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Some Simpson-like Inequalities Involving the (s, m)-Preinvexity

Tarek Chiheb [1], Badreddine Meftah [1], Abdelkader Moumen [2,*], Mouataz Billah Mesmouli [2] and Mohamed Bouye [3]

1 Laboratory of Analysis and Control of Differential Equations "ACED", Department of Mathematics, Facuty MISM, University of 8 May 1945 Guelma, P.O. Box 401, Guelma 24000, Algeria; tarek.chiheb@univ-guelma.dz (T.C.)
2 Department of Mathematics, College of Science, University of Ha'il, P.O. Box 2240, Ha'il 55473, Saudi Arabia; m.mesmouli@uoh.edu.sa
3 Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia; mbmahmad@kku.edu.sa
* Correspondence: mo.abdelkader@uoh.edu.sa

Abstract: In this article, closed Newton–Cotes-type symmetrical inequalities involving four-point functions whose second derivatives are (s, m)-preinvex in the second sense are established. Some applications to quadrature formulas as well as inequalities involving special means are provided.

Keywords: Simpson inequality; Hölder inequality; (s, m)-preinvex functions; discrete power mean inequality; hypergeometric function

MSC: 26D10; 26D15; 26A51

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1. Introduction

One of the important concepts of real analysis and mathematical programming is convexity. Due to its various applications in various fields such as applied mathematics, engineering sciences and other fields, this notion has been extended and generalized in several directions and in various ways.

In [1], Meftah introduced the class of (s, m)-preinvex functions.

Definition 1 ([1])**.** *Function $\mathcal{C} : K \subset [0, b^*] \to \mathbb{R}$ is said to be (s, m)-preinvex with respect to η, where $\eta(.,.) : K \times K \to \mathbb{R}, b^* > 0$ and $s, m \in (0, 1]$, if*

$$\mathcal{C}(l + \jmath\eta(e, l)) \leq (1 - \jmath)^s \mathcal{C}(l) + m\jmath^s \mathcal{C}\left(\tfrac{e}{m}\right)$$

holds for all $l, e \in K$, and $\jmath \in [0, 1]$.

The previous class of function includes several classes depending on the values of s, m and $\eta(.,.)$. Among these classes, we have convex functions [2], m-convex functions [3] , s-convex functions in the second sense [4], (s, m)-convex functions in the second sense [5], preinvex functions [6], m-preinvex functions [7] and s-preinvex functions in the second sense [8].

In numerical analysis, numerous quadrature rules have been established to approximate the integrals defined under the aforementioned convexity classes; see [9–20].

The following inequalities are well known in the literature as Simpson's inequalities:

$$\left| \frac{1}{6}\left(\mathcal{C}(l) + 4\mathcal{C}\left(\frac{l+e}{2}\right) + \mathcal{C}(e)\right) - \frac{1}{e-l}\int_l^e \mathcal{C}(u)du \right| \leq \frac{1}{2880}\left\|\mathcal{C}^{(4)}\right\|_\infty (e-l)^4,$$

and

$$\left|\frac{1}{8}\left(\mathcal{C}(l)+3\mathcal{C}\left(\frac{2l+e}{3}\right)+3\mathcal{C}\left(\frac{l+2e}{3}\right)+\mathcal{C}(e)\right)-\frac{1}{e-l}\int_{l}^{e}\mathcal{C}(u)du\right|\leq\frac{1}{6480}\left\|\mathcal{C}^{(4)}\right\|_{\infty}(e-l)^{4},$$

where f is a four times continuously differentiable function on $[l,e]$, and $\left\|\mathcal{C}^{(4)}\right\|_{\infty}=\sup_{x\in[l,e]}\left|\mathcal{C}^{(4)}(x)\right|$.

As the above inequalities are very popular in estimating errors of quadrature rules, many researchers have massively studied them, as well as similar inequalities; one can consult, for example, [21–30] and references therein.

In [31], Hua et al. offered the following Simpson-type inequalities:

$$\begin{aligned}&\left|\frac{1}{6}\left(\mathcal{C}(l)+2\mathcal{C}\left(\frac{2l+e}{3}\right)+2\mathcal{C}\left(\frac{l+2e}{3}\right)+\mathcal{C}(e)\right)-\frac{1}{e-l}\int_{l}^{e}\mathcal{C}(u)du\right|\\ \leq\ &\frac{6^{\frac{1}{q}}(e-l)^{2}}{324}\left(\left(\frac{(s-3)3^{2+s}+(7+s)2^{2+s}}{3^{s}(1+s)(2+s)(3+s)}\left|\mathcal{C}''(l)\right|^{q}+\frac{1}{3^{s}(2+s)(3+s)}\left|\mathcal{C}''(e)\right|^{q}-c\frac{(e-l)^{2}}{45}\right)^{\frac{1}{q}}\right.\\&+\left(\frac{(s-1)2^{2+s}+5+s}{3^{s}(1+s)(2+s)(3+s)}\left(\left|\mathcal{C}''(l)\right|^{q}+\left|\mathcal{C}''(e)\right|^{q}\right)-c\frac{11(e-l)^{2}}{270}\right)^{\frac{1}{q}}\\&\left.+\left(\frac{1}{3^{s}(2+s)(3+s)}\left|\mathcal{C}''(l)\right|^{q}+\frac{(s-3)3^{2+s}+(7+s)2^{2+s}}{3^{s}(1+s)(2+s)(3+s)}\left|\mathcal{C}''(e)\right|^{q}-c\frac{(e-l)^{2}}{45}\right)^{\frac{1}{q}}\right)\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{6}\left(\mathcal{C}(l)+2\mathcal{C}\left(\frac{2l+e}{3}\right)+2\mathcal{C}\left(\frac{l+2e}{3}\right)+\mathcal{C}(e)\right)-\frac{1}{e-l}\int_{l}^{e}\mathcal{C}(u)du\right|\\ \leq\ &\frac{(e-l)^{2}}{54}\left(\frac{1}{3^{s}(1+s)}\right)^{\frac{1}{q}}\left(B\left(\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right)^{1-\frac{1}{q}}\\&\times\left(\left(\left(3^{1+s}-2^{1+s}\right)\left|\mathcal{C}''(l)\right|^{q}+\left|\mathcal{C}''(e)\right|^{q}-c\frac{7(e-l)^{2}3^{s}(1+s)}{54}\right)^{\frac{1}{q}}\right.\\&+\left(\left(2^{1+s}-1\right)\left(\left|\mathcal{C}''(l)\right|^{q}+\left|\mathcal{C}''(e)\right|^{q}\right)-c\frac{13(e-l)^{2}3^{s}(1+s)}{54}\right)^{\frac{1}{q}}\\&\left.+\left(\left|\mathcal{C}''(l)\right|^{q}+\left(3^{1+s}-2^{1+s}\right)\left|\mathcal{C}''(e)\right|^{q}-c\frac{7(e-l)^{2}3^{s}(1+s)}{54}\right)^{\frac{1}{q}}\right).\end{aligned}$$

