/10.3390/sym15051077

Academic Editors: Juan Luis García Guirao, Yanlin Li and Tiehong Zhao

Received: 19 April 2023
Revised: 3 May 2023
Accepted: 10 May 2023
Published: 12 May 2023

1. Introduction

Sasakian manifolds are a natural generalization of Kähler manifolds to odd-dimensional manifolds. For (α, β) trans-Sasakian structures, if $\beta = 0$, that manifold will become an α-Sasakian manifolds. Sasakian manifolds appear as examples of α-Sasakian manifolds with $\alpha = 1$ and $\beta = 0$ in [1]. In 2005, Yildiz and Murathan introduced the notions of Lorentzian α-Sasakian manifolds [2]. Subsequently, many researchers started to study a class of Lorentzian α-Sasakian manifolds [3], Ricci solitons [4], Gauss–Bonnet theorem [5], and η-Ricci solitons [6] in Lorentzian α-Sasakian manifolds. Over the last few years, an increasing number of scholars have had an increasing interest in the geometric properties and symmetry of contact magnetic curves and slant curves. The notion of using slant curves as a generalization of Legendre curves was introduced in [7]. Abdul and Rajendra introduced slant null curves in [8]. Moreover, Inoguchi studied slant curves in normal almost contact metric 3-manifolds and magnetic curves in quasi-Sasakian 3-manifolds in [9,10]. Furthermore, Lee studied slant curves with CR structures in contact Lorentzian manifolds [11] and supported the usefulness of contact magnetic curves and slant curves in Lorentzian Sasakian 3-manifolds [12].

One motivation for this paper was our observation in [12]. That is, in Lorentzian Sasakian 3-manifolds, the ratio relationship between κ_c and $\tau_c - 1$ along a non-geodesic Frenet slant curve has been proven to be constant . In this paper, we try to derive the ratio relationship between curvature and torsion along a non-geodesic Frenet slant curve and along a non-geodesic Frenet contact magnetic curve in Lorentzian α-Sasakian 3-manifolds. Moreover, we try to find the type of slant curve of non-geodesic Frennet contact magnetic curve. To do this, we first define the notion of Lorentzian cross product on almost contact Lorentzian manifolds and derive the six properties of this Lorentzian cross product. Then, we prove that the ratio relationship between κ_c and $\tau_c - \alpha$ along a non-geodesic Frenet slant curve is constant, and the ratio relationship between κ_c and $\tau_c + \alpha$ along a non-geodesic Frenet contact magnetic curve is also constant, by using the method of Lorentzian cross product in Lorentzian α-Sasakian 3-manifolds. We find that the non-geodesic contact magnetic curve C is a slant pseudo-helix. For future research directions, we hope to do some related work on singularity theory and symmetry (see [13–20]).

The paper is organized in the following way: In Section 2, we recall some sufficient and necessary concepts that are the basis for our research, including some notations of slant curves, magnetic curves and trans-Sasakian structure of (α, β). In Section 3, we introduce the concept of Lorentzian α-Sasakian manifolds. Then, similar to the cross product in three-dimensional almost contact metric manifolds, we define the notion of Lorentzian cross on almost contact Lorentzian 3-manifolds. Along a non-geodesic Frenet slant curve in Lorentzian α-Sasakian 3-manifolds, by using the methods of Lorentzian cross product, we can prove that the ratio relationship between the κ_c and $\tau_c - \alpha$ is constant . For a null curve, the triad of the vector field $\{C', \xi_\alpha, \phi_\alpha C'\}$ is not used as the basis and there does not exist a non-geodesic null slant curve C in Lorentzian α-Sasakian 3-manifolds. In Section 4, we utilize the property of contact mangnetic curves to characterize the manifold as Lorentzian α-Sasakian. At the same time, we prove that the slant curve type of the non-geodesic Frennet contact magnetic curve is a pseudo-helix and prove that the ratio relationship between the κ_c and $\tau_c + \alpha$ is constant along a non-geodesic Frenet contact magnetic curve in Lorentzian α-Sasakian 3-manifolds. In Section 5, we first give a three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 \colon z > 0\}$ and verify that this manifold is a Lorentzian α-Sasakian 3-manifold, where the Reeb vector field ξ_α is F_3 and the contact form is $\eta_\alpha = \frac{1}{\alpha} dz$. Moreover, we prove that the Frenet slant curve C is geodesic and that there does not exist a non-geodesic Frenet slant in Lorentzian α-Sasakian 3-manifolds. Finally, we take a special curve $\gamma(s) = bs + c$ and derive the expression of contact magnetic curves in Lorentzian α-Sasakian 3-manifolds.

2. Preliminaries

A magnetic curve refers to the path followed by a charged particle moving under the influence of both a magnetic and electric field. A closed 2-form magnetic field F [21] on a semi-Riemannian manifold (M, g) satisfies

$$F(X_\alpha, Y_\alpha) = g(\Phi_\alpha(X_\alpha), Y_\alpha), \tag{1}$$

where $X_\alpha, Y_\alpha \in TM$ and Φ_α is a skew-symmetric (1, 1)-type tensor field, which represents the Lorentz force associated with F. The magnetic trajectories of F are curve C on M that satisfies the Lorentz equation

$$\nabla^L_{C'} C' = \Phi_\alpha(C'), \tag{2}$$

where ∇^L is the Levi–Civita connection associated with g. The geodesic of M naturally satisfies the Lorentz equation, that is $\nabla^L_{C'} C' = 0$. The skew-symmetry of the Lorentz force Φ_α gives rise to the property of magnetic curves C

$$\frac{d}{dt} g(C', C') = 2g(\Phi_\alpha(C'), C') = 0.$$

In other words, C has a constant speed curve $|C'| = v_0$. Particularly, if C is parameterized by the arc-length, the magnetic curve C is called a normal curve.

