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A Cosserat Model of Elastic Solids Reinforced by a Family of Curved and Twisted Fibers

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Abstract: A Cosserat theory for fiber-reinforced elastic solids developed in Steigmann (2012) is generalized to accommodate initial curvature and twist of the fibers. The basic variables of the theory are a conventional deformation field and a rotation field that describes the local fiber orientation. Constraints on these fields are introduced to model the materiality of the fibers with respect to the underlying matrix deformation. A variational argument delivers the relevant equilibrium equations and boundary conditions and furnishes the interpretation of the Lagrange multipliers associated with the constraints as shear tractions acting on the fiber cross sections. Finally, the theory of material symmetry for such solids is developed and applied to the classification of some explicit constitutive functions.

Keywords: cosserat elasticity; fiber-reinforced solids; material symmetry

1. Introduction

In the present, work we generalize a theory for fiber-reinforced elastic solids proposed in [1,2] that accounts for the intrinsic flexural and torsional elasticities of the fibers, regarded as continuously distributed spatial rods of the Kirchhoff type in which the kinematics are based on a position field and an orthonormal triad field [3–5]. This model is a special case of the Cosserat theory of nonlinear elasticity [6–12]. We extend this theory to accommodate initially curved and twisted fibers and develop an associated framework for the characterization of material symmetry.

Industrial applications of the mechanics of composite materials reinforced by curvilinear fibers are thoroughly treated in [13]. Further applications to bioelasticity are described in [14]. In this literature, the fibers confer anisotropy to the composite but their instrinsic flexural and torsional elasticities are not taken into account. However, the latter can be expected to play a significant role in local fiber buckling and kink-band failure due to the length scale inherent in the flexural and torsional stiffnesses of the fibers [15]. These stiffnesses are also significant at larger length scales if the fibers are sufficiently stiff relative to the underlying matrix material.

To aid in the interpretation of the theory to be developed, in Section 2 we review the basic elements of Kirchhoff's theory for single rods. This is followed, in Section 3, by a brief outline of nonlinear Cosserat elasticity, specialized to model the effects of a single family of embedded fibers interacting with an elastic matrix. The resulting model is similar in structure to the Kirchhoff theory, with the effects of fiber-matrix interaction manifesting themselves as distributed forces and couples transmitted to the fibers by the matrix material in which they are embedded. Section 4 is devoted to a development of the associated theory of material symmetry, based on an extension to Cosserat elasticity [16] of Noll's concept [17] for simple materials. This is used in Section 5 to discuss some particular constitutive functions for fiber-reinforced solids.

We use standard notation such as $\boldsymbol{A}^t$, $\boldsymbol{A}^{-1}$, $Skw\boldsymbol{A}$, $\det \boldsymbol{A}$ and $tr\boldsymbol{A}$. These are respectively the transpose, the inverse, the skew part, the deteminant and the trace of a tensor $\boldsymbol{A}$, regarded as a linear transformation from a three-dimensional vector space to itself. The axial vector $ax(Skw\boldsymbol{A})$ of $Skw\boldsymbol{A}$ is defined by $ax(Skw\boldsymbol{A}) \times \boldsymbol{v} = (Skw\boldsymbol{A})\boldsymbol{v}$ for any vector $\boldsymbol{v}$. The tensor product of three-vectors is indicated by interposing the symbol $\otimes$, and the Euclidean inner product of tensors $\boldsymbol{A}, \boldsymbol{B}$ is denoted and defined by $\boldsymbol{A} \cdot \boldsymbol{B} = tr(\boldsymbol{A}\boldsymbol{B}^t)$; the induced norm is $|\boldsymbol{A}| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$. The symbol $|\cdot|$ is also used to denote the usual Euclidean norm of three-vectors. Latin and Greek indices take values in $\{1,2,3\}$ and $\{2,3\}$ respectively, and, when repeated, are summed over their ranges. Finally, bold subscripts are used to denote derivatives of scalar functions with respect to their vector or tensor arguments.

2. Kirchhoff Rods

Kirchhoff rods are modelled as spatial curves endowed with an elastic energy density that responds to flexure and twist. According to the derivation from conventional three-dimensional nonlinear elasticity given in [4], this theory also accommodates a small axial strain along the rod. We forego any discussion of the connection between Kirchhoff theory and three-dimensional elasticity and simply regard the rod as a directed curve [5] in which certain a priori constraints are imposed. An accessible discussion of Kirchhoff's theory may be found in [3].

2.1. Kinematics

The basic kinematical variables in the theory are a deformation field $\boldsymbol{r}(s)$, where $s \in [0, l]$ and l is the length of the rod in a reference configuration, and a right-handed, orthonormal triad $\{\boldsymbol{d}_i(s)\}$ in which $\boldsymbol{d}_1 = \boldsymbol{d}$, where $\boldsymbol{d}$ is the unit vector defined by

$$\boldsymbol{r}'(s) = \lambda \boldsymbol{d}, \quad \text{and} \quad \lambda = \left|\boldsymbol{r}'(s)\right|, \tag{1}$$

where λ is the stretch of the rod. Thus $\boldsymbol{d}$ is the unit tangent to the rod in a deformed configuration and $\boldsymbol{d}_\alpha$ $(\alpha = 2,3)$ span its cross-sectional plane at arclength station s.

The central assumption in Kirchhoff's theory is that each cross section deforms as a rigid disc. Thus there is a rotation field $\boldsymbol{R}(s)$ given by

$$\boldsymbol{R} = \boldsymbol{d}_i \otimes \boldsymbol{D}_i, \tag{2}$$

such that $\boldsymbol{d}_i = \boldsymbol{R}\boldsymbol{D}_i$, where $\boldsymbol{D}_i(s)$ are the values of $\boldsymbol{d}_i(s)$ in the reference configuration.

The curvature and twist of the rod are computed from the derivatives $\boldsymbol{d}_i'(s)$, where

$$\boldsymbol{d}_i' = \boldsymbol{R}'\boldsymbol{D}_i + \boldsymbol{R}\boldsymbol{D}_i'. \tag{3}$$

Let $\{\boldsymbol{E}_i\}$ be a fixed right-handed background frame. Then $\boldsymbol{D}_i(s) = \boldsymbol{A}(s)\boldsymbol{E}_i$ for some rotation field $\boldsymbol{A}$, yielding

$$\boldsymbol{d}_i' = \boldsymbol{W}\boldsymbol{d}_i = \boldsymbol{w} \times \boldsymbol{d}_i, \tag{4}$$

where

$$\boldsymbol{W} = \boldsymbol{R}'\boldsymbol{R}^t + \boldsymbol{R}\boldsymbol{A}'\boldsymbol{A}^t\boldsymbol{R}^t \tag{5}$$

is a skew tensor and

$$\boldsymbol{w} = ax\boldsymbol{W} = \kappa_i \boldsymbol{d}_i, \tag{6}$$

with

$$\kappa_i = \frac{1}{2} e_{ijk} \boldsymbol{d}_k \cdot \boldsymbol{d}_j'. \tag{7}$$

Here, e_{ijk} is the permutation symbol ($e_{123} = +1$, etc.), κ_1 is the twist of the rod and κ_α are the curvatures.

2.2. Strain-Energy Function

The strain energy S stored in a rod of length l is assumed to be expressible as

$$S = \int_0^l U ds, \tag{8}$$

where U, the energy per unit initial length, is a function of the list $\{\boldsymbol{R}, \boldsymbol{R}', \boldsymbol{r}'\}$, possibly depending explicitly on s.

We assume U to be Galilean invariant and hence that its values are invariant under $\{\boldsymbol{R}, \boldsymbol{R}', \boldsymbol{r}'\} \to \{\boldsymbol{QR}, \boldsymbol{QR}', \boldsymbol{Qr}'\}$, where $\boldsymbol{Q}$ is an arbitrary uniform rotation. Because U is defined pointwise, to derive a necessary condition we select $\boldsymbol{Q} = \boldsymbol{R}^t_{|s}$ and conclude that U is determined by the list $\{\boldsymbol{R}^t\boldsymbol{R}', \boldsymbol{R}^t\boldsymbol{r}'\} = \{\boldsymbol{R}^t\boldsymbol{W}\boldsymbol{R} - \boldsymbol{A}'\boldsymbol{A}^t, \lambda\boldsymbol{D}\}$, where $\boldsymbol{D} = \boldsymbol{D}_1$ and

$$\boldsymbol{R}^t\boldsymbol{W}\boldsymbol{R} - \boldsymbol{A}'\boldsymbol{A}^t = \boldsymbol{R}^t\boldsymbol{R}' = (\boldsymbol{R}\boldsymbol{D}_i \cdot \boldsymbol{R}'\boldsymbol{D}_j)\boldsymbol{D}_i \otimes \boldsymbol{D}_j, \tag{9}$$

is a Galilean-invariant measure of the relative flexure and twist of the rod due to deformation. This stands in one-to-one relation to its axial vector

$$\boldsymbol{\gamma} = \gamma_i\boldsymbol{D}_i = ax(\boldsymbol{R}^t\boldsymbol{R}'), \tag{10}$$

with components

$$\gamma_i = \frac{1}{2}e_{ijk}\boldsymbol{R}\boldsymbol{D}_k \cdot \boldsymbol{R}'\boldsymbol{D}_j. \tag{11}$$