In [32], Chiheb et al. established some Simpson-type inequalities for functions whose second derivatives are prequasi-invex, the results of which are based on the following identity:

Lemma 1 ([32])**.** *We let $\mathcal{C}:[l,l+\eta(e,l)]\to\mathbb{R}$ be a function such that $\mathcal{C}'$ is absolutely continuous and $\mathcal{C}''$ is integrable on $[l,l+\eta(e,l)]$, then the following equality holds:*

$$\begin{aligned}F(e,l,\mathcal{C})\ =\ &\frac{\eta^{2}(e,l)}{54}\int_{0}^{1}\jmath(1-\jmath)\left\{\mathcal{C}''\left(l+\frac{1-\jmath}{3}\eta(e,l)\right)\right.\\&\left.+\mathcal{C}''\left(l+\frac{2-\jmath}{3}\eta(e,l)\right)+\mathcal{C}''\left(l+\frac{3-\jmath}{3}\eta(e,l)\right)\right\}d\jmath,\qquad(1)\end{aligned}$$

where

$$\digamma(e,l,\mathcal{C}) = \frac{1}{6}\left(\mathcal{C}(l)+2\mathcal{C}\left(\frac{3l+\eta(e,l)}{3}\right)+2\mathcal{C}\left(\frac{3l+2\eta(e,l)}{3}\right)+\mathcal{C}(l+\eta(e,l))\right) - \frac{1}{\eta(e,l)}\int_{l}^{l+\eta(e,l)}\mathcal{C}(u)du. \quad (2)$$

In this paper, using the identity declared in [32], we establish some Simpson-type inequalities for twice differentiable (s,m)-preinvex functions. Some special cases are derived. Applications to quadrature formulas and inequalities involving means are provided.

2. Main Results

The following special functions as well as the algebraic inequality are useful to our study.

Definition 2 ([33])**.** *The beta function is defined for Rel>0 and Ree>0 as follows:*

$$B(l,e) = \int_0^1 \jmath^{l-1}(1-\jmath)^{e-1}d\jmath.$$

Definition 3 ([34])**.** *The hypergeometric function is defined for Rec $>$ Ree >0 and $|z|<1$ as follows:*

$${}_2F_1(l,e,c;z) = \frac{1}{B(e,c-e)}\int_0^1 \jmath^{e-1}(1-\jmath)^{c-e-1}(1-z\jmath)^{-l}d\jmath.$$

Lemma 2 ([35] Discrete Power mean inequality)**.** *For any $l,e>0$ and $0\le\zeta\le1$, we have*

$$l^{\zeta}+e^{\zeta}\le 2^{1-\zeta}(l+e)^{\zeta}.$$

Theorem 1. *We let $\mathcal{C}:[l,l+\eta(e,l)]\subset(0,\infty)\to\mathbb{R}$ be a twice differentiable function such that $\mathcal{C}''\in L\,[l,l+\eta(e,l)]$. If $|\mathcal{C}''|$ is (s,m)-preinvex for some fixed $s,m\in(0,1]$, we have*

$$|\digamma(e,l,\mathcal{C})| \le \frac{\eta^2(e,l)}{54}\left(\frac{\left(2+2^{s+3}+3^{s+2}\right)s+\left(6+3\times2^{s+3}-3^{s+3}\right)}{3^s(s+1)(s+2)(s+3)}\right)\left(|\mathcal{C}''(l)|+m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|\right),$$

where $\digamma(e,l,\mathcal{C})$ is defined as in (2).