For a contact Lorentzian manifold, if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4 [22], a trans-Sasakian structure [23] is an almost contact metric structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ on M, where J is an almost complex structure on $M \times \mathbb{R}$ [1,24–26], defined by

$$J(X_\alpha, f\frac{d}{dt}) = (\phi_\alpha X_\alpha - f\xi_\alpha, \eta_\alpha(X_\alpha)\frac{d}{dt}),$$

where $X_\alpha \in TM$, f is a smooth function on $M \times \mathbb{R}$, and G is the product metric on $M \times \mathbb{R}$. If the almost complex structure J is integrable, then the contact Lorentzian manifold M is either normal or Sasakian. The satisfaction of certain conditions is necessary and sufficient for M to be considered normal. It is known that a contact Lorentzian manifold M is normal if and only if M satisfies

$$N_{\phi_\alpha}(X_\alpha, Y_\alpha) + 2d\eta_\alpha(X_\alpha, Y_\alpha)\xi_\alpha = 0,\ \forall X_\alpha, Y_\alpha \in TM,$$

where N_{ϕ_α} is the Nijenhuis torsion of ϕ_α. This may be expressed by the condition

$$(\nabla^L_{X_\alpha}\phi_\alpha)Y_\alpha = \alpha(g(X_\alpha, Y_\alpha)\xi_\alpha - \eta_\alpha(Y_\alpha)X_\alpha) + \beta(g(\phi_\alpha X_\alpha, Y_\alpha)\xi_\alpha - \eta_\alpha(Y_\alpha)\phi_\alpha X_\alpha), \quad (3)$$

where α and β are smooth functions on M, and we say that the trans-Sasakian structure [27,28] is type (α, β). For (3), it follows that

$$\nabla^L_{X_\alpha}\xi_\alpha = -\alpha\phi_\alpha X_\alpha + \beta(X_\alpha - \eta_\alpha(X_\alpha)\xi_\alpha), \quad (4)$$

$$(\nabla^L_{X_\alpha}\eta_\alpha)Y_\alpha = -\alpha g(\phi_\alpha X_\alpha, Y_\alpha) + \beta g(\phi_\alpha X_\alpha, \phi_\alpha Y_\alpha). \quad (5)$$

Definition 1 ([29]). *A trans-Sasakian structure of (α, β) is α-Sasakian if $\beta = 0$ and α is a non-zero constant.*

In this case, α becomes a constant. If $\alpha = 1$, then the α-Sasakian manifold is a Sasakian manifold.

3. Slant Curves of the Lorentzian α-Sasakian 3-Manifolds

The concept of slant curves, which extends the idea of Legendre curves, was presented in [7] for contact Riemannian 3-manifolds. Such as in the case of these manifolds, a curve is considered to be slant on a contact Lorentzian manifold if its tangent vector field maintains a constant angle with the Reeb vector field (i.e., $g(C', \xi_\alpha) = constant$). In particular, if $g(C', \xi_\alpha) = 0$, then C is a Legendre curve where the contact angle is $\frac{\pi}{2}$. On the other hand, the Reeb flow corresponds to a curve of contact angle 0.

3.1. Lorentzian α-Sasakian Manifold

A Lorentzian α-Sasakian manifold [2] is a connected (2n+1)-dimensional differentiable manifold equipped with a metric g, contact form η_α, 1-form ϕ_α, and vector field ξ_α satisfying certain conditions:

$$\phi_\alpha{}^2 = I + \eta_\alpha \otimes \xi_\alpha,\ \eta_\alpha(\xi_\alpha) = -1, \quad (6)$$

$$g(\phi_\alpha X_\alpha, \phi_\alpha Y_\alpha) = g(X_\alpha, Y_\alpha) + \eta_\alpha(X_\alpha)\eta_\alpha(Y_\alpha), \quad (7)$$

$$g(X_\alpha, \xi_\alpha) = \eta_\alpha(X_\alpha), \quad (8)$$

$$\eta_\alpha \circ \phi_\alpha = 0,\ \phi_\alpha\xi_\alpha = 0. \quad (9)$$

For all $X_\alpha, Y_\alpha \in TM$.

From (4) and (5) , a Lorentzian α-Sasakian manifold M satisfies the following conditions:

$$\nabla^L_{X_\alpha}\xi_\alpha = -\alpha\phi_\alpha X_\alpha, \quad (10)$$

$$(\nabla^L_{X_\alpha}\eta_\alpha)Y_\alpha = -\alpha g(\phi_\alpha X_\alpha, Y_\alpha), \quad (11)$$

where ∇^L represents the covariant difference operator on the Lorentzian metric g.

When the manifold is Lorentzian α-Sasakian, by (3) and $\beta = 0$, we can obtain the following:

Proposition 1. *If $(M^{2n+1}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is an almost contact Lorentzian manifold, then it is α-Sasakian if and only if*

$$(\nabla^L_{X_\alpha} \phi_\alpha) Y_\alpha = \alpha(g(X_\alpha, Y_\alpha)\xi_\alpha - \eta_\alpha(Y_\alpha) X_\alpha). \tag{12}$$

Applying the same parameters and computations as in [30], we obtain the following:

Proposition 2. *Let $(M^{2n+1}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a Lorentzian α-Sasakian manifold. Then,*

$$\nabla^L_{X_\alpha} \xi_\alpha = -\alpha\phi_\alpha X_\alpha - \alpha\phi_\alpha h X_\alpha, \tag{13}$$

where $h = \frac{1}{2}\pounds_{\xi_\alpha}\phi_\alpha$.

If the Lorentzian metric g has a Killing vector field ξ_α, that is, M^{2n+1} is a K-contact Lorentzian manifold, then

$$\nabla^L_{X_\alpha} \xi_\alpha = -\alpha\phi_\alpha X_\alpha. \tag{14}$$

3.2. Frenet Slant Curves

Let $C : I \to M^3$ be a unit speed curve in a Lorentzian 3-manifold with $g(C', C') = \epsilon_1 = \pm 1$. The casual character of C is given by the constant ϵ_1. Furthermore, the second and third casual characters of C are defined as $g(N_c, N_c) = \epsilon_2$ and $g(B_c, B_c) = \epsilon_3$, respectively. It then follows that the relation $\epsilon_1\epsilon_2 = -\epsilon_3$ holds.

