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Comparison of Noether Symmetries and First Integrals of Two-Dimensional Systems of Second Order Ordinary Differential Equations by Real and Complex Methods

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Received: 27 June 2019; Accepted: 19 August 2019; Published: 17 September 2019

Abstract: Noether symmetries and first integrals of a class of two-dimensional systems of second order ordinary differential equations (ODEs) are investigated using real and complex methods. We show that first integrals of systems of two second order ODEs derived by the complex Noether approach cannot be obtained by the real methods. Furthermore, it is proved that a complex method can be extended to larger systems and higher order.

Keywords: systems of ODEs; Noether operators; Noether symmetries; first integrals

1. Introduction

Lie developed a symmetry method for solving differential equations (DEs) [1–4]. Noether [5] used these methods to prove that, for DEs obtained from a variational principle, for each symmetry generator there is a corresponding invariant, first integral. These symmetries are called Noether and, if they exist, then Noether's theorem readily provides the associated first integrals. Since they provide a double reduction of the order of the equation, and a sufficient number can actually be used to solve the equation, it is worthwhile to obtain them. Furthermore, they are useful for studying the physical aspects of the dynamical systems, like time translational symmetry gives energy conservation, spatial translation provides momentum conservation and rotational symmetry implies conservation of angular momentum. For a scalar ODE, the corresponding Lagrangian has a five-dimensional maximal Noether symmetry algebra, as guaranteed by a theorem [6], and all the lower dimensions (obviously except 4).

Though Lie methods involved complex functions of complex variables, they did not make explicit use of the Cauchy–Riemann (CR) equations. These conditions provide an auxiliary system of DEs satisfied by the corresponding system of DEs obtained by splitting the complex functions of the scalar or systems of DEs into the two real ones. One obtains either a system of partial differential equations (PDEs), if the independent variable is complex or a system of ODEs if it is real. The explicit use of complex functions of complex or real variables is demonstrated in [7–10] where solvability of systems of DEs is achieved through Noether symmetries and corresponding first integrals. Furthermore, by employing complex symmetry procedures: the energy stored in the field of a coupled harmonic oscillator was studied in [11] and linearizability of systems of two second order ODEs was addressed in [12,13]. The complex procedure, indeed, has been extended to higher dimensional systems of second order ODEs [14] and two-dimensional, systems of third order ODEs [15].

In this paper, we extend the use of complex symmetry methods further to obtain invariants of systems of ODEs and demonstrate that we can obtain new invariants not obtainable by the usual, non-complex, methods. The new invariants for systems arise due to complex Lagrangians and first integrals of the base ODEs involving complex dependent functions of the real independent variables. Complex symmetries have already been used to construct first integrals through Noether symmetries and derive invariants for two-dimensional, systems of second order ODEs [8–10]. We first compare the usual (real) and complex Noether approaches developed to derive first integrals for systems of two second order ODEs. We find that the latter yields more first integrals than the former for these systems. The first integrals derived using a complex procedure also satisfy the conditions of the real Noether's theorem that exists for systems of ODEs. Next, we prove that Lagrangians and corresponding first integrals of the complex scalar ODEs will always split into two real Lagrangians and first integrals for the corresponding system of two equations. For this purpose, we use the CR-equations, which are satisfied by the Lagrangians and first integrals provided by the complex procedure. Furthermore, we show that the complex Noether symmetries do not, in general, split into two Noether symmetries of the corresponding systems. The thrust is not to find directly applicable invariants, which could turn up but to demonstrate how a complex method can provide new invariants and insights into Noether symmetries and first integrals. This work also suggests that the class of systems presented here should, indeed, be singled out when classifying systems of ODEs on the basis of their Noether symmetries and first integrals as it may not follow the classifications presented by employing real symmetry methods. Theorems and their proofs in the later part of this paper show that the method adopted here can trivially be extended to higher dimensions and order of ODEs.

The plan of the paper is as follows: the next section gives the procedures to derive Noether symmetries, operators and corresponding first integrals for systems of two second order ODEs. In the third section, we obtain Noether symmetries and first integrals for two-dimensional, systems of second order ODEs using a real symmetry method. In the subsequent section, first integrals for these systems are derived by employing complex procedures. We end with a concluding section, which also gives the proofs of the claims given in the previous section.

2. Preliminaries

For a system of two coupled (in general) nonlinear ODEs

$$y'' = S_1(x,y,z,y',z'), \quad z'' = S_2(x,y,z,y',z'), \tag{1}$$

where prime denotes derivative with respect to x, and the point symmetry generator is

$$\mathbf{X} = \xi(x,y,z)\partial_x + \eta_1(x,y,z)\partial_y + \eta_2(x,y,z)\partial_z, \tag{2}$$

where ξ, η_1, and η_2, are the functions that appear in the infinitesimal coordinate transformations of the dependent and independent variables, $\partial_x = \partial/\partial x$, etc. The first extension of $\mathbf{X}$ is

$$\mathbf{X}^{[1]} = \mathbf{X} + \left(\frac{d}{dx}\eta_1 - y'\frac{d}{dx}\xi\right)\partial_{y'} + \left(\frac{d}{dx}\eta_2 - z'\frac{d}{dx}\xi\right)\partial_{z'}, \tag{3}$$

where $d/dx = \partial_x + y'\partial_y + z'\partial_z + \cdots$. If system (1) admits a Lagrangian $L(x,y,z,y',z')$, then it is equivalent to the Euler–Lagrange equations

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0, \quad \frac{d}{dx}\left(\frac{\partial L}{\partial z'}\right) - \frac{\partial L}{\partial z} = 0. \tag{4}$$