Proof. From Lemma 1, properties of modulus and (s,m)-preinvexity of $|\mathcal{C}''|$ on $[l,l+\eta(e,l)]$, we have

$$\begin{aligned}
&|\digamma(e,l,\mathcal{C})| \\
\le\ & \frac{\eta^2(e,l)}{54}\left(\int_0^1 \jmath(1-\jmath)\left|f''\left(l+\frac{1-\jmath}{3}\eta(e,l)\right)\right|d\jmath\right. \\
&\left.+\int_0^1 \jmath(1-\jmath)\left|f''\left(l+\frac{2-\jmath}{3}\eta(e,l)\right)\right|d\jmath+\int_0^1 \jmath(1-\jmath)\left|f''\left(l+\frac{3-\jmath}{3}\eta(e,l)\right)\right|d\jmath\right) \\
\le\ & \frac{\eta^2(e,l)}{54}\left(\int_0^1 \jmath(1-\jmath)\left(\left(\frac{2+\jmath}{3}\right)^s|f''(l)|+m\left(\frac{1-\jmath}{3}\right)^s\left|f''\left(\frac{e}{m}\right)\right|\right)d\jmath\right.
\end{aligned}$$

$$
\begin{aligned}
& +\int_0^1 \jmath(1-\jmath)\left(\left(\tfrac{1+\jmath}{3}\right)^s |f''(l)| + m\left(\tfrac{2-\jmath}{3}\right)^s \left|f''\left(\tfrac{e}{m}\right)\right|\right) d\jmath \\
& +\int_0^1 \jmath(1-\jmath)\left(\left(\tfrac{\jmath}{3}\right)^s |f''(l)| + m\left(\tfrac{3-\jmath}{3}\right)^s \left|f''\left(\tfrac{e}{m}\right)\right|\right) d\jmath\Bigg) \\
= \; & \tfrac{\eta^2(e,l)}{54}\Bigg(|f''(l)|\int_0^1 \jmath(1-\jmath)\left(\tfrac{2+\jmath}{3}\right)^s d\jmath + m\left|f''\left(\tfrac{e}{m}\right)\right|\int_0^1 \jmath(1-\jmath)\left(\tfrac{1-\jmath}{3}\right)^s d\jmath \\
& +|f''(l)|\int_0^1 \jmath(1-\jmath)\left(\tfrac{1+\jmath}{3}\right)^s d\jmath + m\left|f''\left(\tfrac{e}{m}\right)\right|\int_0^1 \jmath(1-\jmath)\left(\tfrac{2-\jmath}{3}\right)^s d\jmath \\
& +|f''(l)|\int_0^1 \jmath(1-\jmath)\left(\tfrac{\jmath}{3}\right)^s d\jmath + m\left|f''\left(\tfrac{e}{m}\right)\right|\int_0^1 \jmath(1-\jmath)\left(\tfrac{3-\jmath}{3}\right)^s d\jmath\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= \; & \tfrac{\eta^2(e,l)}{54}\left(|f''(l)| + m\left|f''\left(\tfrac{e}{m}\right)\right|\right) \\
& \times\left(\int_0^1 \jmath(1-\jmath)\left(\tfrac{2+\jmath}{3}\right)^s d\jmath + \int_0^1 \jmath(1-\jmath)\left(\tfrac{1+\jmath}{3}\right)^s d\jmath + \int_0^1 \jmath(1-\jmath)\left(\tfrac{\jmath}{3}\right)^s d\jmath\right) \\
= \; & \tfrac{\eta^2(e,l)}{54}\left(|f''(l)| + m\left|f''\left(\tfrac{e}{m}\right)\right|\right) \\
& \times\left(\tfrac{\left(2^{s+2}+3^{s+2}\right)s+7\times 2^{s+2}-3^{s+3}}{3^s(s+1)(s+2)(s+3)} + \tfrac{\left(2^{s+2}+1\right)s+5-2^{s+2}}{3^s(s+1)(s+2)(s+3)} + \tfrac{s+1}{3^s(s+1)(s+2)(s+3)}\right) \\
= \; & \tfrac{\eta^2(e,l)}{54}\left(\tfrac{\left(2+2^{s+3}+3^{s+2}\right)s+\left(6+3\times 2^{s+3}-3^{s+3}\right)}{3^s(s+1)(s+2)(s+3)}\right)\left(|f''(l)| + m\left|f''\left(\tfrac{e}{m}\right)\right|\right),
\end{aligned}
$$

where we used the facts that

$$
\begin{aligned}
\int_0^1 \jmath(1-\jmath)\left(\tfrac{2+\jmath}{3}\right)^s d\jmath &= \int_0^1 \jmath(1-\jmath)\left(\tfrac{3-\jmath}{3}\right)^s d\jmath \\
&= \tfrac{\left(2^{s+2}+3^{s+2}\right)s+7\times 2^{s+2}-3^{s+3}}{3^s(s+1)(s+2)(s+3)},
\end{aligned} \tag{3}
$$

$$
\int_0^1 \jmath(1-\jmath)\left(\tfrac{1+\jmath}{3}\right)^s d\jmath = \int_0^1 \jmath(1-\jmath)\left(\tfrac{2-\jmath}{3}\right)^s d\jmath = \tfrac{\left(2^{s+2}+1\right)s+5-2^{s+2}}{3^s(s+1)(s+2)(s+3)} \tag{4}
$$

and

$$
\int_0^1 \jmath(1-\jmath)\left(\tfrac{\jmath}{3}\right)^s d\jmath = \int_0^1 \jmath(1-\jmath)\left(\tfrac{1-\jmath}{3}\right)^s d\jmath = \tfrac{1}{3^s(s+2)(s+3)}. \tag{5}
$$