If the casual character of a unit speed curve C is 1, it is said to be spacelike; if it is -1, it is said to be timelike. A Frenet curve is a unit speed curve C for which $g(C'', C'') \neq 0$. Along a Frenet curve C, there exists an orthonormal frame field $\{T_c = C', N_c, B_c\}$. The Frenet–Serret equations are given by:

$$\begin{cases} \nabla^L_{C'} T_c = \epsilon_2 \kappa_c N_c, \\ \nabla^L_{C'} N_c = -\epsilon_1 \kappa_c T_c - \epsilon_3 \tau_c B_c, \\ \nabla^L_{C'} B_c = \epsilon_2 \tau_c N_c. \end{cases} \tag{15}$$

where $\kappa_c = |\nabla^L_{C'} C'|$ is the geodesic curvature and τ_c is the geodesic torsion of C. The vector fields T_c, N_c, and B_c are referred to as the tangent vector field, principal vector field, and binormal vector field of the curve C, respectively.

A Frenet curve C is a geodesic if and only if its curvature κ_c is zero. A Frenet curve with constant geodesic curvature and zero geodesic torsion is a pseudo-circle. A Frenet curve C with constant geodesic curvature and geodesic torsion is a pseudo-helix.

In a three-dimensional almost contact metric manifold, the cross product is defined as

$$X_\alpha \Lambda Y_\alpha = -g(X_\alpha, \phi_\alpha Y_\alpha)\xi_\alpha - \eta_\alpha(Y_\alpha)\phi_\alpha X_\alpha + \eta_\alpha(X_\alpha)\phi_\alpha Y_\alpha,\ \forall X_\alpha, Y_\alpha \in TM.$$

Proposition 3. *Let $\{T_c, N_c, B_c\}$ be an orthonormal Frame fields in a Lorentzian 3-manifold. Then,*

$$T_c \Lambda_L N_c = \epsilon_3 B_c,\ N_c \Lambda_L B_c = \epsilon_1 T_c,\ B_c \Lambda_L T_c = \epsilon_2 N_c. \tag{16}$$

Therefore, in an almost contact Lorentzian 3-manifold, we define the Lorentzian cross product as follows:

Definition 2. *Let $(M^3, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost contact Lorentzian 3-manifold. The Lorentzian cross product Λ_L is defined as*

$$X_\alpha \Lambda_L Y_\alpha = -g(X_\alpha, \phi_\alpha Y_\alpha)\xi_\alpha - \eta_\alpha(Y_\alpha)\phi_\alpha X_\alpha + \eta_\alpha(X_\alpha)\phi_\alpha Y_\alpha,\ \forall X_\alpha, Y_\alpha \in TM. \tag{17}$$

The Lorentzian cross product Λ_L has the following properties:

Proposition 4. *The Lorentzian cross product Λ_L in an almost contact Lorentzian 3-manifold $(M^3, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ has the following properties for all $X_\alpha, Y_\alpha, Z_\alpha \in TM$:*
(1) The Lorentzian cross product is bilinear and anti-symmetric;
(2) $X_\alpha \Lambda_L Y_\alpha$ is perpendicular to both X_α and Y_α;
(3) $X_\alpha \Lambda_L \phi_\alpha Y_\alpha = -g(X_\alpha, Y_\alpha)\xi_\alpha + \eta_\alpha(X_\alpha)Y_\alpha$;
(4) $\phi_\alpha X_\alpha = -\xi_\alpha \Lambda_L X_\alpha$;
(5) Define a mixed product by $det(X_\alpha, Y_\alpha, Z_\alpha) = g(X_\alpha \Lambda_L Y_\alpha, Z_\alpha)$, we have

$$det(X_\alpha, Y_\alpha, Z_\alpha) = -g(X_\alpha, \phi_\alpha Y_\alpha)\eta_\alpha(Z_\alpha) - g(Y_\alpha, \phi_\alpha Z_\alpha)\eta_\alpha(X_\alpha) - g(Z_\alpha, \phi_\alpha X_\alpha)\eta_\alpha(Y_\alpha)$$

and

$$det(X_\alpha, Y_\alpha, Z_\alpha) = det(Y_\alpha, Z_\alpha, X_\alpha) = det(Z_\alpha, X_\alpha, Y_\alpha);$$

(6) $g(X_\alpha, \phi_\alpha Y_\alpha)Z_\alpha + g(Y_\alpha, \phi_\alpha Z_\alpha)X_\alpha + g(Z_\alpha, \phi_\alpha X_\alpha)Y_\alpha = (X_\alpha, Y_\alpha, Z_\alpha)\xi_\alpha$.

Proof. (We can prove it in a similar way [31])
Property (1) and Property (2) are trivial.
For Property (3), by using (6), (8), and (17), we obtain

$$\begin{aligned} X_\alpha \Lambda_L \phi_\alpha Y_\alpha &= -g(X_\alpha, {\phi_\alpha}^2 Y_\alpha)\xi_\alpha - \eta_\alpha(\phi_\alpha Y_\alpha)\phi_\alpha X_\alpha + \eta_\alpha(X_\alpha){\phi_\alpha}^2 Y_\alpha \\ &= -g(X_\alpha, Y_\alpha + \eta_\alpha(Y_\alpha)\xi_\alpha)\xi_\alpha - \eta_\alpha(\phi_\alpha Y_\alpha)\phi_\alpha X_\alpha + \eta_\alpha(X_\alpha)(Y_\alpha + \eta_\alpha(Y_\alpha)\xi_\alpha) \\ &= -g(X_\alpha, Y_\alpha)\xi_\alpha + \eta_\alpha(X_\alpha)Y_\alpha. \end{aligned}$$

For Property (4), by using (6), (9), and (17), we have

$$\begin{aligned} -\xi_\alpha \Lambda_L X_\alpha &= -g(-\xi_\alpha, \phi_\alpha X_\alpha)\xi_\alpha - \eta_\alpha(X_\alpha)\phi_\alpha(-\xi_\alpha) + \eta_\alpha(-\xi_\alpha)\phi_\alpha X_\alpha \\ &= \phi_\alpha X_\alpha. \end{aligned}$$