Accordingly, because $\boldsymbol{D}(s)$ is independent of the deformation, the list $\{\boldsymbol{R}^t\boldsymbol{R}', \boldsymbol{R}^t\boldsymbol{r}'\}$ is equivalent to the list $\{\boldsymbol{\gamma}, \lambda\}$, and the strain energy may therefore be written in the form

$$U = w(\lambda, \boldsymbol{\gamma}; s). \tag{12}$$

Using (7) and (11) with $\boldsymbol{R}'\boldsymbol{D}_j = \boldsymbol{d}_j' - \boldsymbol{R}\boldsymbol{D}_j'$ yields

$$\gamma_i = \kappa_i - \kappa_i^0, \tag{13}$$

where

$$\kappa_i^0 = \frac{1}{2}e_{ijk}\boldsymbol{D}_k \cdot \boldsymbol{D}_j' \tag{14}$$

are the components of the initial curvature-twist vector

$$\boldsymbol{\kappa}^0 = \kappa_i^0\boldsymbol{D}_i = ax(\boldsymbol{A}'\boldsymbol{A}^t). \tag{15}$$

Thus,

$$\boldsymbol{\gamma} = \boldsymbol{\kappa} - \boldsymbol{\kappa}^0, \tag{16}$$

where

$$\boldsymbol{\kappa} = \boldsymbol{R}^t\boldsymbol{w} = \kappa_i\boldsymbol{D}_i = ax(\boldsymbol{R}^t\boldsymbol{W}\boldsymbol{R}). \tag{17}$$

For example, in the classical theory [4,18] of initially straight and untwisted rods ($\boldsymbol{\kappa}^0 = \boldsymbol{0}$), the strain-energy function for an isotropic rod of circular cross section is

$$w(\lambda, \boldsymbol{\kappa}; s) = \frac{1}{2}A(s)\varepsilon^2 + \frac{1}{2}T(s)\tau^2 + \frac{1}{2}F(s)\kappa_\alpha\kappa_\alpha, \tag{18}$$

where $\varepsilon = \lambda - 1$ is the extensional strain, $\tau = \kappa_1$ is the twist, A is the extensional stiffness (Young's modulus E times the cross-sectional area); F is the flexural stiffness (Young's modulus times the 2nd moment of area I of the cross section); and T is the torsional stiffness (the shear modulus G times the polar moment J of the cross section).

The homogeneous quadratic dependence of the energy on the bending-twist strain may be understood in terms of the local length scale furnished by the diameter of a cross section. The curvature-twist vector, when non-dimensionalized by this local scale, is typically small in applications. For example, the minimum radius of curvature of a bent rod is typically much larger than its diameter. If the bending and twisting moments vanish when the rod is straight and untwisted, then the leading-order contribution of the curvature-twist vector to the strain energy is quadratic. In general the flexural and torsional stiffnesses in this expression may depend on fiber stretch, but in the small-extensional-strain regime they are approximated at leading order by functions of s alone.

2.3. Equilibrium Theory

We recall the variational derivation of the equilibrium equations of the Kirchhoff theory here to provide context for the discussion of Cosserat elasticity in Section 3 [18]. Equilibria are assumed to satisfy the virtual-power statement

$$\dot{S} = P, \tag{19}$$

where P is the virtual power of the loads—the explicit form of which is deduced below—and the superposed dot is used to identify a variational derivative. These are induced by the derivatives, with respect to ϵ, of the one-parameter deformation and rotation fields $\boldsymbol{r}(s;\epsilon)$ and $\boldsymbol{R}(s;\epsilon)$ respectively, where $\boldsymbol{r}(s) = \boldsymbol{r}(s;0)$ and $\boldsymbol{R}(s) = \boldsymbol{R}(s;0)$ are equilibrium fields. Thus,

$$\dot{U} = \dot{w} = w_\lambda \dot{\lambda} + m_i \dot{\gamma}_i, \tag{20}$$

where

$$w_\lambda = \partial w / \partial \lambda \quad \text{and} \quad m_i = \partial w / \partial \gamma_i \tag{21}$$

are evaluated at $\epsilon = 0$.

From (1) we have that

$$\dot{\lambda}\boldsymbol{d} + \boldsymbol{\omega} \times \boldsymbol{r}' = \boldsymbol{u}', \tag{22}$$

where $\boldsymbol{u}(s) = \dot{\boldsymbol{r}}$ is the virtual translational velocity and $\boldsymbol{\omega}(s) = ax(\dot{\boldsymbol{R}}\boldsymbol{R}^t)$ is the virtual rotational velocity. That is, $\dot{\boldsymbol{d}}_i = \dot{\boldsymbol{R}}\boldsymbol{R}^t \boldsymbol{d}_i$, which is equivalent to

$$\dot{\boldsymbol{d}}_i = \boldsymbol{\omega} \times \boldsymbol{d}_i. \tag{23}$$

From (9) and (16) it follows that

$$\begin{aligned} \dot{\kappa}_i &= \frac{1}{2} e_{ijk} (\dot{\boldsymbol{d}}_k \cdot \boldsymbol{d}_j' + \boldsymbol{d}_k \cdot \dot{\boldsymbol{d}}_j') \\ &= \frac{1}{2} e_{ijk} [\boldsymbol{\omega} \times \boldsymbol{d}_k \cdot \boldsymbol{d}_j' + \boldsymbol{d}_k \cdot (\boldsymbol{\omega}' \times \boldsymbol{d}_j + \boldsymbol{\omega} \times \boldsymbol{d}_j')], \end{aligned} \tag{24}$$

in which the terms involving $\boldsymbol{\omega}$ cancel; the $e - \delta$ identity $\frac{1}{2} e_{ijk} e_{mjk} = \delta_{im}$ (the Kronecker delta), combined with $\boldsymbol{d}_j \times \boldsymbol{d}_k = e_{mjk} \boldsymbol{d}_m$, results in

$$\dot{\gamma}_i = \dot{\kappa}_i = \boldsymbol{d}_i \cdot \boldsymbol{\omega}'. \tag{25}$$

Thus,

$$\dot{S} = \int_0^l (w_\lambda \boldsymbol{d} \cdot \boldsymbol{u}' + \boldsymbol{m} \cdot \boldsymbol{\omega}') ds, \tag{26}$$

with

$$\boldsymbol{m} = m_i \boldsymbol{d}_i. \tag{27}$$

Further, (1) implies that

$$\boldsymbol{r}' \cdot \boldsymbol{d}_\alpha = 0; \quad \alpha = 2, 3. \tag{28}$$

To accommodate these constraints in the virtual-power statement, we relax them and introduce the extended energy

$$E = S + \int_0^l f_\alpha \boldsymbol{r}' \cdot \boldsymbol{d}_\alpha ds, \tag{29}$$

where $f_\alpha(s)$ are Lagrange multipliers. The extended variational problem is

$$\dot{E} = P, \tag{30}$$

where

$$\dot{E} = \int_0^l [(w_\lambda \boldsymbol{d} + f_\alpha \boldsymbol{d}_\alpha) \cdot \boldsymbol{u}' + \boldsymbol{m} \cdot \boldsymbol{\omega}' + f_\alpha \boldsymbol{d}_\alpha \times \boldsymbol{r}' \cdot \boldsymbol{\omega} + \dot{f}_\alpha \boldsymbol{r}' \cdot \boldsymbol{d}_\alpha] ds \tag{31}$$

The variations $\dot{f}_\alpha$ simply return the constraints (28), and an integration by parts gives

$$\dot{E} = (\boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{m} \cdot \boldsymbol{\omega}) \,|_0^l - \int_0^l [\boldsymbol{u} \cdot \boldsymbol{f}' + \boldsymbol{\omega} \cdot (\boldsymbol{m}' - \boldsymbol{f} \times \boldsymbol{r}')] ds, \tag{32}$$

where

$$\boldsymbol{f} = w_\lambda \boldsymbol{d} + f_\alpha \boldsymbol{d}_\alpha. \tag{33}$$

This implies that the virtual power is expressible in the form

$$P = (\boldsymbol{t} \cdot \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega}) \,|_0^l + \int_0^l (\boldsymbol{u} \cdot \boldsymbol{g} + \boldsymbol{\omega} \cdot \boldsymbol{\pi}) ds, \tag{34}$$

in which $\boldsymbol{t}$ and $\boldsymbol{c}$ represent forces and couples acting at the ends of the segment and $\boldsymbol{g}$ and $\boldsymbol{\pi}$ are force and couple distributions acting in the interior.

By the Fundamental Lemma, the Euler equations holding at points in the interior of the rod are

$$\boldsymbol{f}' + \boldsymbol{g} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{m}' + \boldsymbol{\pi} = \boldsymbol{f} \times \boldsymbol{r}', \tag{35}$$

and the endpoint conditions are

$$\boldsymbol{f} = \boldsymbol{t} \quad \text{and} \quad \boldsymbol{m} = \boldsymbol{c}, \tag{36}$$

provided that neither position nor orientation is assigned at the endpoints. These are the equilibrium conditions of classical rod theory in which $\boldsymbol{f}$ and $\boldsymbol{m}$ respectively are the cross-sectional force and moment transmitted by the segment $(s, l]$ on the part $[0, s]$. We observe, from (33) and $(36)_1$, that the Lagrange multipliers f_α play the role of constitutively indeterminate transverse shear forces acting on a fiber cross section.