The proof is finished. □

Corollary 1. *Taking $s = m = 1$ and $\eta(e,l) = e - l$, Theorem 1 becomes*

$$
\begin{aligned}
& \left|\tfrac{1}{6}\left(\mathcal{C}(l) + 2\mathcal{C}\left(\tfrac{2l+e}{3}\right) + 2\mathcal{C}\left(\tfrac{l+2e}{3}\right) + \mathcal{C}(e)\right) - \tfrac{1}{e-l}\int_l^e \mathcal{C}(u)du\right| \\
\leq \; & \tfrac{(e-l)^2}{216}\left(|\mathcal{C}''(l)| + |\mathcal{C}''(e)|\right).
\end{aligned}
$$

Theorem 2. *Under the assumptions of Theorem 1, if $|\mathcal{C}''|^q$ where $q>1$, is (s,m)-preinvex in the second sense for some fixed $s,m\in(0,1]$, with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\begin{aligned}
&|\digamma(l,e,\mathcal{C})|\\
\leq\ &\frac{\eta^2(e,l)}{54}(B(p+1,p+1))^{\frac{1}{p}}\left(\left(\frac{\left(3^{s+1}-2^{s+1}\right)|\mathcal{C}''(l)|^q+m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q}{3^s(1+s)}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{\left(2^{s+1}-1\right)\left(|\mathcal{C}''(l)|^q+m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)}{3^s(1+s)}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{C}''(l)|^q+m\left(3^{s+1}-2^{s+1}\right)\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q}{3^s(1+s)}\right)^{\frac{1}{q}}\right),
\end{aligned}$$

where $\digamma(l,e,\mathcal{C})$ is defined as in (1.2).

Proof. From Lemma 1, properties of modulus, Hölder's inequality and (s,m)-preinvexity of $|\mathcal{C}''|^q$ on $[l,l+\eta(e,l)]$, we have

$$\begin{aligned}
&|\digamma(l,e,\mathcal{C})|\\
\leq\ &\frac{\eta^2(e,l)}{54}\left(\int_0^1 \jmath(1-\jmath)\left|\mathcal{C}''\left(l+\frac{1-\jmath}{3}\eta(e,l)\right)\right|d\jmath\right.\\
&\left.+\int_0^1 \jmath(1-\jmath)\left|\mathcal{C}''\left(l+\frac{2-\jmath}{3}\eta(e,l)\right)\right|d\jmath+\int_0^1 t(1-t)\left|\mathcal{C}''\left(l+\frac{3-\jmath}{3}\eta(e,l)\right)\right|d\jmath\right)\\
\leq\ &\frac{\eta^2(e,l)}{54}\left(\int_0^1 \jmath^p(1-\jmath)^p d\jmath\right)^{\frac{1}{p}}\left(\left(\int_0^1\left(\left(\frac{2+\jmath}{3}\right)^s|\mathcal{C}''(l)|^q+m\left(\frac{1-\jmath}{3}\right)^s\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\left(\left(\frac{1+\jmath}{3}\right)^s|\mathcal{C}''(l)|^q+m\left(\frac{2-\jmath}{3}\right)^s\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1\left(\left(\frac{\jmath}{3}\right)^s|\mathcal{C}''(l)|^q+m\left(\frac{3-\jmath}{3}\right)^s\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\right)\\
=\ &\frac{\eta^2(e,l)}{54\times3^{\frac{s}{q}}}\left(\int_0^1 \jmath^p(1-\jmath)^p d\jmath\right)^{\frac{1}{p}}\\
&\times\left(\left(|\mathcal{C}''(l)|^q\int_0^1(2+\jmath)^s d\jmath+m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\int_0^1(1-\jmath)^s d\jmath\right)^{\frac{1}{q}}\right.\\
&+\left(|\mathcal{C}''(l)|^q\int_0^1(1+\jmath)^s d\jmath+m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\int_0^1(2-\jmath)^s d\jmath\right)^{\frac{1}{q}}\\
&\left.+\left(|\mathcal{C}''(l)|^q\int_0^1\jmath^s d\jmath+m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\int_0^1(3-\jmath)^s d\jmath\right)^{\frac{1}{q}}\right)\\
=\ &\frac{\eta^2(e,l)}{54}(B(p+1,p+1))^{\frac{1}{p}}\left(\left(\frac{\left(3^{s+1}-2^{s+1}\right)|\mathcal{C}''(l)|^q+m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q}{3^s(1+s)}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{\left(2^{s+1}-1\right)\left(|\mathcal{C}''(l)|^q+m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)}{3^s(1+s)}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{C}''(l)|^q+m\left(3^{s+1}-2^{s+1}\right)\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q}{3^s(1+s)}\right)^{\frac{1}{q}}\right),
\end{aligned}$$

which completes the proof. □

Corollary 2. *Taking $s = m = 1$ and $\eta(e,l) = e - l$, Theorem 2 becomes*

$$\begin{aligned}
&\left|\frac{1}{6}\left(\mathcal{C}(l) + 2\mathcal{C}\left(\frac{2l+e}{3}\right) + 2\mathcal{C}\left(\frac{l+2e}{3}\right) + \mathcal{C}(e)\right) - \frac{1}{e-l}\int_l^e \mathcal{C}(u)du\right| \\
\leq\ & \frac{(e-l)^2}{54}(B(p+1,p+1))^{\frac{1}{p}}\left(\left(\frac{5|\mathcal{C}''(l)|^q + |\mathcal{C}''(e)|^q}{6}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{|\mathcal{C}''(l)|^q + |\mathcal{C}''(e)|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{C}''(l)|^q + 5|\mathcal{C}''(e)|^q}{6}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Corollary 3. *In Corollary 2, using the discrete power mean inequality, we obtain*