For Property (5), from (8), (17), and ϕ_α being a skew-symmetric (1, 1)-type tensor field, we obtain

$$\begin{aligned} g(X_\alpha \Lambda_L Y_\alpha, Z_\alpha) &= g(-g(X_\alpha, \phi_\alpha Y_\alpha)\xi_\alpha - \eta_\alpha(Y_\alpha)\phi_\alpha X_\alpha + \eta_\alpha(X_\alpha)\phi_\alpha Y_\alpha, Z_\alpha) \\ &= -g(X_\alpha, \phi_\alpha Y_\alpha)g(\xi_\alpha, Z_\alpha) - \eta_\alpha(Y_\alpha)g(\phi_\alpha X_\alpha, Z_\alpha) + \eta_\alpha(X_\alpha)g(\phi_\alpha Y_\alpha, Z_\alpha) \\ &= -g(X_\alpha, \phi_\alpha Y_\alpha)\eta_\alpha(Z_\alpha) - g(Y_\alpha, \phi_\alpha Z_\alpha)\eta_\alpha(X_\alpha) - g(Z_\alpha, \phi_\alpha X_\alpha)\eta_\alpha(Y_\alpha). \end{aligned}$$

Moreover, one can easily verify that

$$det(X_\alpha, Y_\alpha, Z_\alpha) = det(Y_\alpha, Z_\alpha, X_\alpha) = det(Z_\alpha, X_\alpha, Y_\alpha).$$

Property (6) is easily obtained from Property (5). □

Proposition 5. *Let $(M^3, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a Lorentzian α-Sasakian 3-manifold. Then, we have*

$$\nabla^L_{Z_\alpha}(X_\alpha \Lambda_L Y_\alpha) = (\nabla^L_{Z_\alpha} X_\alpha)\Lambda_L Y_\alpha + X_\alpha \Lambda_L(\nabla^L_{Z_\alpha} Y_\alpha), \tag{18}$$

for all $X_\alpha, Y_\alpha, Z_\alpha \in TM$.

Proof. By $\eta_\alpha(X_\alpha) = g(X_\alpha, \xi_\alpha)$, $\eta_\alpha(Y_\alpha) = g(Y_\alpha, \xi_\alpha)$ and (17), we have

$$\begin{aligned}
\nabla^L_{Z_\alpha}(X_\alpha \Lambda_L Y_\alpha) =& \nabla^L_{Z_\alpha}(-g(X_\alpha, \phi_\alpha Y_\alpha)\xi_\alpha - \eta_\alpha(Y_\alpha)\phi_\alpha X_\alpha + \eta_\alpha(X_\alpha)\phi_\alpha Y_\alpha) \\
=& -g(\nabla^L_{Z_\alpha} X_\alpha, \phi_\alpha Y_\alpha)\xi_\alpha - g(X_\alpha, (\nabla^L_{Z_\alpha}\phi_\alpha)Y_\alpha)\xi_\alpha - g(X_\alpha, \phi_\alpha(\nabla^L_{Z_\alpha} Y_\alpha))\xi_\alpha \\
& - g(X_\alpha, \phi_\alpha Y_\alpha)\nabla^L_{Z_\alpha}\xi_\alpha - g(\nabla^L_{Z_\alpha} Y_\alpha, \xi_\alpha)\phi_\alpha X_\alpha - g(Y_\alpha, \nabla^L_{Z_\alpha}\xi_\alpha)\phi_\alpha X_\alpha \\
& - \eta_\alpha(Y_\alpha)(\nabla^L_{Z_\alpha}\phi_\alpha)X_\alpha - \eta_\alpha(Y_\alpha)\phi_\alpha(\nabla^L_{Z_\alpha} X_\alpha) + g(\nabla^L_{Z_\alpha} X_\alpha, \xi_\alpha)\phi_\alpha Y_\alpha \\
& + g(X_\alpha, \nabla^L_{Z_\alpha}\xi_\alpha)\phi_\alpha Y_\alpha + \eta_\alpha(X_\alpha)(\nabla^L_{Z_\alpha}\phi_\alpha)Y_\alpha + \eta_\alpha(X_\alpha)\phi_\alpha(\nabla^L_{Z_\alpha} Y_\alpha) \\
=& (\nabla^L_{Z_\alpha} X_\alpha)\Lambda_L Y_\alpha + X_\alpha \Lambda_L(\nabla^L_{Z_\alpha} Y_\alpha) + A(X_\alpha, Y_\alpha, Z_\alpha),
\end{aligned}$$

where

$$(\nabla^L_{Z_\alpha} X_\alpha)\Lambda_L Y_\alpha = -g(\nabla^L_{Z_\alpha} X_\alpha, \phi_\alpha Y_\alpha)\xi_\alpha - \eta_\alpha(Y_\alpha)\phi_\alpha(\nabla^L_{Z_\alpha} X_\alpha) + \eta_\alpha(\nabla^L_{Z_\alpha} X_\alpha)\phi_\alpha Y_\alpha,$$

$$\begin{aligned}
X_\alpha \Lambda_L(\nabla^L_{Z_\alpha} Y_\alpha) =& -g(X_\alpha, \phi_\alpha(\nabla^L_{Z_\alpha} Y_\alpha))\xi_\alpha - \eta_\alpha(\nabla^L_{Z_\alpha} Y_\alpha)\phi_\alpha X_\alpha + \eta_\alpha(X_\alpha)\phi_\alpha(\nabla^L_{Z_\alpha} Y_\alpha), \\
A(X_\alpha, Y_\alpha, Z_\alpha) =& -g(X_\alpha, (\nabla^L_{Z_\alpha}\phi_\alpha)Y_\alpha)\xi_\alpha - g(X_\alpha, \phi_\alpha Y_\alpha)\nabla^L_{Z_\alpha}\xi_\alpha - g(Y_\alpha, \nabla^L_{Z_\alpha}\xi_\alpha)\phi_\alpha X_\alpha \\
& - \eta_\alpha(Y_\alpha)(\nabla^L_{Z_\alpha}\phi_\alpha)X_\alpha + g(X_\alpha, \nabla^L_{Z_\alpha}\xi_\alpha)\phi_\alpha Y_\alpha + \eta_\alpha(X_\alpha)(\nabla^L_{Z_\alpha}\phi_\alpha)Y_\alpha.
\end{aligned}$$

As M^3 is a Lorentzian α-Sasakian 3-manifold, it satisfies (12) and (14). Hence, we have

$$A(X_\alpha, Y_\alpha, Z_\alpha) = \alpha[g(X_\alpha, \phi_\alpha Y_\alpha)\phi_\alpha Z_\alpha + g(Y_\alpha, \phi_\alpha Z_\alpha)\phi_\alpha X_\alpha + g(Z_\alpha, \phi_\alpha X_\alpha)\phi_\alpha Y_\alpha].$$