The vector field (2) is called a *Noether symmetry generator* corresponding to the Lagrangian $L(x,y,z,y',z')$ for system (1) if there exists a *gauge function* $B(x,y,z)$, such that

$$\mathbf{X}^{[1]}(L) + D(\xi)L = D(B), \tag{5}$$

where D is the total differentiation operator defined by

$$D = \partial_x + y'\partial_y + z'\partial_z + y''\partial_{y'} + z''\partial_{z'} + \cdots . \tag{6}$$

Theorem 1. *If* $\mathbf{X}$ *is a Noether point symmetry generator corresponding to a Lagrangian* $L(x,y,z,y',z')$ *of* (1)*, then the corresponding first integral is:*

$$I = \xi L + (\eta_1 - \xi y')\frac{\partial L}{\partial y'} + (\eta_2 - \xi z')\frac{\partial L}{\partial z'} - B. \tag{7}$$

For a first integral I, of system (1), the following equations

$$\mathbf{X}^{[1]} I = 0, \tag{8}$$

$$DI = 0, \tag{9}$$

where $\mathbf{X}^{[1]}$, and D, given in (3) and (6), are satisfied identically. Though the construction of the variational form of (1) along with Noether symmetries to determine the conserved quantities is nontrivial, the complex method converts a class of systems (1) into variational form trivially [8,10,11], which is obtainable from a single (base) scalar complex equation $u'' = S(x,u,u')$. This class is derived by considering $u(x) = y(x) + \imath z(x)$, and $S(x,u,u') = S_1(x,y,z,y',z') + \imath S_2(x,y,z,y',z')$. Such system admits a pair of Lagrangians $L_1(x,y,z,y',z')$ and $L_2(x,y,z,y',z')$ as the Lagrangian $L(x,u,u')$, of the complex base equation also involves the complex function $u(x)$ and its derivative, hence $L = L_1 + \imath L_2$. With these assumptions, (1) can be obtained from

$$\frac{\partial L_1}{\partial y} + \frac{\partial L_2}{\partial z} - \frac{d}{dx}\left(\frac{\partial L_1}{\partial y'} + \frac{\partial L_2}{\partial z'}\right) = 0, \quad \frac{\partial L_2}{\partial y} - \frac{\partial L_1}{\partial z} - \frac{d}{dx}\left(\frac{\partial L_2}{\partial y'} - \frac{\partial L_1}{\partial z'}\right) = 0. \tag{10}$$

These are obtained by splitting the complex Euler–Lagrange equation of the scalar complex second order ODEs. They are different from (4); however, in the later part of this work, their reduction to (4) is done. The operators

$$\begin{aligned} \mathbf{X}^{[1]} &= \xi_1\partial_x + \tfrac{1}{2}(\eta_1\partial_y + \eta_2\partial_z + \eta_1'\partial_{y'} + \eta_2'\partial_{z'}), \\ \mathbf{Y}^{[1]} &= \xi_2\partial_x + \tfrac{1}{2}(\eta_2\partial_y - \eta_1\partial_z + \eta_2'\partial_{y'} - \eta_1'\partial_{z'}) \end{aligned} \tag{11}$$

are said to be *Noether operators* corresponding to $L_1(x,y,z,y',z')$ and $L_2(x,y,z,y',z')$ of (1), if there exist gauge functions $B_1(x,y,z)$, and $B_2(x,y,z)$, such that

$$\begin{aligned} \mathbf{X}^{[1]}L_1 - \mathbf{Y}^{[1]}L_2 + (D\xi_1)L_1 - (D\xi_2)L_2 &= DB_1, \\ \mathbf{X}^{[1]}L_2 + \mathbf{Y}^{[1]}L_1 + (D\xi_1)L_2 + (D\xi_2)L_1 &= DB_2. \end{aligned} \tag{12}$$

Theorem 2. *If* $\mathbf{X}^{[1]}$ *and* $\mathbf{Y}^{[1]}$ *are Noether operators corresponding to the Lagrangians* $L_1(x,y,z,y',z')$ *and* $L_2(x,y,z,y',z')$ *of* (1)*, then the first integrals for* (1) *are*

$$\begin{aligned} I_1 = {} & \xi_1 L_1 - \xi_2 L_2 + \tfrac{1}{2}(\eta_1 - y'\xi_1 + z'\xi_2)(\tfrac{\partial L_1}{\partial y'} + \tfrac{\partial L_2}{\partial z'}) \\ & - \tfrac{1}{2}(\eta_2 - y'\xi_2 - z'\xi_1)(\tfrac{\partial L_2}{\partial y'} - \tfrac{\partial L_1}{\partial z'}) - B_1, \end{aligned} \tag{13}$$

$$I_2 = \xi_1 L_2 + \xi_2 L_1 + \tfrac{1}{2}(\eta_1 - y'\xi_1 + z'\xi_2)(\tfrac{\partial L_2}{\partial y'} - \tfrac{\partial L_1}{\partial z'}) \\ + \tfrac{1}{2}(\eta_2 - y'\xi_2 - z'\xi_1)(\tfrac{\partial L_1}{\partial y'} + \tfrac{\partial L_2}{\partial z'}) - B_2. \tag{14}$$

Theorem 3. *The first integrals I_1 and I_2, associated with the Noether operators $\boldsymbol{X}^{[1]}$ and $\boldsymbol{Y}^{[1]}$, satisfy*

$$\boldsymbol{X}^{[1]} I_1 - \boldsymbol{Y}^{[1]} I_2 = 0, \quad \boldsymbol{X}^{[1]} I_2 + \boldsymbol{Y}^{[1]} I_1 = 0, \tag{15}$$

and

$$D_1 I_1 - D_2 I_2 = 0, \quad D_1 I_2 + D_2 I_1 = 0, \tag{16}$$

where $D_1 = \partial_x + \frac{1}{2}(y'\partial_y + z'\partial_z + y''\partial_{y'} + z''\partial_{z'} + \cdots)$, $D_2 = \partial_x + \frac{1}{2}(z'\partial_y - y'\partial_z + z''\partial_{y'} - y''\partial_{z'} + \cdots)$.