$$\begin{aligned}
&\left|\frac{1}{6}\left(\mathcal{C}(l) + 2\mathcal{C}\left(\frac{2l+e}{3}\right) + 2\mathcal{C}\left(\frac{l+2e}{3}\right) + \mathcal{C}(e)\right) - \frac{1}{e-l}\int_l^e \mathcal{C}(u)du\right| \\
\leq\ & \frac{(e-l)^2}{18}(B(p+1,p+1))^{\frac{1}{p}}\left(\frac{|\mathcal{C}''(l)|^q + |\mathcal{C}''(e)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 3. *Under the assumptions of Theorem 2, if $|\mathcal{C}''|^q$ is (s,m)-preinvex in the second sense for some fixed $s, m \in (0,1]$ and $q \geq 1$, we have*

$$\begin{aligned}
&|F(l,e,\mathcal{C})| \\
\leq\ & \frac{\eta^2(e,l)}{54\times 6^{1-\frac{1}{q}}}\left(\left(\frac{\left(2^{s+2}+3^{s+2}\right)s + 7\times 2^{s+2} - 3^{s+3}}{3^s(s+1)(s+2)(s+3)}|\mathcal{C}''(l)|^q + \frac{m}{3^s(s+2)(s+3)}\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
& + \left(\frac{\left(2^{s+2}+1\right)s + 5 - 2^{s+2}}{3^s(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}\left(|\mathcal{C}''(l)|^q + m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)^{\frac{1}{q}} \\
&\left. + \left(\frac{1}{3^s(s+2)(s+3)}|\mathcal{C}''(l)|^q + m\frac{\left(2^{s+2}+3^{s+2}\right)s + 7\times 2^{s+2} - 3^{s+3}}{3^s(s+1)(s+2)(s+3)}\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)^{\frac{1}{q}}\right),
\end{aligned}$$

where $F(l,e,\mathcal{C})$ is defined as in (1.2).

Proof. From Lemma 1, properties of modulus, power mean inequality and (s,m)-preinvexity of $|f''|^q$ on $[l, l+\eta(e,l)]$, we obtain

$$\begin{aligned}
&|F(l,e,\mathcal{C})| \\
\leq\ & \frac{\eta^2(e,l)}{54}\left(\int_0^1 \jmath(1-\jmath)\left|\mathcal{C}''\left(l + \frac{1-\jmath}{3}\eta(e,l)\right)\right|d\jmath\right. \\
&\left. + \int_0^1 \jmath(1-\jmath)\left|\mathcal{C}''\left(l + \frac{2-\jmath}{3}\eta(e,l)\right)\right|d\jmath + \int_0^1 \jmath(1-\jmath)\left|\mathcal{C}''\left(l + \frac{3-\jmath}{3}\eta(e,l)\right)\right|d\jmath\right) \\
\leq\ & \frac{\eta^2(e,l)}{54}\left(\int_0^1 \jmath(1-\jmath)dt\right)^{1-\frac{1}{q}}\left(\left(\int_0^1 \jmath(1-\jmath)\left|\mathcal{C}''\left(l + \frac{1-\jmath}{3}\eta(e,l)\right)\right|^q d\jmath\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 \jmath(1-\jmath)\left|\mathcal{C}''\left(l+\tfrac{2-\jmath}{3}\eta(e,l)\right)\right|^q d\jmath\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 \jmath(1-\jmath)\left|\mathcal{C}''\left(l+\tfrac{3-\jmath}{3}\eta(e,l)\right)\right|^q d\jmath\right)^{\frac{1}{q}}\Bigg)\\
\leq\ & \frac{\eta^2(e,l)}{54}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\left(\int_0^1 \jmath(1-\jmath)\left(\left(\tfrac{2+\jmath}{3}\right)^s|\mathcal{C}''(l)|^q+m\left(\tfrac{1-\jmath}{3}\right)^s|\mathcal{C}''(\tfrac{e}{m})|^q\right)d\jmath\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1 \jmath(1-\jmath)\left(\left(\tfrac{1+\jmath}{3}\right)^s|\mathcal{C}''(l)|^q+m\left(\tfrac{2-\jmath}{3}\right)^s|\mathcal{C}''(\tfrac{e}{m})|^q\right)d\jmath\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1 \jmath(1-\jmath)\left(\left(\tfrac{\jmath}{3}\right)^s|\mathcal{C}''(l)|^q+m\left(1-\tfrac{\jmath}{3}\right)^s|\mathcal{C}''(\tfrac{e}{m})|^q\right)d\jmath\right)^{\frac{1}{q}}\right)\\
=\ & \frac{\eta^2(e,l)}{54\times 6^{1-\frac{1}{q}}}\left(\left(|\mathcal{C}''(l)|^q\int_0^1 \jmath(1-\jmath)\left(\tfrac{2+\jmath}{3}\right)^s d\jmath+m|\mathcal{C}''(\tfrac{e}{m})|^q\int_0^1 \jmath(1-\jmath)\left(\tfrac{1-\jmath}{3}\right)^s d\jmath\right)^{\frac{1}{q}}\right.\\
&+\left(|\mathcal{C}''(l)|^q\int_0^1 \jmath(1-\jmath)\left(\tfrac{1+\jmath}{3}\right)^s d\jmath+m|\mathcal{C}''(\tfrac{e}{m})|^q\int_0^1 \jmath(1-\jmath)\left(\tfrac{2-\jmath}{3}\right)^s d\jmath\right)^{\frac{1}{q}}\\
&\left.+\left(|\mathcal{C}''(l)|^q\int_0^1 \jmath(1-\jmath)\left(\tfrac{\jmath}{3}\right)^s d\jmath+m|\mathcal{C}''(\tfrac{e}{m})|^q\int_0^1 \jmath(1-\jmath)\left(1-\tfrac{\jmath}{3}\right)^s d\jmath\right)^{\frac{1}{q}}\right)\\
=\ & \frac{\eta^2(e,l)}{54\times 6^{1-\frac{1}{q}}}\left(\left(\frac{\left(2^{s+2}+3^{s+2}\right)s+7\times 2^{s+2}-3^{s+3}}{3^s(s+1)(s+2)(s+3)}|\mathcal{C}''(l)|^q+\frac{m}{3^s(s+2)(s+3)}|\mathcal{C}''(\tfrac{e}{m})|^q\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{\left(2^{s+2}+1\right)s+5-2^{s+2}}{3^s(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}\left(|\mathcal{C}''(l)|^q+m|\mathcal{C}''(\tfrac{e}{m})|^q\right)^{\frac{1}{q}}\\
&\left.+\left(\frac{1}{3^s(s+2)(s+3)}|\mathcal{C}''(l)|^q+m\frac{\left(2^{s+2}+3^{s+2}\right)s+7\times 2^{s+2}-3^{s+3}}{3^s(s+1)(s+2)(s+3)}|\mathcal{C}''(\tfrac{e}{m})|^q\right)^{\frac{1}{q}}\right),
\end{aligned}
$$