Using Property (6) of Proposition 4 and (9), we have $A(X_\alpha, Y_\alpha, Z_\alpha) = 0$. Hence, we obtain (18). □

The Reeb vector field ξ_α is defined as

$$\begin{aligned}
\xi_\alpha =& \epsilon_1 g(\xi_\alpha, T_c)T_c + \epsilon_2 g(\xi_\alpha, N_c)N_c + \epsilon_3 g(\xi_\alpha, B_c)B_c \\
=& \epsilon_1 \eta_\alpha(T_c)T_c + \epsilon_2 \eta_\alpha(N_c)N_c + \epsilon_3 \eta_\alpha(B_c)B_c.
\end{aligned} \tag{19}$$

From Property (4) of Proposition 4, Proposition 3, and (19), we have the following:

Proposition 6. *For a Frenet curve C in an almost contact Lorentzian α-Sasakian 3-manifold M^3, we have*

$$\begin{aligned}
\phi_\alpha T_c =& \epsilon_2\epsilon_3(\eta_\alpha(N_c)B_c - \eta_\alpha(B_c)N_c), \\
\phi_\alpha N_c =& \epsilon_1\epsilon_3(\eta_\alpha(B_c)T_c - \eta_\alpha(T_c)B_c), \\
\phi_\alpha B_c =& \epsilon_1\epsilon_2(\eta_\alpha(T_c)N_c - \eta_\alpha(N_c)T_c).
\end{aligned}$$

By using (13) and (15), we find that differentiating $\eta_\alpha(T_c)$, $\eta_\alpha(N_C)$, and $\eta_\alpha(B_c)$ along a Frenet curve C leads to the following:

$$\begin{aligned}
\eta_\alpha(T_c)' =& \epsilon_2\kappa_c\eta_\alpha(N_c) - \alpha g(T_c, \phi_\alpha h T_c), \\
\eta_\alpha(N_C)' =& -\epsilon_1\kappa_c\eta_\alpha(T_c) - \epsilon_3(\tau_c - \alpha)\eta_\alpha(B_c) - \alpha g(N_c, \phi_\alpha h T_c), \\
\eta_\alpha(B_c)' =& \epsilon_2(\tau_c - \alpha)\eta_\alpha(N_c) - \alpha g(B_c, \phi_\alpha h T_c).
\end{aligned}$$

When M^3 is a Lorentzian α-Sasakian 3-manifold, then

$$\eta_\alpha(T_c)' = \epsilon_2\kappa_c\eta_\alpha(N_c), \tag{20}$$

$$\eta_\alpha(N_c)' = -\epsilon_1\kappa_c\eta_\alpha(T_c) - \epsilon_3(\tau_c - \alpha)\eta_\alpha(B_c), \tag{21}$$

$$\eta_\alpha(B_c)' = \epsilon_2(\tau_c - \alpha)\eta_\alpha(N_c). \tag{22}$$

From (20), if C is a geodesic curve, i.e., $\kappa_c = 0$, in a Lorentzian α-Sasakian 3-manifold, then C is naturally a slant curve. Considering a non-geodesic curve C, we have the following:

Proposition 7. *In a Lorentzian α-Sasakian 3-manifold, a non-geodesic Frenet curve C is considered a slant curve if and only if $\eta_\alpha(N_c) = 0$.*

Proof. From (20) and C being a non-geodesic Frenet curve, we obtain $\kappa_c \neq 0$. On the one hand, if C is a slant curve, we have $\eta_\alpha(C') = \eta_\alpha(T_c) = constant$. Therefore, if $\eta_\alpha(T_c)' = 0$, we obtain $\eta_\alpha(N_c) = 0$. On the other hand, if $\eta_\alpha(N_c) = 0$, we obtain $\eta_\alpha(T_c)' = 0$; therefore $\eta_\alpha(T_c) = constant$. Hence, the non-geodesic Frenet curve C is a slant curve. □

From (20), (22), and Proposition 7, we obtain that $\eta_\alpha(N_c)$ and $\eta_\alpha(B_c)$ are constant. Thus, by using (21), we obtain the following:

Theorem 1. *The ratio relationship between the κ_c and $\tau_c - \alpha$ along a non-geodesic Frenet slant curve is a constant in Lorentzian α-Sasakian 3-manifold M^3.*

We will now consider a Legendre curve C as a spacelike curve with a spacelike normal vector. For the Legendre curve C, $\eta_\alpha(C') = \eta_\alpha(T_c) = 0$, $\eta_\alpha(N_c) = 0$, and $\eta_\alpha(B_c)$ is a constant. Hence, by (21), we have the following:

Corollary 1. *The torsion of a Legendre curve is α in a Lorentzian α-Sasakian 3-manifold M^3.*

It can be observed that the ratio relationship between κ_c and $\tau_c - \alpha$ along a non-geodesic Frenet slant curve is a constant containing a Legendre curve.

3.3. Null Slant Curves

Null curves exhibit distinct properties from spacelike and timelike curves, and their general theory is developed in [32,33]. These curves find significant applications in general relativity.

Suppose a regular curve C is a null(lightlike) curve on M^3, i.e., at each point x of C, we have

$$g(C', C') = 0, \ C' \neq 0.$$

The general Frenet frame along C in a Lorentzian α-Sasakian 3-manifold M^3 is represented by $F = \{T_c = C', N_c, W_c\}$, and it is determined by:

$$g(T_c, N_c) = g(W_c, W_c) = 1, \ g(T_c, T_c) = g(N_c, N_c) = g(N_c, W_c) = g(T_c, W_c) = 0. \tag{23}$$

We can easily show that C is geodesic for a null Legendre curve C. Therefore, we assume that C is non-geodesic for the following research .