3. Noether Symmetries and Corresponding First Integrals

In this section, we reconsider a class of two-dimensional, systems of second order ODEs that is solved using complex methods [13]. There it was shown that, for this class of systems, dimensions of the Lie point symmetry algebra remain less than 5, while the base complex equations in most of the cases possess an eight-dimensional Lie and a five-dimensional Noether algebra. This will help us here in showing that the number of first integrals for such systems using the real Noether approach remains less than that generated through the complex procedures. This class of systems of two second order cubically semi-linear ODEs reads as:

$$y'' = A_{10}y'^3 - 3A_{20}y'^2z' - 3A_{10}y'z'^2 + A_{20}z'^3 + B_{10}y'^2 - 2B_{20}y'z' - B_{10}z'^2 + C_{10}y' - C_{20}z' + D_{10}, \\ z'' = A_{20}y'^3 + 3A_{10}y'^2z' - 3A_{20}y'z'^2 - A_{10}z'^3 + B_{20}y'^2 + 2B_{10}y'z' - B_{20}z'^2 + C_{20}y' + C_{10}z' + D_{20}, \tag{17}$$

where $A_{j0}, B_{j0}, C_{j0}, D_{j0}$, $(j = 1, 2)$, are analytic functions of x, y, and z. In order to apply the complex Noether approach, we establish correspondence of the above system with the complex scalar second order ODE

$$u'' = A_0(x,u)u'^3 + B_0(x,u)u'^2 + C_0(x,u)u' + D_0(x,u), \tag{18}$$

by considering $y(x) + \imath z(x) = u(x)$, $A_{10} + \imath A_{20} = A_0$, $B_{10} + \imath B_{20} = B_0$, $C_{10} + \imath C_{20} = C_0$, and $D_{10} + \imath D_{20} = D_0$.

Example 1. *The system of two second order quadratically semi-linear ODEs*

$$y'' = x(y'^2 - z'^2), \quad z'' = 2xy'z' \tag{19}$$

admits a three-dimensional Lie algebra spanned by the following symmetry generators

$$\boldsymbol{X}_1 = \partial_y, \;\; \boldsymbol{X}_2 = \partial_z, \;\; \boldsymbol{X}_3 = -x\partial_x + y\partial_y + z\partial_z, \tag{20}$$

and the Lagrangians

$$L_1 = x + \frac{x^2y'}{2} + \frac{1}{2}\ln(y'^2 - z'^2), \quad L_2 = \frac{x^2z'}{2} + \arctan(z', y'). \tag{21}$$

These Lagrangians yield the first integrals

$$I_1^r = \frac{1}{2}x^2 + \frac{y'}{y'^2 + z'^2}, \quad I_2^r = -\frac{z'}{y'^2 + z'^2}, \tag{22}$$

corresponding to $\boldsymbol{X}_1$, and $\boldsymbol{X}_2$.

Example 2. *For the system of two second order semi-linear ODEs,*

$$y'' = -3yy' + 3zz' - y^3 + 3yz^2, \quad z'' = -3yz' - 3zy' - 3y^2z + z^3, \tag{23}$$

there are two Lagrangians

$$L_1 = \frac{3y' + 3y^2 - 3z^2}{(3y' + 3y^2 - 3z^2)^2 + (3z' + 6yz)^2}, \quad L_2 = \frac{3z' + 6yz}{(3y' + 3y^2 - 3z^2)^2 + (3z' + 6yz)^2}, \tag{24}$$

coming from the base complex equation

$$u'' = -3uu' - u^3, \tag{25}$$

and its Lagrangian $L = \frac{1}{3(u'+u^2)}$. *System (23) has three Lie point symmetries*

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x\partial_x - y\partial_y - z\partial_z, \quad \mathbf{X}_3 = \frac{x^2}{2}\partial_x + (1 - xy)\partial_y - xz\partial_z. \tag{26}$$

Only one first integral for (23), corresponding to $\mathbf{X}_1$, *exists*

$$\begin{aligned} I_1^r \;&= \tfrac{1}{3\delta_1}\{y^6 + (z^2 + 4y')y^4 + 8y^3zz' + (5y'^2 - 8y'z^2 - z^4 + 3z'^2)y^2 - 8z(z^2 - \tfrac{1}{2}y')yz' \\ &-z^6 + 4z^4y' - (5y'^2 + 3z'^2)z^2 + 2y'z'^2 + 2y'^3\}, \end{aligned} \tag{27}$$

where

$$\delta_1 = (y^4 + (2z^2 + 2y')y^2 + 4yzz' + y'^2 + z^4 - 2y'z^2 + z'^2)^2. \tag{28}$$

Example 3. *For the system of two second order cubically semi-linear ODEs*

$$y'' = y'^3 - 3y'z'^2, \quad z'' = 3y'^2z' - z'^3, \tag{29}$$

the Lagrangians are

$$L_1 = 2y + \frac{y'}{y'^2 + z'^2}, \quad L_2 = 2z - \frac{z'}{y'^2 + z'^2}. \tag{30}$$

The above system possesses four Lie point symmetries

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_y, \quad \mathbf{X}_3 = \partial_z, \quad \mathbf{X}_4 = 2x\partial_x + y\partial_y + z\partial_z. \tag{31}$$

There are three gauge functions, $B_1 = C_1$, $B_2 = 2x$, $B_3 = C_2$, *for* $\mathbf{X}_1$, $\mathbf{X}_2$, *and* $\mathbf{X}_3$, *respectively, for* L_1. *Similarly,* L_2 *generates* $B_1 = C_3$, $B_2 = C_4$, $B_3 = 2x$, *with the same point symmetries as mentioned above. Thus, there is a three-dimensional Noether algebra for (29) associated with* L_1,

$$\begin{aligned} &I_1^r = 2y + \tfrac{2y'}{y'^2+z'^2} - C_1, \quad I_2^r = \tfrac{1}{y'^2+z'^2} - \tfrac{2y'^2}{(y'^2+z'^2)^2} - 2x, \\ &I_3^r = \tfrac{-2y'z'}{(y'^2+z'^2)^2} - C_2. \end{aligned} \tag{32}$$