For the strain-energy function (18) we have $m_1 \boldsymbol{d}_1 = T\tau \boldsymbol{d}$ and $m_\alpha \boldsymbol{d}_\alpha = F\kappa_\alpha \boldsymbol{d}_\alpha$. To reduce the second expression we use (7), together with $\boldsymbol{d} \cdot \boldsymbol{d}'_\mu = -\boldsymbol{d}_\mu \cdot \boldsymbol{d}'$, to derive $\kappa_\alpha = e_{\alpha 1 \mu} \boldsymbol{d}_\mu \cdot \boldsymbol{d}'$. From $\boldsymbol{d} \cdot \boldsymbol{d}' = 0$ it follows that $\boldsymbol{d}' = (\boldsymbol{d}_\alpha \cdot \boldsymbol{d}') \boldsymbol{d}_\alpha$ and $\boldsymbol{d} \times \boldsymbol{d}' = (\boldsymbol{d}_\alpha \cdot \boldsymbol{d}') \boldsymbol{d} \times \boldsymbol{d}_\alpha = (e_{\beta 1 \alpha} \boldsymbol{d}_\alpha \cdot \boldsymbol{d}') \boldsymbol{d}_\beta$; thus $\kappa_\beta \boldsymbol{d}_\beta = \boldsymbol{d} \times \boldsymbol{d}'$ and (21) yields [3]

$$\boldsymbol{m} = T\tau \boldsymbol{d} + F\boldsymbol{d} \times \boldsymbol{d}'. \tag{37}$$

3. Cosserat Elasticity of Fiber-Reinforced Materials

Cosserat elasticity theory emerges as the natural setting for elastic solids with embedded fibers—modelled as continuously distributed Kirchhoff rods—that support bending and twisting moments. To motivate our kinematical hypotheses, we suppose the fibers and matrix to be perfectly bonded and assume that both may be modelled at the microscale as conventional elastic solids. The interface between the matrix and fiber is then convected by the deformation as a material surface, and Hadamard's compatibility condition requires that $\boldsymbol{F}^+ - \boldsymbol{F}^- = \boldsymbol{a} \otimes \boldsymbol{N}$ for some vector $\boldsymbol{a}$, where $\boldsymbol{N}$ is a unit normal to the interface and $\boldsymbol{F}^{\pm}$ are the values of the deformation gradients in the fiber and matrix at the interface. In particular, if $\boldsymbol{D}(=\boldsymbol{D}_1)$ is the unit tangent to the centerline of an untapered fiber, then $\boldsymbol{N} \in Span\{\boldsymbol{D}_\alpha\}$, where $\boldsymbol{D}_\alpha; \alpha = 2,3$, are orthonormal unit vectors in the fiber cross section. It follows that

$$\boldsymbol{F}^{+}\boldsymbol{D} = \boldsymbol{F}^{-}\boldsymbol{D}, \quad \text{but} \quad \boldsymbol{F}^{+}\boldsymbol{D}_\alpha \neq \boldsymbol{F}^{-}\boldsymbol{D}_\alpha, \tag{38}$$

and hence that the deformation gradients in the fiber and matrix may be unequal. This stands in contrast to a model proposed in [19] on the basis of a single deformation field.

If a fiber is sufficiently stiff relative to the matrix, then its deformation gradient is approximated by a rotation field $\boldsymbol{R}$. In Dill's interpretation of the Kirchhoff theory [4] this is accompanied by a small axial strain. Thus we interpret $(38)_1$ in the form

$$\boldsymbol{F}\boldsymbol{D} = \lambda \boldsymbol{d}, \quad \text{where} \quad \boldsymbol{d} = \boldsymbol{R}\boldsymbol{D} \quad \text{and} \quad \lambda = |\boldsymbol{F}\boldsymbol{D}|\,, \tag{39}$$

where $\lambda(=|\boldsymbol{F}\boldsymbol{D}|)$ is the fiber stretch and $\boldsymbol{F}$ is the matrix deformation gradient. The fields $\boldsymbol{F}$ and $\boldsymbol{R}$ are otherwise independent in accordance with $(38)_2$. These in turn furnish $\boldsymbol{R}^t\boldsymbol{F}\boldsymbol{D} = \lambda\boldsymbol{D}$ and hence two constraints

$$\boldsymbol{D}_\alpha \cdot \boldsymbol{R}^t\boldsymbol{F}\boldsymbol{D} = 0; \quad \alpha = 2,3, \tag{40}$$

analogous to (28), involving the fiber rotation and matrix deformation.

Equation (39) implies that the fibers are convected as material curves relative to the matrix. The cross-sectional vectors $\boldsymbol{D}_\alpha$ are embedded in the fiber but not in the matrix, and so their images $\boldsymbol{d}_\alpha$ in the current configuration are free to shear relative to the matrix while remaining mutually orthogonal and perpendicular to $\boldsymbol{d}$.

3.1. Kinematical and Constitutive Variables in Cosserat Elasticity

The basic kinematical variables of a Cosserat continuum are a rotation field $\boldsymbol{R}(\boldsymbol{X})$ and a deformation field $\boldsymbol{\chi}(\boldsymbol{X})$. Naturally these may depend on time, but such dependence is not relevant to our development and is not made explicit.

The constitutive response of an elastic Cosserat continuum is embodied in a strain-energy density $U(\boldsymbol{F},\boldsymbol{R},\nabla\boldsymbol{R};\boldsymbol{X})$, per unit reference volume, where $\boldsymbol{F} = \nabla\boldsymbol{\chi}$ is the usual deformation gradient and $\nabla\boldsymbol{R}$ is the rotation gradient. In Cartesian index notation, these are

$$\boldsymbol{F} = F_{iA}\boldsymbol{e}_i \otimes \boldsymbol{E}_A, \quad \boldsymbol{R} = R_{iA}\boldsymbol{e}_i \otimes \boldsymbol{E}_A \quad \text{and} \quad \nabla\boldsymbol{R} = R_{iA,B}\boldsymbol{e}_i \otimes \boldsymbol{E}_A \otimes \boldsymbol{E}_B \tag{41}$$

with

$$F_{iA} = \chi_{i,A}, \tag{42}$$

where $(\cdot)_{,A} = \partial(\cdot)/\partial X_A$ and where $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{E}_A\}$ are fixed orthonormal bases associated with the Cartesian coordinates x_i and X_A, with $x_i = \chi_i(X_A)$.

We again suppose the strain energy to be Galilean-invariant and thus require

$$U(\boldsymbol{F},\boldsymbol{R},\nabla\boldsymbol{R};\boldsymbol{X}) = U(\boldsymbol{Q}\boldsymbol{F},\boldsymbol{Q}\boldsymbol{R},\boldsymbol{Q}\nabla\boldsymbol{R};\boldsymbol{X}), \tag{43}$$

where $\boldsymbol{Q}$ is an arbitrary spatially uniform rotation with $(\boldsymbol{Q}\nabla\boldsymbol{R})_{iAB} = (Q_{ij}R_{jA})_{,B} = Q_{ij}R_{jA,B}$. The restriction

$$U(\boldsymbol{F},\boldsymbol{R},\nabla\boldsymbol{R};\boldsymbol{X}) = W(\boldsymbol{E},\boldsymbol{\Gamma};\boldsymbol{X}), \tag{44}$$

with [11,16]

$$\boldsymbol{E} = \boldsymbol{R}^t\boldsymbol{F} = E_{AB}\boldsymbol{E}_A \otimes \boldsymbol{E}_B; \quad E_{AB} = R_{iA}F_{iB}, \tag{45}$$

$$\boldsymbol{\Gamma} = \Gamma_{DC}\boldsymbol{E}_D \otimes \boldsymbol{E}_C; \quad \Gamma_{DC} = \frac{1}{2}e_{BAD}R_{iA}R_{iB,C}, \tag{46}$$

where W is the reduced strain-energy function and e_{ABC} is the permutation symbol, is both necessary and sufficient for Galilean invariance. Sufficiency is obvious, whereas necessity follows by choosing $\boldsymbol{Q} = \boldsymbol{R}^t_{|X}$, where X is the material point in question, and making use of the fact that, for each fixed $C \in \{1,2,3\}$, the matrix $R_{iA}R_{iB,C}$ is skew. This follows by differentiating $R_{iA}R_{iB} = \delta_{AB}$ (the Kronecker delta). The axial vectors $\boldsymbol{\Gamma}_C$ associated with this skew matrix have components

$$\Gamma_{D(C)} = \frac{1}{2} e_{BAD} R_{iA} R_{iB,C}, \tag{47}$$

yielding [11]

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_C \otimes \boldsymbol{E}_C, \tag{48}$$

and so $\boldsymbol{\Gamma}$—the second order wryness tensor—stands in one-to-one relation to the third order Cosserat strain measure $\boldsymbol{R}^t \nabla \boldsymbol{R}$.

3.2. Virtual Power and Equilibrium

As in Section 2, we define equilibria to be states that satisfy the virtual-power statement

$$\dot{E} = P, \tag{49}$$

where P is the virtual power of the loads acting on the body,

$$E = S + \int_\kappa \Lambda_\alpha \boldsymbol{D}_\alpha \cdot \boldsymbol{E}\boldsymbol{D} dv, \tag{50}$$

is the extended energy, Λ_α are Lagrange multipliers accompanying the constraints (40),

$$S = \int_\kappa U dv \tag{51}$$

is the total strain energy, and, as before, superposed dots identify variational derivatives. Thus,

$$\dot{U} = \dot{W} = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{E}} + \boldsymbol{\mu} \cdot \dot{\boldsymbol{\Gamma}}, \tag{52}$$

where

$$\boldsymbol{\sigma} = W_{\boldsymbol{E}} \quad \text{and} \quad \boldsymbol{\mu} = W_{\boldsymbol{\Gamma}} \tag{53}$$

are evaluated at equilibrium. Further,

$$(\boldsymbol{D}_\alpha \cdot \boldsymbol{E}\boldsymbol{D})^{\cdot} = \boldsymbol{D}_\alpha \otimes \boldsymbol{D} \cdot \dot{\boldsymbol{E}} \tag{54}$$

so that

$$\dot{E} = \int_\kappa [(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \cdot \dot{\boldsymbol{E}} + \boldsymbol{\mu} \cdot \dot{\boldsymbol{\Gamma}} + \dot{\Lambda}_\alpha \boldsymbol{D}_\alpha \cdot \boldsymbol{E}\boldsymbol{D}] dv, \tag{55}$$

where

$$\boldsymbol{\Lambda} = \Lambda_\alpha \boldsymbol{D}_\alpha. \tag{56}$$

It follows from (45) that

$$\dot{\boldsymbol{E}} = \boldsymbol{R}^t(\nabla \boldsymbol{u} - \boldsymbol{\Omega}\boldsymbol{F}), \quad \text{where} \quad \boldsymbol{u} = \dot{\chi} \quad \text{and} \quad \boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^t. \tag{57}$$