where we used (2.1)–(2.3). The proof is completed. □

Corollary 4. *Taking $s=m=1$ and $\eta(e,l)=e-l$, Theorem 3 becomes*

$$
\begin{aligned}
&\left|\frac{1}{6}\left(\mathcal{C}(l)+2\mathcal{C}\left(\frac{2l+e}{3}\right)+2\mathcal{C}\left(\frac{l+2e}{3}\right)+\mathcal{C}(e)\right)-\frac{1}{e-l}\int_l^e \mathcal{C}(u)du\right|\\
\leq\ & \frac{(e-l)^2}{324}\left(\left(\frac{5|\mathcal{C}''(l)|^q+|\mathcal{C}''(e)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{C}''(l)|^q+|\mathcal{C}''(e)|^q}{2}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{|\mathcal{C}''(l)|^q+5|\mathcal{C}''(e)|^q}{6}\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

Corollary 5. *In Corollary 4, using the discrete power mean inequality, we obtain*

$$
\begin{aligned}
& \left| \frac{1}{6}\left(\mathcal{C}(l) + 2\mathcal{C}\left(\frac{2l+e}{3}\right) + 2\mathcal{C}\left(\frac{l+2e}{3}\right) + \mathcal{C}(e)\right) - \frac{1}{e-l}\int_l^e \mathcal{C}(u)du \right| \\
\leq \quad & \frac{(e-l)^2}{108}\left(\frac{|\mathcal{C}''(l)|^q + |\mathcal{C}''(e)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 4. *Under the assumptions of Theorem 2, we have the following inequality:*

$$
\begin{aligned}
& |\digamma(l,e,\mathcal{C})| \\
\leq \quad & \frac{\eta^2(e,l)}{54}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{{}_2F_1\left(-s,1,q+2;\frac{1}{3}\right)}{q+1}|\mathcal{C}''(l)|^q + m\frac{B(q+1,s+1)}{3^s}\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
& + \left(\frac{2^s \times {}_2F_1\left(-s,1,q+2;\frac{1}{2}\right)}{3^s(q+1)}|\mathcal{C}''(l)|^q + m\frac{2^s \times {}_2F_1\left(-s,q+1,q+2;\frac{1}{2}\right)}{3^s(q+1)}\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)^{\frac{1}{q}} \\
& \left. + \left(\frac{1}{3^s(q+s+1)}|\mathcal{C}''(l)|^q + m\frac{{}_2F_1\left(-s,q+1,q+2;\frac{1}{3}\right)}{q+1}\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)^{\frac{1}{q}}\right),
\end{aligned}
$$

where $\digamma(l,e,\mathcal{C})$ *is defined as in (1.2) and* B *and* ${}_2F_1$ *are beta and hypergeometric functions, respectively.*

Proof. From Lemma 1, properties of modulus, Hölder's inequality and (s,m)-preinvexity of $|f''|^q$ on $[a, a+\eta(b,a)]$, we have