Similarly, for L_2, *the first integrals are*

$$\begin{aligned} &I_1^r = 2z - \tfrac{2z'}{y'^2+z'^2} - C_3, \quad I_2^r = \tfrac{2y'z'}{(y'^2+z'^2)^2} - C_4, \\ &I_3^r = \tfrac{-1}{y'^2+z'^2} + \tfrac{2z'^2}{(y'^2+z'^2)^2} - 2x. \end{aligned} \tag{33}$$

Example 4. *A nonlinear system of two second order cubically semi-linear ODEs*

$$y'' = \alpha^2xy'^3 - 3\alpha^2xy'z'^2, \quad z'' = 3\alpha^2xy'^2z' - \alpha^2xz'^3, \tag{34}$$

where α is a constant and possesses the Lagrangians

$$L_1 = 2\alpha^2 xy + \frac{y'}{y'^2 + z'^2}, \;\; L_2 = 2\alpha^2 xy - \frac{z'}{y'^2 + z'^2}. \tag{35}$$

The above system (34) has three Lie point symmetries

$$\mathbf{X}_1 = \partial_y, \;\; \mathbf{X}_2 = \partial_z, \;\; \mathbf{X}_3 = x\partial_x. \tag{36}$$

For $\mathbf{X}_1$, and $\mathbf{X}_2$, we obtain gauge functions with L_1, and L_2 that are $B_1 = C_1 x^2$, $B_2 = C_2$, and $B_1 = C_2$, $B_2 = C_1 x^2$, respectively. Thus, two-dimensional Noether algebra is found to exist for (34) and the first integrals corresponding to L_1, and L_2, are

$$I_1^r = \frac{1}{y'^2 + z'^2} - \frac{2y'^2}{(y'^2 + z'^2)^2} - C_1 x^2, \quad I_2^r = \frac{-2y'z'}{(y'^2 + z'^2)^2} - C_2, \tag{37}$$

and

$$I_1^r = \frac{2y'z'}{(y'^2 + z'^2)^2} - C_2, \quad I_2^r = \frac{-1}{y'^2 + z'^2} + \frac{2z'^2}{(y'^2 + z'^2)^2} - C_1 x^2, \tag{38}$$

respectively.

Example 5. *The system of two second order cubically semi-linear ODEs*

$$\begin{aligned} y'' &= \alpha y y'^3 - 3\alpha z y'^2 z' - 3\alpha y y' z'^2 + \alpha z z'^3, \\ z'' &= \alpha z y'^3 + 3\alpha y y'^2 z' - 3\alpha z y' z'^2 - \alpha y z'^3 \end{aligned} \tag{39}$$

has the Lagrangians

$$L_1 = \alpha y^2 - \alpha z^2 + \frac{y'}{y'^2 + z'^2}, \quad L_2 = 2\alpha yz - \frac{z'}{y'^2 + z'^2}, \tag{40}$$

where α is a constant. This system admits a two-dimensional Lie point symmetry algebra

$$\mathbf{X}_1 = \partial_x, \;\; \mathbf{X}_2 = 3x\partial_x + y\partial_y + z\partial_z. \tag{41}$$

The gauge term for $\mathbf{X}_1$, with both L_1, L_2, is $B = C_1$. Thus, there is a one-dimensional Noether algebra that provides the following first integrals

$$\begin{aligned} I_1^r &= \alpha(y^2 - z^2) + \tfrac{2y'}{y'^2+z'^2} - C_1, \\ I_2^r &= 2\alpha yz - \tfrac{2z'}{y'^2+z'^2} - C_2. \end{aligned} \tag{42}$$

Example 6. *The system of two second order cubically semi-linear ODEs*

$$\begin{aligned} y'' &= xyy'^3 - 3xzy'^2 z' - 3xyy'z'^2 + xzz'^3, \\ z'' &= xzy'^3 + 3xyy'^2 z' - 3xzy'z'^2 - xyz'^3 \end{aligned} \tag{43}$$

has two Lagrangians

$$L_1 = xy^2 - xz^2 + \frac{y'}{y'^2 + z'^2}, \quad L_2 = 2xyz - \frac{z'}{y'^2 + z'^2}, \tag{44}$$

and one Lie point symmetry

$$\mathbf{X}_1 = x\partial_x. \tag{45}$$

There is no gauge function corresponding to $\mathbf{X}_1$, *for* L_1, *or* L_2, *which implies that there is a* 0*-dimensional Noether algebra. Hence, no first integral exists by a real method.*

4. Noether Operators and Corresponding First Integrals

In this section, we obtain first integrals for all systems considered in the previous section, by employing complex Noether procedure.

Example 7. *By considering* $y(x) + \imath z(x) = u(x)$, *system (19) corresponds to a scalar ODE*

$$u'' = xu'^2, \tag{46}$$

which has only one symmetry $\mathbf{Z}_1 = \partial_u$. *A complex Lagrangian,* $L = x + \frac{x^2 u'}{2} + \ln u'$, *is admitted by (46), yielding the first integral*

$$I_1 = \frac{x^2}{2} + \frac{1}{u'}. \tag{47}$$

This complex first integral splits into the following real first integrals

$$I_1^c = \frac{1}{2}x^2 + \frac{y'}{y'^2 + z'^2}, \quad I_2^c = -\frac{z'}{y'^2 + z'^2} \tag{48}$$

of system (19). Notice that both these first integrals are the same as those obtained earlier in (22) by the real method. It shows an agreement between the complex and real Noether approaches.