Then,

$$(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \cdot \dot{\boldsymbol{E}} = \boldsymbol{R}(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\Omega} \cdot Skw[\boldsymbol{R}(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D})\boldsymbol{F}^t]. \tag{58}$$

If $\boldsymbol{\alpha}$ is a skew tensor, then $\boldsymbol{\Omega} \cdot \boldsymbol{\alpha} = 2\boldsymbol{\omega} \cdot \boldsymbol{a}$, where $\boldsymbol{\omega} = ax\boldsymbol{\Omega}$ and $\boldsymbol{a} = ax\boldsymbol{\alpha}$. Further, for any tensor $\boldsymbol{A}$ we have $\boldsymbol{RAF}^t = \boldsymbol{RAE}^t\boldsymbol{R}^t$ and $Skw(\boldsymbol{RAE}^t\boldsymbol{R}^t) = \boldsymbol{R}Skw(\boldsymbol{AE}^t)\boldsymbol{R}^t$, and therefore

$$(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \cdot \dot{\boldsymbol{E}} = \boldsymbol{R}(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \cdot \nabla \boldsymbol{u} - 2ax\{\boldsymbol{R}Skw[(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D})\boldsymbol{E}^t]\boldsymbol{R}^t\} \cdot \boldsymbol{\omega}. \tag{59}$$

The reduction

$$\dot{\boldsymbol{\Gamma}} = \boldsymbol{R}^t \nabla \boldsymbol{\omega}, \tag{60}$$

which is somewhat more involved, is detailed in Appendix A. Accordingly,

$$\boldsymbol{\mu} \cdot \dot{\boldsymbol{\Gamma}} = \boldsymbol{R\mu} \cdot \nabla \boldsymbol{\omega}, \tag{61}$$

and application of the divergence theorem to (55), with (59) and (61), gives

$$\begin{aligned} \dot{E} \;=\; & \int_{\partial\kappa} [(\boldsymbol{R\sigma} + \boldsymbol{\lambda} \otimes \boldsymbol{D})\boldsymbol{\nu} \cdot \boldsymbol{u} + (\boldsymbol{R\mu})\boldsymbol{\nu} \cdot \boldsymbol{\omega}]da + \int_{\kappa} \dot{\Lambda}_\alpha \boldsymbol{D}_\alpha \cdot \boldsymbol{ED} dv \\ & - \int_{\kappa} \{\boldsymbol{u} \cdot Div(\boldsymbol{R\sigma} + \boldsymbol{\lambda} \otimes \boldsymbol{D}) + \boldsymbol{\omega} \cdot [Div(\boldsymbol{R\mu}) + 2ax(\boldsymbol{R}Skw[(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D})\boldsymbol{E}^t]\boldsymbol{R}^t)]\}dv, \end{aligned} \tag{62}$$

where $\boldsymbol{\nu}$ is the exterior unit normal to the (piecewise smooth) surface $\partial\kappa$, and

$$\boldsymbol{\lambda} = \boldsymbol{R\Lambda} = \Lambda_\alpha \boldsymbol{d}_\alpha. \tag{63}$$

The virtual power is thus of the form

$$P = \int_{\partial\kappa} (\boldsymbol{t} \cdot \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega})da + \int_{\kappa} (\boldsymbol{g} \cdot \boldsymbol{u} + \boldsymbol{\pi} \cdot \boldsymbol{\omega})dv, \tag{64}$$

where $\boldsymbol{t}$ and $\boldsymbol{c}$ are densities of force and couple acting on $\partial\kappa$, and $\boldsymbol{g}$ and $\boldsymbol{\pi}$ are densities of force and couple acting in κ.

The fundamental lemma delivers the constraints (40) together with the differential equations

$$\boldsymbol{g} = -Div(\boldsymbol{R\sigma} + \boldsymbol{\lambda} \otimes \boldsymbol{D}) \quad \text{and} \quad \boldsymbol{\pi} = -Div(\boldsymbol{R\mu}) - 2ax\{\boldsymbol{R}Skw[(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D})\boldsymbol{E}^t]\boldsymbol{R}^t\} \quad \text{in} \quad \kappa, \tag{65}$$

and the natural boundary conditions

$$\boldsymbol{t} = (\boldsymbol{R\sigma} + \boldsymbol{\lambda} \otimes \boldsymbol{D})\boldsymbol{\nu} \quad \text{on} \quad \partial\kappa_t \quad \text{and} \quad \boldsymbol{c} = (\boldsymbol{R\mu})\boldsymbol{\nu} \quad \text{on} \quad \partial\kappa_c, \tag{66}$$

where $\partial\kappa_t$ is a part of $\partial\kappa$ where position is not assigned and $\partial\kappa_c$ is a part where rotation is not assigned. We assume position to be assigned on $\partial\kappa \setminus \partial\kappa_t$ ($\boldsymbol{u} = \boldsymbol{0}$), and rotation to be assigned on $\partial\kappa \setminus \partial\kappa_c$ ($\boldsymbol{\omega} = \boldsymbol{0}$).

3.3. Fiber-Matrix Interaction

Pursuant to the discussion at the start of this section, we assume that Cosserat elasticity is conferred by the mechanical interaction between an elastomeric matrix and a single family of embedded fibers. The relative curvature-twist vector $\boldsymbol{\gamma} = ax(\boldsymbol{R}^t\boldsymbol{R}')$ of a fiber initially oriented along a unit-vector field $\boldsymbol{D}(\boldsymbol{X})$, where $(\cdot)'$ is the directional derivative along $\boldsymbol{D}$, is (cf. (11))

$$\boldsymbol{\gamma} = \gamma_i \boldsymbol{D}_i \quad \text{with} \quad \gamma_i = \frac{1}{2} e_{ijk} \boldsymbol{D}_k \cdot \boldsymbol{R}^t\boldsymbol{R}'\boldsymbol{D}_j. \tag{67}$$

Thus, with $R'_{iA} = R_{iA,B}D_B$ we derive (cf. (46))

$$\boldsymbol{R}^t\boldsymbol{R}' = R_{iC}R_{iA,B}D_B\boldsymbol{E}_C \otimes \boldsymbol{E}_A = e_{ACD}\Gamma_{DB}D_B\boldsymbol{E}_C \otimes \boldsymbol{E}_A, \tag{68}$$

and conclude that γ is determined by $\boldsymbol{\Gamma}$ via $\boldsymbol{\Gamma D}$. Here the director fields $\boldsymbol{D}_i(\boldsymbol{X})$ form a positively oriented orthonormal triad with $\boldsymbol{D}_1 = \boldsymbol{D}$, and the Cosserat rotation field is given simply by

$$\boldsymbol{R} = \boldsymbol{d}_i \otimes \boldsymbol{D}_i, \tag{69}$$

as in (2), where $\boldsymbol{d}_i = \boldsymbol{R}\boldsymbol{D}_i$ are the images of directors in the deformed body, with $\boldsymbol{d} = \boldsymbol{d}_1$ the field of unit tangents to the deformed fibers. Then, as in Section 2, we may express the strain energy in the form

$$W(\boldsymbol{E}, \boldsymbol{\Gamma}; \boldsymbol{X}) = w(\boldsymbol{E}, \gamma; \boldsymbol{X}), \tag{70}$$

where w is now the strain energy per unit reference volume. Thus,

$$\sigma = w_{\boldsymbol{E}}. \tag{71}$$

To obtain the couple stress $\boldsymbol{\mu}$ we use (25) in the form

$$\dot{\gamma}_i = \boldsymbol{R}\boldsymbol{D}_i \cdot (\nabla\boldsymbol{\omega})\boldsymbol{D} = \boldsymbol{D}_i \cdot (\boldsymbol{R}^t\nabla\boldsymbol{\omega})\boldsymbol{D} = \boldsymbol{D}_i \otimes \boldsymbol{D} \cdot \dot{\boldsymbol{\Gamma}}. \tag{72}$$

Variation of the energy at fixed $\boldsymbol{E}$ then gives

$$\boldsymbol{\mu} \cdot \dot{\boldsymbol{\Gamma}} = \dot{W} = \dot{w} = w_{\gamma} \cdot \dot{\gamma} = \boldsymbol{M} \otimes \boldsymbol{D} \cdot \dot{\boldsymbol{\Gamma}}, \tag{73}$$

where

$$\boldsymbol{M} = w_{\gamma} = m_i \boldsymbol{D}_i \quad \text{with} \quad m_i = \partial w/\partial \gamma_i, \tag{74}$$

yielding

$$\boldsymbol{\mu} = \boldsymbol{M} \otimes \boldsymbol{D}. \tag{75}$$

The equilibrium equations for this model follow simply from (65) and (66). To facilitate comparison with Kirchhoff's theory we use

$$Div(\boldsymbol{\lambda} \otimes \boldsymbol{D}) = \boldsymbol{\lambda}' + (Div\boldsymbol{D})\boldsymbol{\lambda}, \quad \text{where} \quad \boldsymbol{\lambda}' = (\nabla\boldsymbol{\lambda})\boldsymbol{D} \tag{76}$$

is the fiber-derivative of $\boldsymbol{\lambda}$, the directional derivative along the fiber passing through the point with reference position $\boldsymbol{X}$. Further,