The general Frenet equations with respect to the frame F and the covariant derivative ∇^L on M^3 are known from [8]

$$\begin{aligned} \nabla^L_{T_c} T_c &= hT_c + \kappa_c W_c, \\ \nabla^L_{T_c} N_c &= -hN_c + \tau_c W_c, \\ \nabla^L_{T_c} W_c &= -\tau_c T_c - \kappa_c N_c, \end{aligned} \tag{24}$$

where h is a smooth function, κ_c is the geodesic curvature of C, and τ_c is the torsion of C. There is known to exist a parameter q, called the parameter of a distinguished parameter, for which the function h vanishes in (24). The pair $(C(q), F)$, where F is a Frenet frame along C with respect to a distinguished parameter q, is called a frame null curve (see [33]). Generally, $(C(q), F)$ is not unique as it depends on the distribution of q and the screen. Therefore, under Lorentzian transformations, we seek a Frenet frame that has the minimum number of curvature functions that are invariant. Such a frame is called the Cartan Frenet frame for the null curve C of the Frenet frame. In [33], it is shown that if the null curve $C(q)$ is non-geodesic, such that the following condition for $C'' = \frac{d}{dq}C'$ holds,

$$g(C'', C'') = \kappa_c = 1,$$

then there only exists one Cartan Frenet frame F that satisfies the following Frenet equations:

$$\begin{aligned} \nabla^L_{T_c} T_c &= W_c, \\ \nabla^L_{T_c} N_c &= \tau_c W_c, \\ \nabla^L_{T_c} W_c &= -\tau_c T_c - N_c. \end{aligned} \tag{25}$$

Lemma 1. *Suppose C is a null curve on a three-dimensional almost contact Lorentzian α-Sasakian 3-manifold. It follows that the triad of vector fields $\{C', \xi_\alpha, \phi_\alpha C'\}$ cannot be considered as a basis of $T_x M^3$ at $x \in C$.*

Proof. From (6) and (23) at each point x of C, we find that ξ_α is a timelike vector field and C' is a lightlike vector field. Hence, it is easy to verify that ξ_α and C' are linearly independent vector fields along C. If we assume $\phi_\alpha C'$ belongs to the plane $H = span\{\xi_\alpha, C'\}$, in the case of functions m and n, we have $\phi_\alpha C' = m\xi_\alpha + nC'$. We also obtain $C' + \eta_\alpha(C')\xi_\alpha = mn\xi_\alpha + n^2 C'$ by applying ϕ_α to both sides of this equality, which implies $n^2 = 1$. Thus, $\{C', \xi_\alpha, \phi_\alpha C'\}$ are linear vector fields along C, which confirms our assertion. □

In summary, in Lorentzian α-Sasakian 3-manifolds, it is impossible for a non-geodesic null slant curve C to exist.

4. Contact Magnetic Curves

The Reeb vector field ξ_α is Killing, in a Lorentzian α-Sasakian 3-manifold. Hence, the 2-form Φ_α is $d\eta_\alpha$, i.e., $d\eta_\alpha(X_\alpha, Y_\alpha) = g(X_\alpha, \phi_\alpha Y_\alpha)$, for all $X_\alpha, Y_\alpha \in TM$.

Let $C : I \to M^3$ be a smooth curve on a contact Lorentzian manifold $(M^3, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. Then, we define a magnetic field on M^3 by

$$F_{\xi_\alpha, q}(X_\alpha, Y_\alpha) = -q d\eta_\alpha(X_\alpha, Y_\alpha), \tag{26}$$

where $X_\alpha, Y_\alpha \in TM$, q is a nonzero constant, and $F_{\xi_\alpha, q}$ represents the contact magnetic field with strength q.

Using (1), (26), and $d\eta_\alpha(X_\alpha, Y_\alpha) = g(X_\alpha, \phi_\alpha Y_\alpha)$, we obtain $\Phi_\alpha(X_\alpha) = q\phi_\alpha X_\alpha$. Hence, from (2), the Lorentz equation is

$$\nabla^L_{C'} C' = q\phi_\alpha C'. \tag{27}$$

This is the geodesic generalized equation under the arc-length parametrization, i.e., $\nabla^L_{C'} C' = 0$. For $q = 0$, it can be observed that the contact magnetic field vanishes identically and the magnetic curves on M^3 are indeed geodesic. These solutions satisfying Equation (27) are referred to as contact magnetic curves or trajectories.

In contact Lorentzain 3-manifolds, by using (13) and (27), differentiating $g(\xi_\alpha, C')$ along a contact magnetic curve C is

$$\begin{aligned} \frac{d}{dt} g(\xi_\alpha, C') &= g(\nabla^L_{C'} \xi_\alpha, C') + g(\xi_\alpha, \nabla^L_{C'} C') \\ &= g(-\alpha\phi_\alpha C' - \alpha\phi_\alpha h C', C') + g(\xi_\alpha, q\phi_\alpha C') \\ &= -\alpha g(\phi_\alpha h C', C'). \end{aligned}$$

Therefore, we utilize the property of contact magnetic curves to characterize the conditions of contact magnetic curves and slant curves in Lorentzian 3-manifolds in the following theorem.

Theorem 2. *A contact magnetic curve C is a slant curve if and only if M^3 is α-Sasakian in a contact Lorentzian 3-mainfold M^3.*

Proof. On the one hand, if C is a slant curve, $g(C', \xi_\alpha) = constant$, so $\frac{d}{dt} g(\xi_\alpha, C') = 0$ and M is α-Sasakian. On the other hand, if M is α-Sasakian, the manifold is K-contact, so $h = 0$ and $g(C', \xi_\alpha) = constant$. Hence, C is a slant curve. □

Following this along a non-geodesic Frenet contact magnetic curve C, we find the curvature κ_c and torsion τ_c, and we assume that $\eta_\alpha(C') = \eta_\alpha(T_c) = a$, for a constant a. Then, using (7), (15), and (27), we obtain

$$\epsilon_2 {\kappa_c}^2 = q^2 g(\phi_\alpha C', \phi_\alpha C') = q^2(\epsilon_1 + a^2).$$

According to the assumptions and deductions made, it can be concluded that C has constant curvature

$$\kappa_c = |q|\sqrt{\epsilon_2(\epsilon_1 + a^2)}, \tag{28}$$

and from (15), (27), and (28), the principal normal vector field N_c is

$$N_c = \frac{\delta\epsilon_2}{\sqrt{\epsilon_2(\epsilon_1 + a^2)}} \phi_\alpha C', \tag{29}$$

where $\delta = \frac{q}{|q|}$.