Example 8. *The base scalar ODE (25) has a five-dimensional Noether symmetry algebra spanned by*

$$\begin{array}{l} \mathbf{Z}_1 = \partial_x, \;\; \mathbf{Z}_2 = u\partial_x - u^3\partial_u, \;\; \mathbf{Z}_3 = xu\partial_x + (u^2 - xu^3)\partial_u, \\ \mathbf{Z}_4 = (x - \frac{3x^2u}{2})\partial_x + (2u - 3xu^2 + \frac{3x^2u^3}{2})\partial_u, \\ \mathbf{Z}_5 = (\frac{x^3u}{2} - \frac{x^2}{2})\partial_x + (1 - 2xu + \frac{3x^2u^2}{2} - \frac{x^3u^3}{2})\partial_u. \end{array} \tag{49}$$

The first integrals corresponding to the above Noether symmetries are

$$\begin{array}{l} I_1 = \frac{2u'+u^2}{3(u^2+u')^2}, \;\; I_2 = x - \frac{u}{u^2+u'}, \;\; I_3 = \frac{(-u+xu^2+xu')^2}{(u^2+u')^2}, \\ I_4 = \frac{1}{3}\frac{((u'+u^2)x-u)(2+(u'+u^2)x^2-2xu)}{(u^2+u')^2}, \\ I_5 = \frac{3(2-2xu+x^2u^2+x^2u')}{u^2+u'}. \end{array} \tag{50}$$

Putting $u(x) = y(x) + \imath z(x)$, *these complex first integrals split to provide the first integrals for the system (23)*

$$\begin{array}{l} I_1^c = \frac{1}{3}\frac{(2y'+y^2-z^2)J_1+(2z'+2yz)J_2}{J_1^2+J_2^2}, \;\; I_2^c = \frac{1}{3}\frac{(2z'+2yz)J_1-(2y'+y^2-z^2)J_2}{J_1^2+J_2^2}, \\ I_3^c = x - \frac{yJ_3-zJ_4}{J_3^2+J_4^2}, \;\; I_4^c = \frac{yJ_4-zJ_3}{J_3^2+J_4^2}, \;\; I_5^c = \frac{J_5J_6+J_7J_8}{J_6^2+J_8^2}, \;\; I_6^c = \frac{J_6J_7-J_5J_8}{J_6^2+J_8^2}, \\ I_7^c = \frac{1}{3}\frac{(J_9J_{10}-J_{11}J_{12})J_{13}+(J_{10}J_{11}+J_9J_{12})J_{14}}{J_{13}^2+J_{14}^2}, \\ I_8^c = \frac{1}{3}\frac{(J_{10}J_{11}+J_9J_{12})J_{13}-(J_9J_{10}-J_{11}J_{12})J_{14}}{J_{13}^2+J_{14}^2}, \\ I_9^c = 3\frac{J_3J_{15}+J_4J_{16}}{J_3^2+J_4^2}, \;\; I_{10}^c = 3\frac{J_3J_{16}-J_4J_{15}}{J_3^2+J_4^2}, \end{array} \tag{51}$$

where

$$
\begin{aligned}
J_1 &= -4yzz' - 6y^2z^2 + z^4 + y^4 + 2y^2y' - 2z^2y' + y'^2 - z'^2,\\
J_2 &= 4yzy' - 4yz^3 + 2y'z' + 4y^3z + 2y^2z' - 2z^2z',\\
J_3 &= y' + y^2 - z^2,\\
J_4 &= z' + 2yz,\\
J_5 &= -4x^2yzz' + 2xzz' + y^2 - z^2 + 6xyz^2 - 2xyy' - 6x^2y^2z^2 + 2x^2y^2y'\\
&\quad - 2x^2z^2y' - 2xy^3 + x^2y^4 + x^2z^4 + x^2y'^2 - x^2z'^2,\\
J_6 &= y^4 + z^4 + y'^2 - 6y^2z^2 - 2z^2y' + 2y^2y' - 4yzz' - z'^2,\\
J_7 &= 2x^2y'z' + 4x^2y^3z + 2x^2y^2z' - 2x^2z^2z' - 4x^2yz^3 - 2xzy' - 6xy^2z\\
&\quad - 2xyz' + 2yz + 2xz^3 + 4x^2yzy',\\
J_8 &= 4y^3z + 2y^2z' - 4yz^3 - 2z^2z' + 2y'z' + 4yzy',\\
J_9 &= (y^2 - z^2 + y')x - y,\\
J_{10} &= 2 + (y^2 - z^2 + y')x^2 - 2xy,\\
J_{11} &= (2yz + z')x - z,\\
J_{12} &= (2yz + z')x^2 - 2xz,\\
J_{13} &= -6y^2z^2 - 4yzz' + 2y^2y' - 2z^2y' + y'^2 - z'^2 + y^4 + z^4,\\
J_{14} &= -2z^2z' - 4yz^3 + 2y^2z' + 4y^3z + 4yzy' + 2y'z',\\
J_{15} &= 2 - 2xy + x^2(y^2 - z^2) + x^2y',\\
J_{16} &= -2xz + 2x^2yz + x^2z'.
\end{aligned}
\tag{52}
$$

In the following examples, we show that a complex symmetry approach provides 10 first integrals for systems of two second order ODEs. In particular, there is a system (43) that has a 0-dimensional Noether symmetry algebra, but 10 first integrals are generated by Noether operators obtained by complex methods.