$$\begin{aligned} 2ax\{\boldsymbol{R}Skw[(\boldsymbol{\Lambda} \otimes \boldsymbol{D})\boldsymbol{E}^t]\boldsymbol{R}^t\} &= 2ax\{Skw[\boldsymbol{R}(\boldsymbol{\Lambda} \otimes \boldsymbol{D})\boldsymbol{E}^t\boldsymbol{R}^t]\} \\ &= 2ax[Skw(\boldsymbol{\lambda} \otimes \boldsymbol{F}\boldsymbol{D})] \\ &= ax(\boldsymbol{\lambda} \otimes \boldsymbol{\chi}' - \boldsymbol{\chi}' \otimes \boldsymbol{\lambda}) \\ &= \boldsymbol{\chi}' \times \boldsymbol{\lambda}, \end{aligned} \tag{77}$$

where

$$\boldsymbol{\chi}' = (\nabla\boldsymbol{\chi})\boldsymbol{D} \tag{78}$$

is the fiber derivative of the deformation. Lastly,

$$Div(\boldsymbol{R}\boldsymbol{\mu}) = \boldsymbol{m}' + (Div\boldsymbol{D})\boldsymbol{m}, \quad \text{where} \quad \boldsymbol{m} = \boldsymbol{R}\boldsymbol{M} = m_i \boldsymbol{d}_i, \tag{79}$$

with fiber derivative $\boldsymbol{m}' = (\nabla\boldsymbol{m})\boldsymbol{D}$. The equilibrium conditions (65) holding in κ thus specialize to

$$\boldsymbol{\lambda}' + (Div\boldsymbol{D})\boldsymbol{\lambda} + Div(\boldsymbol{R}\sigma) + \boldsymbol{g} = \boldsymbol{0} \tag{80}$$

and

$$\boldsymbol{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} + (Div\boldsymbol{D})\boldsymbol{m} + 2ax[\boldsymbol{R}Skw(\sigma\boldsymbol{E}^t)\boldsymbol{R}^t] + \boldsymbol{\pi} = \boldsymbol{0}, \tag{81}$$

with
$$\boldsymbol{t} = (\boldsymbol{R}\boldsymbol{\sigma})\boldsymbol{\nu} + (\boldsymbol{D}\cdot\boldsymbol{\nu})\boldsymbol{\lambda} \quad \text{and} \quad \boldsymbol{c} = (\boldsymbol{D}\cdot\boldsymbol{\nu})\boldsymbol{m} \tag{82}$$
holding on $\partial\kappa_t$ and $\partial\kappa_c$, respectively. From (35) and (36) these conditions yield the interpretation of $\boldsymbol{\lambda}$ and $\boldsymbol{m}$ respectively as shear force and moment densities acting on fiber 'cross sections', i.e., on surfaces that intersect fibers orthogonally ($\boldsymbol{D}\cdot\boldsymbol{\nu} = \pm 1$). We observe that no solution exists if a non-zero couple is specified on a part of $\partial\kappa_c$ containing $\boldsymbol{D}$ as a tangent vector ($\boldsymbol{D}\cdot\boldsymbol{\nu} = 0$). Additionally, comparison of (80) with (35) furnishes the interpretation of $Div(\boldsymbol{R}\boldsymbol{\sigma})$ as a distributed density of force transmitted to a fiber by the matrix in which it is embedded, whereas comparison of (81) and (35) implies that $2ax[\boldsymbol{R}Skw(\boldsymbol{\sigma}\boldsymbol{E}^t)\boldsymbol{R}^t]$ is a density of distributed couple transmitted by the matrix to the fiber. The derivation of the system (80)–(82) simplifies and and generalizes that of a similar system for initially straight fibers presented in [1,2].

The dependence of the strain-energy function on $\boldsymbol{\gamma}$ (or $\boldsymbol{\Gamma}$) introduces a natural length scale, L say, into the constitutive theory which is on the order of that of the microstructure and hence of the diameter of a fiber cross section or the spacing between adjacent fibers. Using the larger of these to define the dimensionless curvature-twist vector $L\boldsymbol{\gamma}$, supposing that $|L\boldsymbol{\gamma}| \ll 1$ in typical applications and assuming that the fibers transmit no moments when $\boldsymbol{\gamma}$ vanishes, we find that w is given to leading order by
$$w(\boldsymbol{E},\boldsymbol{\gamma};\boldsymbol{X}) = \varpi(\boldsymbol{E};\boldsymbol{X}) + \frac{1}{2}\boldsymbol{\gamma}\cdot\boldsymbol{K}(\boldsymbol{E};\boldsymbol{X})\boldsymbol{\gamma}, \tag{83}$$
where $\varpi(\boldsymbol{E};\boldsymbol{X}) = w(\boldsymbol{E},\boldsymbol{0};\boldsymbol{X})$ and $\boldsymbol{K}(\boldsymbol{E};\boldsymbol{X}) = w_{\boldsymbol{\gamma}\boldsymbol{\gamma}|\boldsymbol{\gamma}=\boldsymbol{0}}$. For small $\boldsymbol{E}$ we have $\boldsymbol{K}(\boldsymbol{E};\boldsymbol{X}) = \boldsymbol{K}(\boldsymbol{0};\boldsymbol{X}) + O(|\boldsymbol{E}|)$, provided that $\boldsymbol{K}(\cdot;\boldsymbol{X})$ is differentiable. Then the energy is approximated, as in (18), by the decoupled energy
$$w(\boldsymbol{E},\boldsymbol{\gamma};\boldsymbol{X}) = \varpi(\boldsymbol{E};\boldsymbol{X}) + \varphi(\boldsymbol{\gamma};\boldsymbol{X}), \tag{84}$$
for some homogeneous quadratic function $\varphi(\cdot;\boldsymbol{X})$.

4. Material Symmetry

In this section we develop the theory of material symmetry for elastic Cosserat materials subjected to (40). A comprehensive study of material symmetry in the setting of Cosserat elasticity, extending Noll's concept [17] for simple materials, is given in [16]. The concept of material symmetry in rods is discussed in [20–22]. As a preliminary step we first describe the manner in which the constitutive function for the strain energy may be computed for any choice of reference configuration when that pertaining to any particular choice is given.

4.1. Change of Reference Configuration

Let κ and μ be two reference configurations, and let $\boldsymbol{Y} = \boldsymbol{\Pi}(\boldsymbol{X})$ be a diffeomorphism mapping points in κ to points in μ. The deformation gradients relative to κ and μ, denoted by $\boldsymbol{F}_\kappa$ and $\boldsymbol{F}_\mu$ respectively, are related by
$$\boldsymbol{F}_\kappa = \boldsymbol{F}_\mu\boldsymbol{H}, \quad \text{where} \quad \boldsymbol{H} = \nabla\boldsymbol{\Pi}. \tag{85}$$

We restrict attention to transformations $\boldsymbol{\Pi}$ with $\det\boldsymbol{H} = 1$, for reasons that are well known in conventional elasticity [8,16], and also impose $\boldsymbol{\Pi}(\boldsymbol{X}_0) = \boldsymbol{X}_0$. The specification of this pivot point removes an inessential translational degree of freedom. Henceforth, we are concerned with the properties of the map $\boldsymbol{\Pi}(\boldsymbol{X})$ in a neighborhood $N_\kappa(\boldsymbol{X}_0) \subset \kappa$ of the pivot point. This neighborhood is mapped by $\boldsymbol{\Pi}$ to the neighborhood $N_\mu(\boldsymbol{X}_0)$ of the pivot.

The Cosserat rotation $\boldsymbol{R}_\kappa$ relative to κ is such that $\boldsymbol{d}_i = \boldsymbol{R}_\kappa\boldsymbol{D}_i$. In the same way there is a rotation $\boldsymbol{R}_\mu$ such that $\boldsymbol{d}_i = \boldsymbol{R}_\mu\boldsymbol{G}_i$, where $\{\boldsymbol{G}_i(\boldsymbol{Y})\}$ is the positively-oriented orthonormal director field defined in μ. Thus,
$$\boldsymbol{R}_\kappa = \boldsymbol{R}_\mu\boldsymbol{L}, \tag{86}$$

where

$$\boldsymbol{L} = \boldsymbol{G}_i \otimes \boldsymbol{D}_i \tag{87}$$

is the rotation field that maps the directors in κ to their images in μ. We have $\boldsymbol{d} = \boldsymbol{R}_\kappa \boldsymbol{D} = \boldsymbol{R}_\mu \boldsymbol{G}$, where $\boldsymbol{G}(= \boldsymbol{G}_1)$ is the unit-tangent field to fibers in μ, so that $\boldsymbol{G} = \boldsymbol{L}\boldsymbol{D}$. To ensure that $\boldsymbol{D}$ remains a material vector relative to the matrix (cf. (39)) under the change of reference, it is necessary to impose

$$\boldsymbol{H}\boldsymbol{D} = |\boldsymbol{H}\boldsymbol{D}|\, \boldsymbol{L}\boldsymbol{D}. \tag{88}$$

Following the characterization of solids in [8], we assume the existence of an undistorted reference and suppose κ to be one of these. Thus we confine attention to proper-orthogonal $\boldsymbol{H}$. Further, we remove an inessential orientational degree of freedom in the local change of reference by requiring that it preserve the pivotal axis $\boldsymbol{D}$ at the point $\boldsymbol{X}_0$; thus, $|\boldsymbol{H}\boldsymbol{D}| = 1$ and

$$\boldsymbol{D} = \boldsymbol{H}\boldsymbol{D} = \boldsymbol{L}\boldsymbol{D}. \tag{89}$$

With $\boldsymbol{G}_\alpha = \boldsymbol{L}\boldsymbol{D}_\alpha$ and $\boldsymbol{G} = \boldsymbol{H}\boldsymbol{D}$, this in turn implies that

$$\boldsymbol{G}_\alpha \cdot \boldsymbol{G} = \boldsymbol{L}\boldsymbol{D}_\alpha \cdot \boldsymbol{H}\boldsymbol{D} = \boldsymbol{L}\boldsymbol{D}_\alpha \cdot \boldsymbol{L}\boldsymbol{D} = \boldsymbol{D}_\alpha \cdot \boldsymbol{D}, \tag{90}$$

and hence that the constraints $\boldsymbol{G}_\alpha \cdot \boldsymbol{G} = 0$ are automatically satisfied (cf. (40)). Accordingly, $\boldsymbol{L}, \boldsymbol{H} \in S$, where

$$S = \{\boldsymbol{Q} \mid \ \boldsymbol{Q} \text{ is a rotation with axis } \boldsymbol{D}\}. \tag{91}$$