By utilizing Proposition 3, (17), and (29), the binormal B_c is calculated as

$$\begin{aligned} \epsilon_3 B_c &= T_c \Lambda_L N_c \\ &= C' \Lambda_L \Big(\frac{\delta\epsilon_2}{\sqrt{\epsilon_2(\epsilon_1 + a^2)}} \phi_\alpha C' \Big) \\ &= \frac{\delta\epsilon_2}{\sqrt{\epsilon_2(\epsilon_1 + a^2)}} (-\epsilon_1 \xi_\alpha + aC'). \end{aligned}$$

Differentiating the binormal vector field B_c, we obtain

$$\begin{aligned} \nabla^L_{C'} B_c &= \frac{\delta\epsilon_2\epsilon_3}{\sqrt{\epsilon_2(\epsilon_1 + a^2)}} \nabla^L_{C'}(-\epsilon_1 \xi_\alpha + aC') \\ &= \frac{\delta\epsilon_2\epsilon_3}{\sqrt{\epsilon_2(\epsilon_1 + a^2)}} \nabla^L_{C'}(\alpha\epsilon_1 + aq)\phi_\alpha C'. \end{aligned} \tag{30}$$

On another aspect of the binormal vector field, by (15), we obtain

$$\nabla^L_{C'} B_c = \epsilon_2 \tau_c N_c = \tau_c \frac{\delta\phi_\alpha C'}{\sqrt{\epsilon_2(\epsilon_1 + a^2)}}. \tag{31}$$

From (30) and (31), as $\epsilon_1\epsilon_2\epsilon_3 = -1$, we obtain

$$\tau_c = -\alpha - \epsilon_1 a q.$$

Furthermore, if C is a non-geodesic curve, then

$$\frac{\tau_c + \alpha}{\kappa_c} = -\frac{\delta\epsilon_1 a}{\sqrt{\epsilon_2(\epsilon_1 + a^2)}}.$$

Therefore, we obtain the following:

Theorem 3. *Let C be a non-geodesic Frenet curve in a Lorentzian α-Sasakian 3-manifold. If C is a contact magnetic curve and α is a constant, then it is a slant pseudo-helix with curvature $\kappa_c = |q|\sqrt{\epsilon_2(\epsilon_1 + a^2)}$ and $\tau_c = -\alpha - \epsilon_1 aq$. Moreover, the ratio of κ_c and $\tau_c + \alpha$ is a constant along a non-geodesic Frenet contact magnetic curve.*

Assuming a Legendre curve is a spacelike curve with a spacelike normal vector field and $\eta_\alpha(C') = a = 0$, we assume that C is a Legendre curve and we can conclude that:

Corollary 2. *If a non-geodesic Legendre curve C is a contact magnetic curve in a Lorentzian α-Sasakian 3-manifold, then it is a Legendre pseudo-helix with curvature $\kappa_c = |q|$ and torsion $\tau_c = -\alpha$.*

Using Equation (28) for the geodesic curvature, if $\epsilon_1 = 1$, then $\eta_\alpha(C') = a$ and $1 \leq 1 + a^2$, and we have $\epsilon_2 = 1$. As $\epsilon_1\epsilon_2\epsilon_3 = -1$, we can deduce that $\epsilon_3 = -1$. If $\epsilon_1 = -1$, then $\eta_\alpha(C') = a = coshx_0$, as C is a geodesic for $a = coshx_0 = 1$. Assuming C is a non-geodesic curve, we have $a^2 > 0$, which implies that $-1 + a^2 > 0$, and we can deduce that $\epsilon_2 = \epsilon_3 = 1$. Therefore, we obtain the following:

Theorem 4. *If a non-geodesic Frenet curve C is a contact magnetic curve in a Lorentzian α-Sasakian 3-manifold M^3, then C is one of the following:*
(1) a spacelike curve with a spacelike normal vector field or
(2) a timelike curve.

Moreover, we have the following content:

Corollary 3. *There is no spacelike curve with a timelike normal vector field in Lorentzian α-Sasakian 3-manifold M^3 if C is a non-geodesic Frenet curve.*

5. Example of Lorentzian α-Sasakian 3-Manifolds

In this section, we present an example to validate our results.

Example 1. *We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ in [6], where (x, y, z) are the standard coordinates of $\mathbb{R}^3$. Let*

$$F_1 = e^{-z}\frac{\partial}{\partial y},\ F_2 = e^{-z}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}),\ F_3 = \alpha\frac{\partial}{\partial z},$$

where these vector fields are linearly independent at each point of M, and we take the contact form $\eta_\alpha = \frac{1}{\alpha}dz$ and the Reeb vector field of η_α is $\xi_\alpha = F_3 = \alpha\frac{\partial}{\partial z}$.

Let g be the Lorentzian metric

$$g = e^{2z}dx^2 + e^{2z}(-dx + dy)^2 - \frac{1}{\alpha^2}dz^2$$

and

$$g(F_1, F_1) = g(F_2, F_2) = 1,\ g(F_3, F_3) = -1,\ g(F_1, F_2) = g(F_1, F_3) = g(F_2, F_3) = 0.$$

Then, we have

$$[F_1, F_2] = 0,\ [F_1, F_3] = \alpha F_1,\ [F_2, F_3] = \alpha F_2.$$

The (1,1)-type field ϕ_α is defined by

$$\phi_\alpha F_1 = -F_1,\ \phi_\alpha F_2 = -F_2,\ \phi_\alpha F_3 = 0. \tag{32}$$

By applying linearity of ϕ_α and g, we have

$$\eta_\alpha(\xi_\alpha) = g(\xi_\alpha, \xi_\alpha) = -1, {\phi_\alpha}^2 X_\alpha = X_\alpha + \eta_\alpha(X_\alpha)\xi_\alpha, g(\phi_\alpha X_\alpha, \phi_\alpha Y_\alpha) = g(X_\alpha, Y_\alpha) + \eta_\alpha(X_\alpha)\eta_\alpha(Y_\alpha)$$

for all $X_\alpha, Y_\alpha \in TM$. The Levi–Civita connection ∇^L of M is

$$\begin{aligned}
&\nabla^L_{F_1}F_1 = \alpha F_3,\ \nabla^L_{F_1}F_2 = 0,\ \nabla^L_{F_1}F_3 = \alpha F_1,\\
&\nabla^L_{F_2}F_1 = 0,\ \nabla^L_{F_2}F_2 = \alpha F_3,\ \nabla^L_{F_2}F_3 = \alpha F_2,\\
&\nabla^L_{F_3}F_1 = 0,\ \nabla^L_{F_3}F_2 = 0,\ \nabla^L_{F_3}F_3 = 0.
\end{aligned} \tag{33}$$

The contact form η_α satisfies $d\eta_\alpha(X_\alpha, Y_\alpha) = g(X_\alpha, \phi_\alpha Y_\alpha)$. Moreover, the structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is α-Sasakian.