Example 9. *The base complex scalar ODE*

$$u'' = u'^3, \tag{53}$$

for system (29) has the Lagrangian $L = 2u + \frac{1}{u'}$. For the five Lie point symmetries

$$
\begin{aligned}
&\mathbf{Z}_1 = \partial_x,\ \mathbf{Z}_2 = \partial_u,\ \mathbf{Z}_3 = u\partial_x,\ \mathbf{Z}_4 = (u^3 - 2xu)\partial_x - 2u^2\partial_u,\\
&\mathbf{Z}_5 = (3u^2 - 2x)\partial_x - 4u\partial_u
\end{aligned}
\tag{54}
$$

of the ODE (53), the gauge terms found are $B_1 = C$, $B_2 = 2x$, $B_3 = 2x + u^2$, $B_4 = \frac{3u^4}{2} - 2xu^2 - 2x^2$, $B_5 = 4u^3$, respectively. The corresponding first integrals are

$$
\begin{aligned}
&I_1 = 2u + \tfrac{2}{u'} - C,\ \ I_2 = \tfrac{-1}{u'^2} - 2x,\ \ I_3 = u^2 - 2x + \tfrac{2u}{u'},\\
&I_4 = \tfrac{1}{2u'^2}((u^2 - 2x)u' + 2u)^2,\ \ I_5 = 2u^3 - 4xu + \tfrac{(6u^2 - 4x)}{u'} + \tfrac{4u}{u'^2}.
\end{aligned}
\tag{55}
$$

Splitting (54) into real and imaginary parts yields the 10 Noether operators that provide the following first integrals:

$$
\begin{aligned}
&I_1^c = 2y + \tfrac{2y'}{y'^2 + z'^2} - C_1,\ \ I_2^c = 2z - \tfrac{2z'}{y'^2 + z'^2} - C_2,\\
&I_3^c = -2x - \tfrac{y'^2 - z'^2}{(y'^2 + z'^2)^2},\ \ I_4^c = \tfrac{2y'z'}{(y'^2 + z'^2)^2},\\
&I_5^c = y^2 - z^2 - 2x + 2\tfrac{yy' + zz'}{y'^2 + z'^2},\ \ I_6^c = 2\left(yz + \tfrac{y'z - yz'}{y'^2 + z'^2}\right),\\
&I_7^c = y^4 + z^4 - 6y^2z^2 + 4x^2 - 4x(y^2 - z^2) + 4\tfrac{(y^3 - 3yz^2 - 2xy)y' + (3y^2z - z^3 - 2xz)z'}{y'^2 + z'^2}\\
&\quad + 4\tfrac{(y^2 - z^2)(y'^2 - z'^2) + 4yzy'z'}{(y'^2 + z'^2)^2},
\end{aligned}
\tag{56}
$$

$$\begin{aligned}
I_8^c &= 4yz(y^2-z^2) - 8xyz + 4\tfrac{(3y^2z-z^3-2xz)y'-(y^3-3yz^2-2xy)z'}{y'^2+z'^2} \\
&\quad + 4\tfrac{2yz(y'^2-z'^2)-2(y^2-z^2)y'z'}{(y'^2+z'^2)^2}, \\
I_9^c &= y^3 - 3yz^2 - 2xy + \tfrac{(3y^2-3z^2-2x)y'+6yzz'}{y'^2+z'^2} + 2\tfrac{y(y'^2-z'^2)+2zy'z'}{(y'^2+z'^2)^2}, \\
I_{10}^c &= 3y^2z - z^3 - 2xz + \tfrac{6yzy'+(3z^2-3y^2+2x)z'}{y'^2+z'^2} + 2\tfrac{z(y'^2-z'^2)-2yy'z'}{(y'^2+z'^2)^2}.
\end{aligned} \tag{57}$$

Example 10. *System (34) is obtainable from a scalar second order complex ODE*

$$u'' = \alpha^2 x u'^3. \tag{58}$$

The Lagrangian associated with this equation is $L = 2\alpha^2 xu + \frac{1}{u'}$. The following gauge functions

$$\begin{aligned}
&B_1 = \alpha^2x^2,\ \ B_2 = 2\alpha^2xu\sin(\alpha u) + 2\alpha x\cos(\alpha u),\ \ B_3 = 2\alpha^2xu\cos(\alpha u) - 2\alpha x\sin(\alpha u), \\
&B_4 = \alpha^2x^2(2\alpha u\cos(2\alpha u) - \sin(2\alpha u)),\ \ B_5 = \alpha^2x^2(2\alpha u\sin(2\alpha u) + \cos(2\alpha u)),
\end{aligned} \tag{59}$$

correspond to the respective Lie point symmetries

$$\begin{aligned}
&\mathbf{Z}_1 = \partial_u,\ \ \mathbf{Z}_2 = \sin(\alpha u)\partial_x,\ \ \mathbf{Z}_3 = \cos(\alpha u)\partial_x, \\
&\mathbf{Z}_4 = \alpha x\cos(2\alpha u)\partial_x + \sin(2\alpha u)\partial_u,\ \ \mathbf{Z}_5 = \alpha x\sin(2\alpha u)\partial_x - \cos(2\alpha u)\partial_u.
\end{aligned} \tag{60}$$

Hence, there are five Noether symmetries and corresponding first integrals

$$\begin{aligned}
I_1 &= \tfrac{-1}{u'^2} - \alpha^2x^2, \\
I_2 &= 2\left(\tfrac{\sin(\alpha u)}{u'} - \alpha x\cos(\alpha u)\right), \\
I_3 &= 2\left(\tfrac{\cos(\alpha u)}{u'} + \alpha x\sin(\alpha u)\right), \\
I_4 &= \alpha^2x^2\sin(2\alpha u) - \tfrac{\sin(2\alpha u)}{u'^2} + 2\tfrac{\alpha x\cos(2\alpha u)}{u'}, \\
I_5 &= -\alpha^2x^2\cos(2\alpha u) + \tfrac{\cos(2\alpha u)}{u'^2} + 2\tfrac{\alpha x\sin(2\alpha u)}{u'}.
\end{aligned} \tag{61}$$