For the model discussed in Section 3, the strain energy depends on the Cosserat rotation and its gradient via $\boldsymbol{\gamma} = \gamma_i \boldsymbol{D}_i$, where γ_i is given by (67) in which the prime refers to the fiber derivative in the reference configuration κ. In particular, for any function f we have $f' = \nabla_\kappa f \cdot \boldsymbol{D} = \boldsymbol{H}^t(\nabla_\mu f) \cdot \boldsymbol{D} = \nabla_\mu f \cdot \boldsymbol{H}\boldsymbol{D}$, where the subscripts κ and μ identify gradients with respect to $\boldsymbol{X} \in \kappa$ and $\boldsymbol{Y} \in \mu$, respectively. In view of (89) we have $f' = \nabla_\mu f \cdot \boldsymbol{D}$ at the pivot $\boldsymbol{X}_0$, implying that the fiber derivative is invariant under transformations of the reference configuration that preserve the fiber axis. Accordingly, it is immediately apparent that the κ_i, defined by (7) in which the prime is again a fiber derivative, are also invariant.

Alternatively, we may use (67) to derive

$$\begin{aligned}(\kappa_i)_\kappa - (\kappa_i^0)_\kappa &= (\gamma_i)_\kappa \\ &= \frac{1}{2} e_{ijk} \boldsymbol{D}_k \cdot \boldsymbol{R}_\kappa^t \boldsymbol{R}_\kappa' \boldsymbol{D}_j,\end{aligned} \tag{92}$$

in which

$$\boldsymbol{R}_\kappa^t \boldsymbol{R}_\kappa' = \boldsymbol{L}^t(\boldsymbol{R}_\mu^t \boldsymbol{R}_\mu' + \boldsymbol{L}'\boldsymbol{L}^t)\boldsymbol{L}. \tag{93}$$

Thus,

$$(\kappa_i)_\kappa - (\kappa_i^0)_\kappa = (\kappa_i)_\mu - (\kappa_i^0)_\mu + \Delta_i, \tag{94}$$

where

$$\begin{aligned}(\kappa_i)_\mu - (\kappa_i^0)_\mu &= (\gamma_i)_\mu \\ &= \frac{1}{2} e_{ijk} \boldsymbol{G}_k \cdot \boldsymbol{R}_\mu^t \boldsymbol{R}_\mu' \boldsymbol{G}_j\end{aligned} \tag{95}$$

and

$$\Delta_i = \frac{1}{2} e_{ijk} \boldsymbol{G}_k \cdot \boldsymbol{L}'\boldsymbol{L}^t \boldsymbol{G}_j. \tag{96}$$

Noting that

$$
\begin{aligned}
(\kappa_i^0)_\mu &= \frac{1}{2} e_{ijk} \boldsymbol{G}_k \cdot \boldsymbol{G}_j' \\
&= \frac{1}{2} e_{ijk} \boldsymbol{L}\boldsymbol{D}_k \cdot (\boldsymbol{L}'\boldsymbol{D}_j + \boldsymbol{L}\boldsymbol{D}_j') \\
&= \frac{1}{2} e_{ijk} \boldsymbol{G}_k \cdot \boldsymbol{L}'(\boldsymbol{L}^t \boldsymbol{G}_j) + (\kappa_i^0)_\kappa,
\end{aligned}
\tag{97}
$$

and hence that

$$(\kappa_i^0)_\mu = (\kappa_i^0)_\kappa + \Delta_i, \tag{98}$$

we conclude, in accordance with (94), that

$$(\kappa_i)_\kappa = (\kappa_i)_\mu, \tag{99}$$

as claimed.

From (67), the curvature-twist strains γ_κ and γ_μ relative to κ and μ are related by

$$\gamma_\kappa = \boldsymbol{L}^t(\gamma_\mu + \boldsymbol{\Delta}), \tag{100}$$

where

$$\boldsymbol{\Delta} = \Delta_i \boldsymbol{G}_i = ax(\boldsymbol{L}'\boldsymbol{L}^t), \tag{101}$$

whereas the Cosserat strains $\boldsymbol{E}_\kappa$ and $\boldsymbol{E}_\mu$ are related by

$$\boldsymbol{E}_\kappa = \boldsymbol{L}^t \boldsymbol{E}_\mu \boldsymbol{H}. \tag{102}$$

Because the state of the material is not affected by the choice of reference, we require

$$w_\mu(\boldsymbol{E}_\mu, \gamma_\mu; \boldsymbol{X}_0) = w_\kappa(\boldsymbol{E}_\kappa, \gamma_\kappa; \boldsymbol{X}_0) = w_\kappa(\boldsymbol{L}^t \boldsymbol{E}_\mu \boldsymbol{H}, \boldsymbol{L}^t(\gamma_\mu + \boldsymbol{\Delta}); \boldsymbol{X}_0). \tag{103}$$

4.2. Material Symmetry Transformations

According to Noll's theory [17], $N_\kappa(\boldsymbol{X}_0)$ and $N_\mu(\boldsymbol{X}_0)$ are related by material symmetry if their responses to a given experiment are identical at the pivot point. In the present context an experiment consists of a deformation function $\boldsymbol{\chi}(\cdot)$ and rotation function $\boldsymbol{R}(\cdot)$. Thus the experiment acts on $N_\kappa(\boldsymbol{X}_0)$ to produce the fields $\{\boldsymbol{\chi}(\boldsymbol{X}), \boldsymbol{R}(\boldsymbol{X})\}$ for $\boldsymbol{X} \in N_\kappa(\boldsymbol{X}_0)$, and on $N_\mu(\boldsymbol{X}_0)$ to produce $\{\boldsymbol{\chi}(\boldsymbol{Y}), \boldsymbol{R}(\boldsymbol{Y})\}$ for $\boldsymbol{Y} \in N_\mu(\boldsymbol{X}_0)$. Accordingly, $N_\kappa(\boldsymbol{X}_0)$ and $N_\mu(\boldsymbol{X}_0)$ are both subjected to the same pair $\{\boldsymbol{F}, \boldsymbol{R}\}$, and hence the same strain $\boldsymbol{E}$, at $\boldsymbol{X}, \boldsymbol{Y} = \boldsymbol{X}_0$. Moreover, $\nabla_\mu \boldsymbol{R} = (\nabla_\kappa \boldsymbol{R})\boldsymbol{H}$ and with (89) we infer that the fiber derivatives $(\boldsymbol{R}')_\kappa$ and $(\boldsymbol{R}')_\mu$, relative to $N_\kappa(\boldsymbol{X}_0)$ and $N_\mu(\boldsymbol{X}_0)$ respectively, also coincide at $\boldsymbol{X}_0$. This in turn implies that both neighborhoods experience the same bend-twist strain γ at $\boldsymbol{X}_0$, a fact that is most easily appreciated by using a single background frame $\{\boldsymbol{E}_i\}$ to evaluate

$$\gamma = \gamma_i \boldsymbol{E}_i, \quad \text{with} \quad \gamma_i = \frac{1}{2} e_{ijk} \boldsymbol{E}_k \cdot \boldsymbol{R}^t \boldsymbol{R}' \boldsymbol{E}_j, \tag{104}$$

in both neighborhoods.

Accordingly, material symmetry is tantamount to

$$w_\kappa(\boldsymbol{E}, \gamma; \boldsymbol{X}_0) = w_\mu(\boldsymbol{E}, \gamma; \boldsymbol{X}_0), \tag{105}$$

which, when combined with (100), (102) and (103), yields the restriction

$$w_\kappa(\boldsymbol{E}, \gamma; \boldsymbol{X}_0) = w_\kappa(\boldsymbol{L}^t \boldsymbol{E} \boldsymbol{H}, \boldsymbol{L}^t(\gamma + \boldsymbol{\Delta}); \boldsymbol{X}_0) \tag{106}$$

on the single response function w_κ, where the rotations $\boldsymbol{H}$ and $\boldsymbol{L}$ are connected by (89), but otherwise independent, and of course $\boldsymbol{\Delta}$ is evaluated at $\boldsymbol{X}_0$.

For the decoupled energy (84) considered hereafter, this is equivalent to

$$\varpi(\boldsymbol{E};\boldsymbol{X}_0)+\varphi(\gamma;\boldsymbol{X}_0)=\varpi(\boldsymbol{L}^t\boldsymbol{E}\boldsymbol{H};\boldsymbol{X}_0)+\varphi(\boldsymbol{L}^t(\gamma+\boldsymbol{\Delta});\boldsymbol{X}_0) \tag{107}$$

5. Examples

We close with some examples of constitutive functions that conform to (107).