From the above results, we use the curvature tensor $R(X_\alpha, Y_\alpha)Z_\alpha = \alpha^2[g(Y_\alpha, Z_\alpha)X_\alpha - g(X_\alpha, Z_\alpha)Y_\alpha]$. Then, obtaining the components of the curvature tensor is straightforward, as follows:

$$\begin{aligned}
&R(F_1, F_2)F_1 = -\alpha^2 F_2,\ R(F_1, F_3)F_1 = -\alpha^2 F_3,\ R(F_2, F_3)F_1 = 0,\\
&R(F_1, F_2)F_2 = \alpha^2 F_1,\ R(F_1, F_3)F_2 = 0,\ R(F_2, F_3)F_2 = -\alpha^2 F_3,\\
&R(F_1, F_2)F_3 = 0,\ R(F_1, F_3)F_3 = -\alpha^2 F_1,\ R(F_2, F_3)F_3 = -\alpha^2 F_2.
\end{aligned}$$

The sectional curvature [25] is given by

$$\mathcal{K}(\xi_\alpha, F_i) = -R(\xi_\alpha, F_i, \xi_\alpha, F_i) = 1, for\ i = 1, 2$$

and

$$\mathcal{K}(F_1, F_2) = R(F_1, F_2, F_1, F_2) = -\alpha^2.$$

Thus, we see that the Lorentzian α-Sasakian 3-manifold M^3 has constant holomorphic sectional curvature $\mu = -\alpha^2$.

Suppose C is a Frenet slant curve in Lorentzian α-Sasakian 3-manifolds that is parameterized by arc-length. In this case, the tangent vector field takes on the following form:

$$T = C' = \sqrt{\epsilon_1 + a^2} cos\gamma F_1 + \sqrt{\epsilon_1 + a^2} sin\gamma F_2 + aF_3, \tag{34}$$

where $a = constant$, $\gamma = \gamma(s)$. Using (33), we obtain

$$\nabla^L_{C'}C' = \sqrt{\epsilon_1 + a^2}[(-\gamma' sin\gamma + a\alpha cos\gamma)F_1 + (\gamma' cos\gamma + a\alpha sin\gamma)F_2 + \alpha\sqrt{\epsilon_1 + a^2}F_3]. \tag{35}$$

On the one hand, from (27) and (32), we have $\kappa_c = 0$, that is, C is a geodesic curve; in a Lorentzian α-Sasakian 3-manifold, C is naturally a slant curve. From (35), (27), and (32), there does not exist a non-geodesic slant Frenet curve in M.

Let $C(s) = (x(s), y(s), z(s))$ be a curve in Lorentzian α-Sasakian manifolds. Then, the tangent vector field C' of C is

$$C' = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z}.$$

By utilizing the relation:

$$\frac{\partial}{\partial x} = e^z(F_2 - F_1), \frac{\partial}{\partial y} = e^z F_1, \frac{\partial}{\partial z} = \frac{1}{\alpha}F_3,$$

From (34), the system of differential equations for the slant curve C in M is

$$\begin{aligned} \frac{dx}{ds}(s) &= \sqrt{\epsilon_1 + a^2} sin\gamma e^{-z}, \\ \frac{dy}{ds}(s) &= \sqrt{\epsilon_1 + a^2}(sin\gamma + cos\gamma)e^{-z}, \\ \frac{dz}{ds}(s) &= a\alpha. \end{aligned} \tag{36}$$

In this particular case, take a special curve $\gamma(s) = bs + c$, where $b, c \in \mathbb{R}$. Suppose $C : I \to M$ is a non-geodesic curve that is parameterized by arc-length s in the Lorentzian α-Sasakian 3-manifold. If C is a contact magnetic curve, then the parametric equations of C are provided by Equation (36)

$$\begin{cases} x(s) = -\frac{1}{b}e^{-z}\sqrt{\epsilon_1 + a^2}cos(bs + c) + x_0, \\ y(s) = \frac{1}{b}e^{-z}\sqrt{\epsilon_1 + a^2}(-cos(bs + c) + sin(bs + c)) + y_0, \\ z(s) = a\alpha s + z_0. \end{cases}$$

where x_0, y_0, z_0 are constants.

6. Conclusions

This paper solves an interesting question of geometric properties and symmetry of two special types of curves in the Lorentzian α-Sasakian 3-manifolds. We prove the ratio relationship between κ_c and $\tau_c - \alpha$ or κ_c and $\tau_c + \alpha$ along the two special types of curves. We also use the properties of the slant curve and contact magnetic curve to characterize the manifold as a Lorentzian α-Sasakian 3-manifold. Finally, we provide an example and verify the slant curve and contact magnetic curve in the Lorentzian α-Sasakian 3-manifolds. As a future work, we plan to proceed to study some applications of contact magnetic curves and slant curves with singularity theory and submanifold theory, etc. in [34–37] to obtain new results and theorems.

Author Contributions: Conceptualization, X.C. and H.L.; writing—original draft preparation, X.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Special Fund for Scientific and Technological Innovation of Graduate Students in Mudanjiang Normal University (Grant No. kjcx2022-096mdjnu), the Project of Science and Technology of Mudanjiang Normal University (Grant No. GP2022006), the Natural Science Foundation of Heilongjiang Province of China (Grant No. LH2021A020), and by the Reform and Development Foundation for Local Colleges and Universities of the Central Government, (Excellent Young Talents Project of Heilongjiang Province, Grant No. ZYQN2019071).

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare no conflict of interest.

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