The real and imaginary parts of (61) yield 10*, first integrals for (34).*

Example 11. *The system (39) and Lagrangian (40) correspond to the complex scalar linearizable ODE*

$$u'' = \alpha u u'^3, \tag{62}$$

and Lagrangian $L = \alpha u^2 + \frac{1}{u'}$, which have the following gauge functions $B_1 = C$, $B_2 = 2x + \frac{\alpha u^3}{3}$, $B_3 = \frac{\alpha^2u^4}{2}$, $B_4 = \frac{2}{3}\alpha^2u^6 - \alpha xu^3 - 3x^2$,$B_5 = -3\alpha^2u^5$, for the following Lie point symmetries

$$\begin{aligned}
&\mathbf{Z}_1 = \partial_x,\ \ \mathbf{Z}_2 = u\partial_x,\ \ \mathbf{Z}_3 = \alpha u^2\partial_x - 2\partial_u, \\
&\mathbf{Z}_4 = (\alpha u^4 - 3xu)\partial_x - 3u^2\partial_u,\ \ \mathbf{Z}_5 = (-5\alpha u^3 + 6x)\partial_x + 12u\partial_u.
\end{aligned} \tag{63}$$

Thus, there are five complex first integrals

$$\begin{aligned}
&I_1 = \alpha u^2 + 2/u' - C,\ \ I_2 = \tfrac{1}{3}(2\alpha u^3 - 6x) + \tfrac{2u}{u'}, \\
&I_3 = \tfrac{(2+\alpha u^2u')^2}{2u'^2},\ \ I_4 = \tfrac{1}{3}\left(\tfrac{(\alpha u^3-3x)u'+3u}{u'}\right)^2, \\
&I_5 = -2\alpha^2u^5 + 6\alpha xu^2 - 2\tfrac{(5\alpha u^3-6x)}{u'} - 12\tfrac{u}{u'^2},
\end{aligned} \tag{64}$$

which split into 10 *real first integrals for system (39).*

Example 12. *System (43) is obtainable from the complex linearizable ODE*

$$u'' = xuu'^3, \tag{65}$$

with the Lagrangian $L = xu^2 + \frac{1}{u'}$. *It admits a five-dimensional Noether symmetry algebra, which yields 10 first integrals for system (43). In the previous section, we showed that there is no Noether symmetry for system (43) by real methods, but the complex method yields* 10 *first integrals.*

5. Conclusions

In this paper, we demonstrated, by considering explicit examples that the complex methods provide Noether invariants that do not appear by real methods. While the examples, in themselves, do prove the point, one would like to understand why this should be the case. For this purpose, we state and prove the following theorems that summarize our results and provide insight into how the complex methods work and go beyond the real methods.

Theorem 4. *The Lagrangian and associated first integrals of complex* nth *(*$n \geq 2$*) order ODEs with complex dependent and real independent variable provide Lagrangians and first integrals, respectively, for corresponding two-dimensional systems of* n^{th} *order ODEs.*

Proof. We prove the result for $n = 2$, as its extension to higher orders is trivial. In this case, i.e., for a scalar second order ODE, the Euler–Lagrange equation reads as $\frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right) - \frac{\partial L}{\partial u} = 0$, which expands to

$$L_{xu'} + u'L_{uu'} + u''L_{u'u'} - L_u = 0.$$

By considering $u(x) = y(x) + \imath z(x)$, $L(x, u, u') = L_1(x, y, z, y', z') + \imath L_2(x, y, z, y', z')$, in the above equation and splitting it into the real and imaginary parts, one obtains

$$\begin{aligned}
&\frac{1}{2}(L_{1,xy'} + L_{2,xz'}) + \frac{y'}{4}(L_{1,yy'} + L_{2,y'z} + L_{2,yz'} - L_{1,zz'}) - \frac{z'}{4}(L_{2,yy'} - L_{1,y'z} - L_{1,yz'} - L_{2,zz'}) \\
&+\frac{y''}{4}(L_{1,y'y'} + L_{2,y'z'} + L_{2,y'z'} - L_{1,z'z'}) - \frac{z''}{4}(L_{2,y'y'} - L_{1,y'z'} - L_{1,y'z'} - L_{2,z'z'}) - \frac{1}{2}(L_{1,y} + L_{2,z}) = 0, \\
&\frac{1}{2}(L_{2,xy'} - L_{1,xz'}) + \frac{y'}{4}(L_{2,yy'} - L_{1,y'z} - L_{1,yz'} - L_{2,zz'}) + \frac{z'}{4}(L_{1,yy'} + L_{2,y'z} + L_{2,yz'} - L_{1,zz'}) \\
&+\frac{y''}{4}(L_{2,y'y'} - L_{1,y'z'} - L_{1,y'z'} - L_{2,z'z'}) + \frac{z''}{4}(L_{1,y'y'} + L_{2,y'z'} + L_{2,y'z'} - L_{1,z'z'}) - \frac{1}{2}(L_{2,y} - L_{1,z}) = 0.
\end{aligned}$$

Both of the above equations reduce to

$$\begin{aligned}
L_{i,xy'} + y'L_{i,yy'} + z'L_{i,y'z} + y''L_{i,y'y'} + z''L_{i,y'z'} - L_{i,y} = 0, \\
L_{i,xz'} + y'L_{i,yz'} + z'L_{i,zz'} + y''L_{i,y'z'} + z''L_{i,z'z'} - L_{i,z} = 0,
\end{aligned}$$

for $i = 1, 2$, by employing the CR-equations $L_{1,y} = L_{2,z}$, $L_{1,z} = -L_{2,y}$, $L_{1,y'} = L_{2,z'}$, and $L_{1,z'} = -L_{2,y'}$. Notice that these are Euler–Lagrange Equations (4) for two-dimensional systems of second order ODEs. Hence, the real and imaginary parts L_i, for $i = 1, 2$, of a complex Lagrangian $L(x, u, u')$ satisfy the Euler–Lagrange equations for systems obtainable from complex scalar equations. In other words, the Euler–Lagrange Equations (10) become the Euler–Lagrange Equations (4). A similar argument applies to first integrals I_i, for $i = 1, 2$, obtained for a system of two second order ODEs from complex first integral $I(x, u, u')$, of a scalar second order complex equation which satisfy $DI = 0$, i.e.,