$$
\begin{aligned}
& |\digamma(l,e,\mathcal{C})| \\
\leq \quad & \frac{\eta^2(e,l)}{54}\left(\int_0^1 (1-\jmath)^p dt\right)^{\frac{1}{p}}\left(\left(\int_0^1 \jmath^q\left|\mathcal{C}''\left(l + \frac{1-\jmath}{3}\eta(e,l)\right)\right|^q d\jmath\right)^{\frac{1}{q}}\right. \\
& \left. + \left(\int_0^1 \jmath^q\left|\mathcal{C}''\left(l + \frac{2-\jmath}{3}\eta(e,l)\right)\right|^q d\jmath\right)^{\frac{1}{q}} + \left(\int_0^1 \jmath^q\left|\mathcal{C}''\left(l + \frac{3-\jmath}{3}\eta(e,l)\right)\right|^q d\jmath\right)^{\frac{1}{q}}\right) \\
\leq \quad & \frac{\eta^2(e,l)}{54}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(|\mathcal{C}''(l)|^q\int_0^1 \jmath^q\left(\frac{2+\jmath}{3}\right)^s d\jmath + m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\int_0^1 \jmath^q\left(\frac{1-\jmath}{3}\right)^s d\jmath\right)^{\frac{1}{q}}\right. \\
& + \left(|\mathcal{C}''(l)|^q\int_0^1 \jmath^q\left(\frac{1+\jmath}{3}\right)^s d\jmath + m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\int_0^1 \jmath^q\left(\frac{2-\jmath}{3}\right)^s d\jmath\right)^{\frac{1}{q}} \\
& \left. + \left(|\mathcal{C}''(l)|^q\int_0^1 \jmath^q\left(\frac{\jmath}{3}\right)^s d\jmath + m\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\int_0^1 \jmath^q\left(\frac{3-\jmath}{3}\right)^s d\jmath\right)^{\frac{1}{q}}\right) \\
= \quad & \frac{\eta^2(e,l)}{54}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{{}_2F_1\left(-s,1,q+2;\frac{1}{3}\right)}{q+1}|\mathcal{C}''(l)|^q + m\frac{B(q+1,s+1)}{3^s}\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
& + \left(\frac{2^s \times {}_2F_1\left(-s,1,q+2;\frac{1}{2}\right)}{3^s(q+1)}|\mathcal{C}''(l)|^q + m\frac{2^s \times {}_2F_1\left(-s,q+1,q+2;\frac{1}{2}\right)}{3^s(q+1)}\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)^{\frac{1}{q}} \\
& \left. + \left(\frac{1}{3^s(q+s+1)}|\mathcal{C}''(l)|^q + m\frac{{}_2F_1\left(-s,q+1,q+2;\frac{1}{3}\right)}{q+1}\left|\mathcal{C}''\left(\frac{e}{m}\right)\right|^q\right)^{\frac{1}{q}}\right),
\end{aligned}
$$

where we used

$$
\begin{aligned}
\int_0^1 \jmath^q \left(\frac{2+\jmath}{3}\right)^s d\jmath &= \frac{1}{q+1} \cdot {}_2F_1\left(-s, 1, q+2; \frac{1}{3}\right), \\
\int_0^1 \jmath^q \left(\frac{1-\jmath}{3}\right)^s d\jmath &= \left(\frac{1}{3}\right)^s B(q+1, s+1), \\
\int_0^1 \jmath^q \left(\frac{1+\jmath}{3}\right)^s d\jmath &= \left(\frac{2}{3}\right)^s \frac{1}{q+1} \cdot {}_2F_1\left(-s, 1, q+2; \frac{1}{2}\right), \\
\int_0^1 \jmath^q \left(\frac{2-t}{3}\right)^s d\jmath &= \left(\frac{2}{3}\right)^s \frac{1}{q+1} \cdot {}_2F_1\left(-s, q+1, q+2; \frac{1}{2}\right), \\
\int_0^1 \jmath^q \left(\frac{\jmath}{3}\right)^s d\jmath &= \left(\frac{1}{3}\right)^s \frac{1}{q+s+1}, \\
\int_0^1 \jmath^q \left(\frac{3-\jmath}{3}\right)^s d\jmath &= \frac{1}{q+1} \cdot {}_2F_1\left(-s, q+1, q+2; \frac{1}{3}\right).
\end{aligned}
$$

The proof is completed. □

Corollary 6. *In Theorem 4, taking $s = m = 1$ and $\eta(e, l) = e - l$, we obtain*

$$
\begin{aligned}
&\left| \frac{1}{6}\left(\mathcal{C}(l) + 2\mathcal{C}\left(\frac{2l+e}{3}\right) + 2\mathcal{C}\left(\frac{l+2e}{3}\right) + \mathcal{C}(e)\right) - \frac{1}{e-l}\int_l^e \mathcal{C}(u)du \right| \\
\leq\ & \frac{(e-l)^2}{162}\left(\frac{3}{p+1}\right)^{\frac{1}{p}} \left(\left(\frac{(3q+5)|\mathcal{C}''(l)|^q + |\mathcal{C}''(e)|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{(2q+3)|\mathcal{C}''(l)|^q + (q+3)|\mathcal{C}''(e)|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}} + \left(\frac{(q+1)|\mathcal{C}''(l)|^q + (2q+5)|\mathcal{C}''(e)|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}} \right).
\end{aligned}
$$

3. Applications

We let Y be the partition of points $l = x_0 < x_1 < \cdots < x_n = a$ of interval $[l, e]$ and consider quadrature formula

$$
\int_l^e \mathcal{C}(u)du = \lambda(\mathcal{C}, \mathrm{Y}) + R(\mathcal{C}, \mathrm{Y}),
$$

where

$$
\lambda(\mathcal{C}, \mathrm{Y}) = \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{6}\left(\mathcal{C}(x_i) + 2\mathcal{C}\left(\frac{2x_i + x_{i+1}}{3}\right) + 2\mathcal{C}\left(\frac{x_i + 2x_{i+1}}{3}\right) + \mathcal{C}(x_{i+1})\right)
$$

and $R(\mathcal{C}, \mathrm{Y})$ is the error of approximation.