5.1. Matrix Energy

Consider the list [1]

$$I=\{I_1,...,I_9\}, \tag{108}$$

of functionally independent scalar-valued functions of $\boldsymbol{E}$, with

$$\begin{aligned} I_1 &= tr(\boldsymbol{E}^t\boldsymbol{E}),\ I_2=tr[(\boldsymbol{E}^t\boldsymbol{E})^2],\ I_3=\det\boldsymbol{E},\ I_4=\boldsymbol{D}\cdot\boldsymbol{E}\boldsymbol{D},\ I_5=\boldsymbol{D}\cdot(\boldsymbol{E}^t\boldsymbol{E})\boldsymbol{D},\\ I_6 &= \boldsymbol{D}\cdot(\boldsymbol{E}\boldsymbol{E}^t)\boldsymbol{D},\ I_7=\boldsymbol{D}\cdot\boldsymbol{E}^*\boldsymbol{D},\ I_8=\boldsymbol{D}\cdot(\boldsymbol{E}^t\boldsymbol{E})^2\boldsymbol{D},\ I_9=\boldsymbol{D}\cdot(\boldsymbol{E}\boldsymbol{E}^t)^2\boldsymbol{D}, \end{aligned} \tag{109}$$

where $\boldsymbol{E}^*=(\det\boldsymbol{E})\boldsymbol{E}^{-t}$ is the cofactor of $\boldsymbol{E}$. We note that $\det\boldsymbol{E}=\det\boldsymbol{F}$, $\boldsymbol{E}^t\boldsymbol{E}=\boldsymbol{C}$ and $\boldsymbol{E}\boldsymbol{E}^t=\boldsymbol{R}^t\boldsymbol{B}\boldsymbol{R}$, where $\boldsymbol{C}=\boldsymbol{F}^t\boldsymbol{F}$ and $\boldsymbol{B}=\boldsymbol{F}\boldsymbol{F}^t$, respectively, are the right and left Cauchy–Green deformation tensors.

Straightforward calculations show, remarkably, that each member I_k of this list satisfies

$$I_k(\boldsymbol{L}^t\boldsymbol{E}\boldsymbol{H})=I_k(\boldsymbol{E}) \tag{110}$$

for any—hence every—$\boldsymbol{L},\boldsymbol{H}\in S$. Particular matrix energies may thus be obtained by taking

$$\varpi(\boldsymbol{E};\boldsymbol{X})=M(I_1,...,I_9;\boldsymbol{X}), \tag{111}$$

for some function M. This satisfies

$$\varpi(\boldsymbol{E};\boldsymbol{X})=\varpi(\boldsymbol{L}^t\boldsymbol{E}\boldsymbol{H};\boldsymbol{X}) \tag{112}$$

for all distinct $\boldsymbol{L},\boldsymbol{H}\in S$. For example, (111) furnishes energies for transversely hemitropic matrix materials for which $\boldsymbol{H}$ is an arbitrary element of S, without any restrictions on the independent fiber rotations $\boldsymbol{L}\in S$. However, we have not shown that I is a function basis for any particular kind of symmetry.

The stress $\boldsymbol{\sigma}$ associated with this energy is given by

$$\boldsymbol{\sigma}=\varpi_{\boldsymbol{E}}=\sum_j M_j(I_j)_{\boldsymbol{E}},\quad \text{where}\quad M_j=\partial M/\partial I_j \tag{113}$$

and [1]

$$\begin{aligned} (I_1)_{\boldsymbol{E}} &= 2\boldsymbol{E},\quad (I_2)_{\boldsymbol{E}}=4\boldsymbol{E}\boldsymbol{C},\quad (I_3)_{\boldsymbol{E}}=\boldsymbol{E}^*,\quad (I_4)_{\boldsymbol{E}}=\boldsymbol{D}\otimes\boldsymbol{D},\quad (I_5)_{\boldsymbol{E}}=2\boldsymbol{E}(\boldsymbol{D}\otimes\boldsymbol{D}),\\ (I_6)_{\boldsymbol{E}} &= 2(\boldsymbol{D}\otimes\boldsymbol{D})\boldsymbol{E},\quad (I_7)_{\boldsymbol{E}}=I_7\boldsymbol{E}^{-t}-I_3\boldsymbol{E}^{-t}(\boldsymbol{D}\otimes\boldsymbol{D})\boldsymbol{E}^{-t},\quad (I_8)_{\boldsymbol{E}}=2\boldsymbol{E}[(\boldsymbol{D}\otimes\boldsymbol{D})\boldsymbol{C}+\boldsymbol{C}(\boldsymbol{D}\otimes\boldsymbol{D})],\\ (I_9)_{\boldsymbol{E}} &= 2[(\boldsymbol{D}\otimes\boldsymbol{D})\boldsymbol{E}\boldsymbol{C}+\boldsymbol{E}\boldsymbol{E}^t(\boldsymbol{D}\otimes\boldsymbol{D})\boldsymbol{E}]. \end{aligned} \tag{114}$$

We observe that this model yields an asymmetric $\boldsymbol{\sigma}\boldsymbol{E}^t$, and thus makes provision for distributed couples to be transmitted to the fibers by the matrix (cf. (81)), if the energy involves I_4, I_6, I_7 or I_9.

5.2. Fiber Symmetry

For matrix energies satisfying (112), the restriction (107) reduces to

$$\varphi(\gamma; \mathbf{X}) = \varphi(\mathbf{L}^t(\gamma + \mathbf{\Delta}); \mathbf{X}) \tag{115}$$

for some $\mathbf{L}$ and $\mathbf{\Delta} = ax(\mathbf{L}'\mathbf{L}^t)$ such that $\mathbf{L}(\mathbf{X})\mathbf{D}(\mathbf{X}) = \mathbf{D}(\mathbf{X})$ (cf. (89)). The fiber derivative $\mathbf{L}'$ thus satisfies

$$\mathbf{L}'\mathbf{D} + \mathbf{L}\mathbf{D}' = \mathbf{D}', \tag{116}$$

and with $\mathbf{D} = \mathbf{L}^t\mathbf{D}$ this may be cast in the form

$$\mathbf{\Delta} \times \mathbf{D} + (\mathbf{L} - \mathbf{I})\mathbf{D}' = \mathbf{0}. \tag{117}$$

We then have

$$\mathbf{\Delta} = \Delta\mathbf{D} + \mathbf{D} \times (\mathbf{\Delta} \times \mathbf{D}) \tag{118}$$

in which $\Delta (= \mathbf{\Delta} \cdot \mathbf{D})$ is unrestricted, whereas

$$\mathbf{D} \times (\mathbf{\Delta} \times \mathbf{D}) = [(\mathbf{L} - \mathbf{I})\mathbf{D}'] \times \mathbf{D}. \tag{119}$$

Consider the particular fiber with arclength parametrization $\mathbf{X}(s)$ passing through the point $\mathbf{X}_0$ at $s = s_0$, i.e., $\mathbf{X}(s_0) = \mathbf{X}_0$. If the fiber is curved at $\mathbf{X}_0$ then $\mathbf{D}' = K\mathbf{N}$ there, where $K = |\mathbf{D}'| > 0$ is the principal curvature of the fiber and $\mathbf{N}$ is the unique principal normal. In this case the local Frenet triad $\{\mathbf{D}, \mathbf{N}, \mathbf{B}\}$, where $\mathbf{B} = \mathbf{D} \times \mathbf{N}$ is the binormal, is well defined. The Rodrigues representation formula for rotations [23] thus yields

$$\mathbf{L} = \mathbf{D} \otimes \mathbf{D} + \cos\theta(\mathbf{N} \otimes \mathbf{N} + \mathbf{B} \otimes \mathbf{B}) + \sin\theta(\mathbf{B} \otimes \mathbf{N} - \mathbf{N} \otimes \mathbf{B}) \tag{120}$$

for some angle θ, which may be used with (118) and (119) to construct

$$\mathbf{\Delta} = \Delta\mathbf{D} - K(\cos\theta - 1)\mathbf{B} + K\sin\theta\mathbf{N}, \tag{121}$$

and with some effort we derive

$$\mathbf{L}^t(\gamma + \mathbf{\Delta}) = (\gamma_1 + \Delta)\mathbf{D} + \mathbf{v}(\theta), \tag{122}$$

where

$$\mathbf{v}(\theta) = \cos\theta\mathbf{1}\gamma + \sin\theta\mathbf{1}\gamma \times \mathbf{D} + K(\cos\theta - 1)\mathbf{B} + K\sin\theta\mathbf{N} \tag{123}$$

in which

$$\mathbf{1} = \mathbf{I} - \mathbf{D} \otimes \mathbf{D} \tag{124}$$

is the projection onto the fiber cross section. The symmetry condition (115) thus becomes

$$\varphi(\gamma; \mathbf{X}) = \varphi((\gamma_1 + \Delta)\mathbf{D} + \mathbf{v}(\theta); \mathbf{X}). \tag{125}$$

If the fiber is initially straight, then K vanishes and $\mathbf{\Delta} = \Delta\mathbf{D}$, again with Δ arbitrary. The Frenet triad is not well defined in this case and so we use (120) with $\{\mathbf{N}, \mathbf{B}\}$ replaced by $\{\mathbf{D}_\alpha\}$, finding that (115) reduces to

$$\varphi(\gamma; \mathbf{X}) = \varphi((\gamma_1 + \Delta)\mathbf{D} + \gamma_2\mathbf{i}(\theta) + \gamma_3\mathbf{j}(\theta); \mathbf{X}), \tag{126}$$

where

$$\mathbf{i}(\theta) = \cos\theta\mathbf{D}_2 - \sin\theta\mathbf{D}_3 \quad \text{and} \quad \mathbf{j}(\theta) = \sin\theta\mathbf{D}_2 + \cos\theta\mathbf{D}_3 \tag{127}$$

in which θ differs from the angle appearing in (120).

For curved or straight fibers we differentiate (125) or (126) with respect to Δ, concluding that

$$\varphi_\gamma \cdot \boldsymbol{D} = 0, \tag{128}$$

and hence that the fiber is insensitive to the twist strain. In view of the fact that the energies associated with twist and bending of a typical fiber are comparable in magnitude (cf. (18)), we regard such a circumstance as unrealistic and therefore require that $\Delta = 0$. Then, $\boldsymbol{\Delta} = \boldsymbol{D} \times (\boldsymbol{\Delta} \times \boldsymbol{D})$ and (128) is not applicable. In the case of initially straight fibers this implies that $\boldsymbol{L}' = \boldsymbol{0}$, i.e., that $\boldsymbol{L}$ cannot vary along the fibers, although $\nabla \boldsymbol{L}$ need not vanish. A similar observation was made in [16] in the context of conventional Cosserat elasticity.