$$I_x + u'I_u + u''I_{u'} = 0.$$

Considering $u(x) = y(x) + \iota(x)$, $I(x,u,u') = I_1(x,y,z,y',z') + \iota I_2(x,y,z,y',z')$, $D = D_1 + \iota D_2$, and splitting into the real and imaginary parts and employing CR-equations $I_{1,y} = I_{2,z}$, $I_{1,z} = -I_{2,y}$, $I_{1,y'} = I_{2,z'}$, $I_{1,z'} = -I_{2,y'}$, yields

$$I_{i,x} + y' I_{i,y} + z' I_{i,z} + y'' I_{i,y'} + z'' I_{i,z'} = 0, \quad i = 1,2.$$

This is exactly the criterion whose first integrals of a system of two second order ODEs satisfy $DI_1 = DI_2 = 0$, where D is the derivative operator for such systems given in (6). □

The above result is extendable to higher dimensional systems of ODEs of order more than two as the CR-equations and their derivatives establish a connection between Lagrangians and first integrals of the complex base equations and the corresponding systems. Therefore, for those systems (of *nth* order ODEs) that correspond to complex DEs (of the same order), their Lagrangians and first integrals are obtainable from the complex Lagrangian and first integrals of the base equations.

The base complex equations in Examples (2)–(6) and (8)–(12) admit an eight-dimensional Lie and five-dimensional Noether symmetry algebras. It implies that there exist five first integrals for these scalar equations, which, when considered complex, convert into ten first integrals (as guaranteed by above theorem) of the corresponding two-dimensional systems of second order ODEs. Based on these observations, we can state the following result.

Corollary 1. *For two-dimensional systems of second order ODEs with symmetry algebras of dimension d, ($d < 5$) that are obtainable from complex linearizable scalar ODEs, a complex Noether approach provides more first integrals than the real symmetry method.*

Theorem 5. *The real and imaginary parts of the complex Noether symmetries of the complex scalar second order ODEs are not necessarily the Noether symmetries of the corresponding two-dimensional systems of second order ODEs.*

Proof. A complex first integral $I(x,u,u')$ satisfies the invariance criterion $\mathbf{Z}^{[1]} I = 0$, where

$$\mathbf{Z}^{[1]} = \xi \partial_x + \eta' \partial_{u'} + \eta'' \partial_{u''}$$

is the first extension of the Noether symmetry of a second order complex ODE. Splitting it into the real and imaginary parts leads to two invariance conditions (15) that expand to

$$\begin{aligned}
&\xi_1 I_{1,x} - \xi_2 I_{2,x} + \frac{1}{2}\{\eta_1(I_{1,y} + I_{2,z}) - \eta_2(I_{2,y} - I_{1,z}) + \eta_1'(I_{1,y'} + I_{2,z'}) - \eta_2'(I_{2,y'} - I_{1,z'})\} = 0,\\
&\xi_1 I_{2,x} + \xi_2 I_{1,x} + \frac{1}{2}\{\eta_1(I_{2,y} - I_{1,z}) + \eta_2(I_{1,y} + I_{2,z}) + \eta_1'(I_{2,y'} - I_{1,z'}) + \eta_2'(I_{1,y'} + I_{2,z'})\} = 0,
\end{aligned}$$

respectively, where $\mathbf{X}^{[1]}$, and $\mathbf{Y}^{[1]}$, are the operators given in (11). Applying the CR-equations on I_1, and I_2, the above equations become

$$\begin{aligned}
&\xi_1 I_{1,x} - \xi_2 I_{2,x} + \eta_1 I_{1,y} + \eta_2 I_{1,z} + \eta_1' I_{1,y'} + \eta_2' I_{1,z'} = 0,\\
&\xi_1 I_{2,x} + \xi_2 I_{1,x} + \eta_1 I_{2,y} + \eta_2 I_{2,z} + \eta_1' I_{2,y'} + \eta_2' I_{2,z'} = 0,
\end{aligned}$$

while the real invariance criterion for systems reads as $\mathbf{X}^{[1]} I_i = 0$, for $i = 1,2$, which yields two equations

$$\begin{aligned}
&\xi I_{1,x} + \eta_1 I_{1,y} + \eta_2 I_{1,z} + \eta_1' I_{1,y'} + \eta_2' I_{1,z'} = 0,\\
&\xi I_{2,x} + \eta_1 I_{2,y} + \eta_2 I_{2,z} + \eta_1' I_{2,y'} + \eta_2' I_{2,z'} = 0.
\end{aligned}$$

A comparison of these equations with the previous two implies that the real and imaginary parts of a complex Noether symmetry of the base scalar equation split into two Noether symmetries for the corresponding system of

ODEs only if $\xi_2 = 0$, which implies that, if the infinitesimal coordinate ξ, of a complex Noether symmetry is a function of both the real independent variable x, and the complex dependent variable $u(x)$, then it does not split into Noether symmetries for the corresponding system. $\square$

Author Contributions: Conceptualization, M.S. and A.Q.; methodology, M.S. and A.Q.; software, M.S.; validation, M.U.F.; formal analysis, M.U.F. and M.S.; investigation, M.S. and M.U.F.; resources, M.S.; writing—original draft preparation, M.S. and A.Q.; review and editing, A.Q.; supervision, A.Q.

Funding: This research received no external funding.

Acknowledgments: We thank the anonymous referees for their useful suggestions, which helped with refining the results and presentation of the article.

Conflicts of Interest: The authors declare no conflict of interest.

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