Proposition 1. *We let $\mathcal{C}$ be as in Theorem 1 and $n \in \mathbb{N}$. If $|\mathcal{C}''|^q$ is an s-convex function in the second sense for some fixed $s \in (0, 1]$, we have*

$$
|R(\mathcal{C}, \mathrm{Y})| \leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3}{54}\left(\frac{(2 + 2^{s+3} + 3^{s+2})s + \left(6 + 3 \times 2^{s+3} - 3^{s+3}\right)}{3^s(s+1)(s+2)(s+3)}\right)\left(\left|\mathcal{C}''(x_i)\right| + \left|\mathcal{C}''(x_{i+1})\right|\right).
$$

Proof. Applying Theorem 1 with $\eta(e,l) = e - l$ and $m = 1$ on $[x_i, x_{i+1}]$ $(i = 0,1,\dots,n-1)$ of partition Y, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left(\mathcal{C}(x_i) + 2\mathcal{C}\left(\frac{2x_i+x_{i+1}}{3}\right) + 2\mathcal{C}\left(\frac{x_i+2x_{i+1}}{3}\right) + \mathcal{C}(x_{i+1})\right) - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \mathcal{C}(u)du\right| \\ \leq\ & \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^2}{54}\left(\frac{\left(2+2^{s+3}+3^{s+2}\right)s + \left(6+3\times 2^{s+3}-3^{s+3}\right)}{3^s(s+1)(s+2)(s+3)}\right)\left(\left|\mathcal{C}''(x_i)\right| + \left|\mathcal{C}''(x_{i+1})\right|\right).\end{aligned}$$

We add the above inequalities for all $i = 0,1,\dots,n-1$, and then multiply the resulting inequality by $(x_{i+1} - x_i)$. The desired result follows from the triangular inequality. □

Proposition 2. *We let $\mathcal{C}$ be as in Theorem 1 and $n \in \mathbb{N}$. If $|\mathcal{C}''|^q$ is convex function where $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$|R(\mathcal{C}, \mathrm{Y})| \leq \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^3}{18}(B(p+1,p+1))^{\frac{1}{p}}\left(\frac{|\mathcal{C}''(x_i)|^q + |\mathcal{C}''(x_{i+1})|^q}{2}\right).$$

Proof. Applying Corollary 3 on $[x_i, x_{i+1}]$ $(i = 0,1,\dots,n-1)$ of partition Y, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left(\mathcal{C}(x_i) + 2\mathcal{C}\left(\frac{2x_i+x_{i+1}}{3}\right) + 2\mathcal{C}\left(\frac{x_i+2x_{i+1}}{3}\right) + \mathcal{C}(x_{i+1})\right) - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \mathcal{C}(u)du\right| \\ \leq\ & \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^2}{18}(B(p+1,p+1))^{\frac{1}{p}}\left(\frac{|\mathcal{C}''(x_i)|^q + |\mathcal{C}''(x_{i+1})|^q}{2}\right).\end{aligned}$$

We add above inequalities for all $i = 0,1,\dots,n-1$, and then multiply the resulting inequality by $(x_{i+1} - x_i)$. The desired result follows from the triangular inequality. □

Proposition 3. *We let $\mathcal{C}$ be as in Theorem 1 and $n \in \mathbb{N}$. If $|\mathcal{C}''|^q$ is convex function where $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\begin{aligned}|R(\mathcal{C}, \mathrm{Y})| \quad \leq \quad & \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^3}{162}\left(\frac{3}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{(3q+5)|\mathcal{C}''(x_i)|^q + |\mathcal{C}''(x_{i+1})|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}\right. \\ & + \left(\frac{(2q+3)|\mathcal{C}''(x_i)|^q + (q+3)|\mathcal{C}''(x_{i+1})|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}} \\ & \left. + \left(\frac{(q+1)|\mathcal{C}''(x_i)|^q + (2q+5)|\mathcal{C}''(x_{i+1})|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Proof. Applying Corollary 6 on $[x_i, x_{i+1}]$ $(i = 0,1,\dots,n-1)$ of partition Y, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left(\mathcal{C}(x_i) + 2\mathcal{C}\left(\frac{2x_i+x_{i+1}}{3}\right) + 2\mathcal{C}\left(\frac{x_i+2x_{i+1}}{3}\right) + \mathcal{C}(x_{i+1})\right) - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \mathcal{C}(u)du\right| \\ \leq\ & \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^2}{162}\left(\frac{3}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{(3q+5)|\mathcal{C}''(x_i)|^q + |\mathcal{C}''(x_{i+1})|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\frac{(2q+3)|\mathcal{C}''(x_i)|^q + (q+3)|\mathcal{C}''(x_{i+1})|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}} + \left(\frac{(q+1)|\mathcal{C}''(x_i)|^q + (2q+5)|\mathcal{C}''(x_{i+1})|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}\right).\end{aligned}$$

We add above inequalities for all $i = 0,1,\dots,n-1$, and then multiply the resulting inequality by $(x_{i+1} - x_i)$. The desired result follows from the triangular inequality. □

For arbitrary real numbers l, e, t, we have:
The arithmetic mean: $\mathcal{A}(l,e) = \frac{l+e}{2}$ and $\mathcal{A}(l,e,t) = \frac{l+e+t}{3}$.
The p-logarithmic mean: $\mathcal{L}_p(l,e) = \left(\frac{e^{p+1}-l^{p+1}}{(p+1)(e-l)}\right)^{\frac{1}{p}}$, $l, e > 0, l \neq e$ and $p \in \mathbb{R}\backslash\{-1,0\}$.

Proposition 4. *We let $l, e \in \mathbb{R}$ with $0 < l < e$, then we have*

$$\left|\mathcal{A}\left(l^3, e^3\right) + \mathcal{A}^3(l,l,e) + \mathcal{A}^3(l,e,e) - 3\mathcal{L}_3^3(l,e)\right| \leq \frac{(e-l)^2}{6}\left(\frac{l^q+e^q}{2}\right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Corollary 5, with $q \geq 2$, applied to function $f(x) = x^3$. □

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