5.2.1. Transversely Hemitropic Fibers

Transversely hemitropic fibers are defined to be those for which (125) or (126) hold for arbitrary θ. For curved fibers we differentiate the first of these with respect to θ, obtaining

$$\varphi_\gamma \cdot \boldsymbol{v}'(\theta) = 0, \tag{129}$$

where

$$\boldsymbol{v}'(\theta) = \cos\theta(K\boldsymbol{B} + \mathbf{1}\boldsymbol{\gamma}) \times \boldsymbol{D} - \sin\theta(K\boldsymbol{B} + \mathbf{1}\boldsymbol{\gamma}) \tag{130}$$

is an arbitrary vector in the fiber cross section. Accordingly $\mathbf{1}(\varphi_\gamma)$ vanishes and the bending moments m_α vanish. We conclude that a curved fiber can be transversely hemitropic only if it has no flexural elasticity. It is therefore effectively a string with a possible twisting elasticity. Exceptionally, (129) imposes no restrictions on φ if $\mathbf{1}\boldsymbol{\gamma}$ is fixed at the value $-K\boldsymbol{B}$.

For initially straight fibers, we differentiate (126) with respect to θ and evaluate the result at $\theta = 0$, obtaining

$$\gamma_2 \partial\phi/\gamma_3 = \gamma_3 \partial\phi/\partial\gamma_2, \tag{131}$$

where $\phi(\gamma_1, \gamma_2, \gamma_3) = \varphi(\gamma_i \boldsymbol{D}_i; \boldsymbol{X})$. This implies that $\phi(\gamma_1, \gamma_2, \gamma_3) = \psi(\gamma_1, \sqrt{\gamma_2^2 + \gamma_3^2})$ for some function ψ. Moreover, the referential version of (4), $\boldsymbol{D}_i' = \boldsymbol{\kappa}^0 \times \boldsymbol{D}_i$, implies, for $i = 1$, that $K^2 = (\kappa_2^0)^2 + (\kappa_3^0)^2$ and hence that $\kappa_\alpha^0 = 0$ if the fibers are initially straight. Thus,

$$\varphi(\boldsymbol{\gamma}; \boldsymbol{X}) = \mathcal{F}(J_1, J_2; \boldsymbol{X}) \tag{132}$$

for some function $\mathcal{F}$, where

$$J_1 = \boldsymbol{\gamma} \cdot \boldsymbol{D} = \gamma_1 \quad \text{and} \quad J_2 = |\mathbf{1}\boldsymbol{\kappa}| = \sqrt{\kappa_2^2 + \kappa_3^2}. \tag{133}$$

Conversely, (132) satisfies the operative version of the restriction (115) in the present circumstances, namely

$$\varphi(\boldsymbol{\gamma}; \boldsymbol{X}) = \varphi(\boldsymbol{L}^t \boldsymbol{\gamma}; \boldsymbol{X}), \tag{134}$$

for all $\boldsymbol{L}(\boldsymbol{X}) \in \mathcal{S}$, and thus furnishes the general representation for initially straight, transversely hemitropic fibers with bending and twisting resistance.

Homogeneous quadratic energies of this kind are of the form

$$\varphi(\boldsymbol{\gamma}; \boldsymbol{X}) = \frac{1}{2} A(\boldsymbol{X}) J_1^2 + \frac{1}{2} B(\boldsymbol{X}) J_2^2, \tag{135}$$

which may be compared to the classical energy (18) for rods of circular cross section. The product $J_1 J_2$ is excluded because the replacement $\boldsymbol{\gamma} \to t\boldsymbol{\gamma}$ transforms $J_1 J_2$ to $t\,|t|\, J_1 J_2$. Accordingly, $J_1 J_2$ is not a homogeneous quadratic function of $\boldsymbol{\gamma}$.

To derive the response function $\boldsymbol{M}$ (cf. (74)) we require the gradients

$$(J_1)_{\gamma} = \boldsymbol{D} \quad \text{and} \quad (J_2)_{\gamma} = J_2^{-1}\mathbf{1}\boldsymbol{\kappa}. \tag{136}$$

Equations (74) and (135) then deliver

$$\boldsymbol{M} = \varphi_{\gamma} = A\gamma_1\boldsymbol{D} + B(\kappa_\alpha \boldsymbol{D}_\alpha) \tag{137}$$

and (cf. (79))

$$\boldsymbol{m} = A\gamma_1\boldsymbol{d} + B\boldsymbol{d} \times \boldsymbol{d}', \tag{138}$$

where $\boldsymbol{d}' = (\nabla \boldsymbol{d})\boldsymbol{D}$.

5.2.2. Transversely Orthotropic Fibers

Transversely orthotropic fibers are defined to be those for which (125) or (126) are satisfied with $\theta = 0$ and $\theta = \pi$. This models fibers having rectangular cross sections on the microscale. The first alternative corresponds to $\boldsymbol{L}(\boldsymbol{X}) \equiv \boldsymbol{I}$ and $\boldsymbol{\Delta} = \boldsymbol{0}$, for which (115) reduces to an identity. For initially curved fibers, the second alternative, $\theta = \pi$, reduces (125) to the severe restriction

$$\varphi(\gamma_1\boldsymbol{D} + \mathbf{1}\boldsymbol{\gamma}; \boldsymbol{X}) = \varphi(\gamma_1\boldsymbol{D} - \mathbf{1}\boldsymbol{\gamma} - 2K\boldsymbol{B}; \boldsymbol{X}) \tag{139}$$

on the bending response. For an arbitrarily shaped fiber this can be satisfied if the energy depends solely on the twist strain. Thus, as a practical matter, transverse orthotropy is meaningful for initially straight rods, for which (126) reduces to

$$\varphi(\gamma_1\boldsymbol{D} + \mathbf{1}\boldsymbol{\kappa}; \boldsymbol{X}) = \varphi(\gamma_1\boldsymbol{D} - \mathbf{1}\boldsymbol{\kappa}; \boldsymbol{X}), \tag{140}$$

or simply $\phi(\gamma_1, \kappa_2, \kappa_3) = \phi(\gamma_1, -\kappa_2, -\kappa_3)$. Then,

$$\varphi(\boldsymbol{\gamma}; \boldsymbol{X}) = \psi(\gamma_1, \delta_2, \delta_3; \boldsymbol{X}), \quad \text{where} \quad \delta_\alpha = \frac{1}{2}\kappa_\alpha^2, \tag{141}$$

and

$$\boldsymbol{m} = (\partial\psi/\partial\gamma_1)\boldsymbol{d} + (\partial\psi/\partial\delta_2)\kappa_2\boldsymbol{d}_2 + (\partial\psi/\partial\delta_3)\kappa_3\boldsymbol{d}_3. \tag{142}$$

The foregoing considerations lead to the conclusion that initially curved fibers exhibiting elastic resistance to twisting and bending can be expected to have only trivial symmetry, i.e., $\boldsymbol{L}(\boldsymbol{X}) \equiv \boldsymbol{I}$. This conclusion, perhaps unexpected, is due to the fact that in the present theory the initial shape of a fiber contributes to its material properties through the presence of $\boldsymbol{\kappa}^0$ in the strain-energy function. To illustrate the point, imagine a fiber in the shape of a circular arc. If the arc is rotated about its tangent at point $\boldsymbol{X}$ by π, say, then it becomes the mirror image of the original with respect to the plane containing the tangent and lying perpendicular to that containing the fiber. We would not expect both local configurations of the fiber—the one before the rotation and the one after—to exhibit identical response to the same (arbitrary) experiment, unless the fiber is a string with negligible flexural stiffness.

In [2], several boundary value problems are solved explicitly in the context of a simpler version of the present theory in which the fibers are initially straight, untwisted and parallel. These pertain to finite torsion of a cylinder and to finite bending and transverse shearing of a block.

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Appendix A

To verify (60) we combine (46) and (57), obtaining

$$\begin{aligned}
\dot{\Gamma}_{DC} &= \frac{1}{2}e_{BAD}(\dot{R}_{iA}R_{iB,C} + R_{iA}\dot{R}_{iB,C}) \\
&= \frac{1}{2}e_{BAD}[\Omega_{im}R_{mA}R_{iB,C} + R_{mA}(\Omega_{mj,C}R_{jB} + \Omega_{mj}R_{jB,C})] \\
&= \frac{1}{2}e_{BAD}R_{mA}[\Omega_{mi,C}R_{iB} + (\Omega_{mi} + \Omega_{im})R_{iB,C}] \\
&= \frac{1}{2}e_{BAD}R_{iB}R_{mA}\Omega_{mi,C}.
\end{aligned} \tag{A1}$$

On the other hand, because $\det \boldsymbol{R} = 1$ we have

$$e_{BAD}R_{iB}R_{mA}R_{jD} = e_{imj}, \tag{A2}$$

and hence

$$e_{BAC}R_{iB}R_{mA} = e_{imj}R_{jC}, \tag{A3}$$

where we have used $R_{jD}R_{jC} = \delta_{DC}$.

We obtain

$$\begin{aligned}
\dot{\Gamma}_{DC} &= \frac{1}{2}e_{BAD}R_{iB}R_{mA}\Omega_{mi,C} \\
&= R_{jD}\omega_{j,C},
\end{aligned} \tag{A4}$$

where

$$\omega_j = \frac{1}{2}e_{imj}\Omega_{mi} \tag{A5}$$

are the components of $\boldsymbol{\omega} = ax\boldsymbol{\Omega}$, and thus confirm that $\dot{\boldsymbol{\Gamma}} = \boldsymbol{R}^t\nabla\boldsymbol{\omega}$.

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